Lecture 14

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1 Recap

Last week we looked briefly at equilibrium models of galaxy evolution. In particular we saw that if we postulate an equilibrium between gas infall, star formation and gas outflow, then the cosmological infall of baryons into halos predict a simple scaling relation between the star formation rate and the mass of the halo a galaxy resides in. We did note that this gave the wrong trend for the specific star formation rate with stellar mass as compared to observations, but when we included a function to account for the heating of gas by virial shocks the resulting trend was in much better agreement with that seen in observations.

We then moved on to the intergalactic medium and started with a discussion of the physical properties of Lyman-α absorbers, in particular the Lyman-α forest. We argued that the characteristic size of such structures was related to the Jeans length of the medium and through this argument we could estimate the sizes and masses of these structures. We found that the structures associated with faint Ly-α absorbers, $N_{\mathrm{HI}} \sim 10^{14}$ cm^{-2} were large, over 100 kpc in characteristic size with the size scaling as $N_{\mathrm{HI}}^{-1/3}$.

Today we will explore briefly what happens when these absorbers become so numerous that they effectively absorb all light blueward of Ly-α, an effect known as the Gunn-Peterson effect. This will give us useful insight on the properties of the IGM at high redshift and provide us with a path to understanding the process of re-ionisation.

2 The Gunn-Peterson effect

The name of this effect comes from a paper by Gunn & Peterson (1965, ApJ, 142, 1633). At the time, it had become clear that some QSOs lie at very high redshift and that the Ly-α line of hydrogen would be redshifted to optical wavelengths, thus possible to observe. They showed what we will derive here, namely that even very modest amounts of neutral hydrogen is sufficient to absorb a lot of light and it can therefore be used as a probe of the intergalactic medium between us and the high-redshift object.

When light traverses a region with neutral hydrogen gas, it will suffer significant extinction through Ly-α-absorption. If a total flux, F, traverses a length Δl in a region with hydrogen gas density n_H, it will lose an amount,

$$\Delta F = -F n_H \sigma \Delta l, \tag{1}$$

where σ is the cross-section for absorption. Going to infinitesimal quantities we find that the final flux is

$$F^{\rm out} = F_0 e^{-\int n_H \sigma dl} = F_0 e^{-\tau}, \tag{2}$$

where F_0 is the unattenuated flux, and where we have defined

$$\tau \stackrel{\rm def}{=} \int n_H \sigma dl. \tag{3}$$

The cross-section for absorption at a resonance line where the absorption cross-section is sharply peaked, can be written generally as

$$\sigma(\nu) = \frac{\pi e^2}{m_e c} f \phi(\nu), \tag{4}$$

where I have used Gaussian units as is common in astronomy[1]. Here $\phi(\nu)$ is the line profile function which is normalised to an integral of unity, f is the oscillator strength of the line, which is 0.416 for Ly-α, and the other symbols have their usual meaning.

For hydrogen it is convenient to rewrite this in the form

$$\sigma(\nu) = \sigma_0 \nu_0 \phi(\nu - \nu_0), \tag{5}$$

where ν_0 is the wavelength of the line. For relatively weak absorption we can approximate $\phi(\nu)$ with a delta function, δ_D so get

$$\sigma(\nu) \approx \sigma_0 \nu_0 \delta_D(\nu - \nu_0). \tag{6}$$

The dominant line for us is the 1s–2p transition in hydrogen, normally referred to as the Ly-α line. The wavelength of the transition is $\lambda_0 = 1215.7$Å, and its associated energy is 10.2 eV. The cross section is $\sigma_0 \approx 4.5 \times 10^{-18}\,\mathrm{cm}^2$.

The integral in equation (3) is along the line of sight towards us and we have

$$\tau(\nu_{\rm obs}) = \int_0^l n_{\rm HI} \sigma(\nu) dl, \tag{7}$$

which we can rewrite by using $\nu = \nu_{\rm obs}(1+z)$ and hence $d\nu/\nu_{\rm obs} = dz$, to get

$$\tau(\nu_{\rm obs}) = \int_{\nu_{\rm obs}}^{\nu_{\rm obs}(1+z)} n_{\rm HI}(z) \sigma(\nu) \left| \frac{dl}{dz} \right| \frac{d\nu}{\nu_{\rm obs}}. \tag{8}$$

[1]For quantification it is useful to recall that the electron charge is 4.8032042510^{-10} statColoumbs in these units and that a statColoumb is equation to $\mathrm{erg}^{1/2}\,\mathrm{cm}^{1/2}$.

If there is no very strong absorption, so that the details of the line profile are unimportant we can use the simplication of equation (6) which gives

$$\tau(\nu_{\rm obs}) = \frac{\sigma_0 \nu_0}{\nu_{\rm obs}} n_{\rm HI}(z) \left| \frac{dl}{dz} \right|, \tag{9}$$

where z is the given by $\nu_0 = \nu_{\rm obs}(1+z)$. What is immediately clear is that this absorption can be used to trace density along the line of sight.

The transformation above assumed that there is a straight match between frequency and redshift. This is not correct if peculiar velocities are taken into account. However the derivation for the Gunn-Peterson effect normally assumes a uniform medium and ignores the effect of peculiar velocities. In that case we can continue by writing

$$dl = cdt = c\frac{dt}{da}\frac{da}{dz}dz = -\frac{c}{H(z)}\frac{dz}{1+z}, \tag{10}$$

where $H(z) = \dot{a}/a$. Inserting this into equation (9) we get

$$\tau(\nu_{\rm obs}) = \sigma_0 \left(\frac{c}{H_0}\right) \left(\frac{H_0}{H(z)}\right) n_{\rm HI}(z) \tag{11}$$

We now need an expression for $n_{\rm HI}$. The natural procedure is to relate it to the average density along the line of sight, which gives

$$n_{\rm HI} = \frac{n_{\rm HI}}{n_{\rm H}} n_{\rm H} = \frac{n_{\rm HI}}{n_{\rm H}} \frac{n_{\rm H}}{\bar{n}_{\rm H}} \bar{n}_{\rm H}. \tag{12}$$

In this expression $n_{\rm HI}/n_{\rm H}$ quantifies the neutral fraction, while $n_{\rm H}/\bar{n}_{\rm H}$ is a measure of the overdensity, which for the conditions we are considering here is ≈ 1. $\bar{n}_{\rm H}$ denotes the mean hydrogen density in the Universe. This can be related to the baryon density through

$$\bar{n}_{\rm H} = \frac{\bar{\rho}_B X}{m_P} = \frac{X}{m_P} \Omega_b \rho_{\rm crit} (1+z)^3, \tag{13}$$

where X is the hydrogen mass fraction and Ω_b is the mass density in baryons relative to the critical density, $\rho_{\rm crit}$. Putting in numbers we find

$$\bar{n}_{\rm H} \approx 1.2 \times 10^{-5}\ {\rm cm}^{-3} \left(\frac{X}{0.75}\right) \left(\frac{\Omega_b h^2}{0.023}\right) \left(\frac{1+z}{4}\right)^3, \tag{14}$$

which by insertion in equation (11), gives:

$$\tau(\nu_{\rm obs}) \approx 5.2 \times 10^5 h^{-1} \frac{n_{\rm HI}}{n_{\rm H}} \frac{n_{\rm H}}{\bar{n}_{\rm H}} \left(\frac{H_0}{H(z)}\right) \left(\frac{X}{0.75}\right) \left(\frac{\Omega_b h^2}{0.023}\right) \left(\frac{1+z}{4}\right)^3. \tag{15}$$

If we focus on the high redshift Universe, we can then use the approximation

$$H(z)^2 \approx H_0^2 \Omega_m (1+z)^3, \tag{16}$$

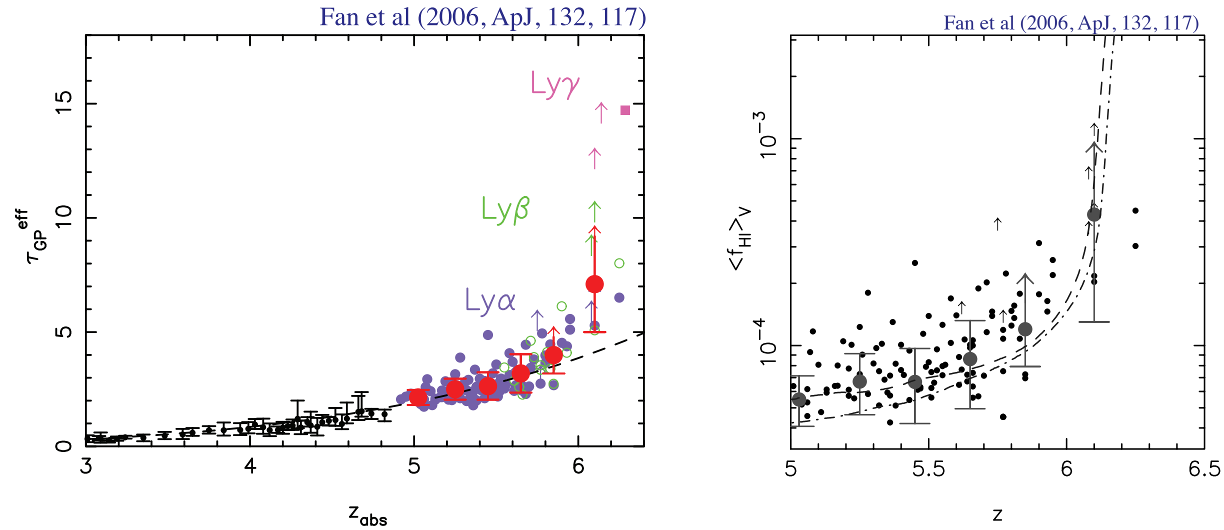

Figure 1: *Left:* The effective Gunn-Peterson optical depth derived for a set of high redshift quasars by Fan et al (2006, ApJ, 132, 117). The different coloured circles and arrows without error bars refer to different Lyman lines and the red circles with error bars to the medians in bins in redshift. *Right:* The average neutral fraction as a function of redshift inferred from the sample on the left.

and thus get

$$\tau(\nu_{\rm obs}) \approx 1.7 \times 10^5 \frac{n_{\rm HI}}{n_{\rm H}} \frac{n_{\rm H}}{\bar{n}_{\rm H}} \left(\frac{\Omega_m h^2}{0.147}\right)^{-1/2} \left(\frac{\Omega_b h^2}{0.023}\right) \left(\frac{1+z}{4}\right)^{3/2} \left(\frac{X}{0.75}\right). \tag{17}$$

Now imagine you have high signal-to-noise (S/N) spectrum, say you can achieve $S/N = 100$. You can then say that a signal is consistent with zero of $F/F_0 < 0.01$. This would be a 1σ limit — a more robust limit would be set by the 3σ limit, $F/F_0 < 0.03$, but in either case the optical depth will be moderate $\tau \gtrsim 4.6$ for the 1σ limit and $\tau \gtrsim 3.5$ for the 3σ case.

Inserting this into equation (17), we find that this limit corresponds to $n_{\rm HI}/n_{\rm H} \gtrsim 2$–$3 \times 10^{-5}$. If we wanted to put constraints on the neutral fraction, $n_{\rm HI}/n_{\rm H}$ of, say, 10%, we would need to measure $\tau > 10^4$ which means flux attenuations of e^{-10^4} which obviously is impossible.

The "problem" here is that Ly-α is very efficient at absorbing photons. It is therefore sufficient with only a very small amount of neutral material to absorb light at and shortwards of Ly-α very efficiently.

It is possible to make some progress using higher order Lyman lines, and also using more complex techniques for measuring the attenuation (see Fan et al 2006, ARA&A, 44, 415 and references therein), but you gain only a factor of ~ 20 by going to Ly-γ, and higher order lines are exceedingly hard to measure accurately because they become blended with lower redshift Ly-α absorption.

By measuring the Gunn-Peterson effect in distant quasars we can nevertheless get some insight into the properties of the IGM at those redshift. From the QSO spectra it is possible to estimate the decrement in flux due to absorption and use equation (17) (or usually some modification of it) to infer the neutral fraction of the Universe. In Figure 1 we shown this process based on a paper by Fan et al (2006, ApJ, 132, 117). The left hand plot shows the effective optical depth measured for a sample of quasars, while the right hand panel shows the inferred neutral fraction. What is obvious from these plots is that the neutral fraction increases rapidly at high redshift and this has been taken by some to mean that we are approaching the epoch of re-ionisation. ionise That is not a robust inference however, as we are still talking about neutral fractions of only $\sim 10^{-3}$, but what can be said is that this is evidence that reionisation at least must have mostly been finished by $z \sim 6$ because the Universe is then mostly ionised.

To improve on the Lyman line absorption, it would in theory also be possible to use metal lines instead of hydrogen lines, as these would saturate at higher densities. However as you go back in time, the abundance of metals is also expected to decrease and separating those two effects, the change strength of lines due to a higher column of absorbing material, and due to a decrease in metal abundance, is not easy to do.

2.1 Polarization of the Cosmic Microwave Background

The Cosmic Microwave Background (CMB) offers a source of photons that cover the whole sky, and this can be used to probe the intervening IGM out to the epoch of reionisation. It is a rather different method

The CMB photons will interact with free electrons in the IGM they pass through (as we will see later, they will also interact with other matter of course). They will scatter due to Thompson scattering and we can calculate the optical depth to Thompson scattering in a very similar way to what we did for the Gunn-Peterson effect above. We replace the absorption cross-section, $\sigma(\nu)$, with the Thompson cross section, σ_T, and we replace the neutral hydrogen density, n_{HI}, with the electron density, n_e. This gives us

$$\tau(z) \approx 0.063 h\Omega_b \int_0^z \frac{H_0}{H(z)}(1+z)^2 \left(\frac{\rho}{\bar{\rho}}\right) f_e(z) dz, \tag{18}$$

where ρ is the background density and $\bar{\rho}$ is the mean background density, thus their ratio is the overdensity, and $f_e(z)$ is the electron fraction.

This Thompson scattering leads firstly to an overall damping of the CMB power spectrum, but arguably more importantly, it causes an overall polarisation of the CMB. To get a quantitative estimate of this it is necessary to have a model for the process of reionisation. If we for simplicity assume that the reionisation happened instantaneously at $z = z_{\mathrm{ri}}$, so that $f_e(z) = 0$ at $z > z_{\mathrm{ri}}$ and 1 later, we get

$$\tau(z_{\mathrm{ri}}) \approx 0.08 \left(\frac{1+z_{\mathrm{ri}}}{1+10}\right)^{3/2}. \tag{19}$$

We can then interpret this as follows (see Zaldarriaga 1997, Phys. Rev. D 55, 1822): If the optical depth is $\tau = 0$, then all photons reaching us today comes from the surface of last scattering, ie. the period of recombination. Correspondingly, if $\tau \approx 1$, then most photons that reach us would have been scattered during the period of re-ionisation. It turns out that the fraction of photons that were scattered at the epoch of re-ionisation is $1 - e^{-\tau(z_{\rm ri})}$. Thus with the numbers in equation (19) we find that about 8% of the photons were scattered during re-ionisation, if it happened at $z = 10$. This leads to a polarisation signal of the order of $\sim 10\%$.

Although in reality the details are a bit more complex than what we covered here, this optical depth to Thompson scattering is eminently measurable by todays CMB experiments and the results from the 7-year WMAP results is $\tau \approx 0.088 \pm 0.015$, which if you insert it into equation (19) above results in a redshift of reionisation of $9.345 < z_{\rm ri} < 12$ (Komatsu et al 2011, ApJS, 192, 1)[2].

2.2 Hydrogen 21 cm absorption and emission

A comprehensive (120 pages) review of the 21 cm transition can be found in Furlanetto, Oh & Briggs (2006, Phys. Rep. 433, 181. A shorter overview can be found in the Fan et al (2006, ARA&A, 44, 415) article

Photons originating at high redshift that lie in the radio domain, in particular those of the CMB, will also interact with foreground neutral matter. In particular with the 21 cm transition of hydrogen. This is fast becoming one of the main probes of the period of reionisation and high redshift IGM although it is still observationally very challenging.

The 21 cm radiation comes from a transition where the electron in hydrogen flips its spin relative to the nucleus. This transition has a very low energy threshold of $\Delta E \approx 5.9 \times 10^{-6}$eV and is widely used in radio astronomy. For our purposes a very crucial aspect of this line is that it does not saturate, thus it can be used, at least in principle, to probe high neutral fractions.

Again the same derivation can be carried out estimate the optical depth to the 21 cm line, although in this case we have to also include the contribution from stimulated emission. In this case one finds that

$$\tau \approx 0.3\, T_S^{-1}\, \frac{n_{\rm HI}}{n_{\rm H}} \left(\frac{\rho}{\bar{\rho}}\right) \left(\frac{1+z}{1+9}\right)^{3/2} \tag{20}$$

where T_S is the spin temperature. Like the derivation for the Gunn-Peterson effect for Ly-α-absorption, this equation assumes no peculiar velocities.

The spin temperature is defined by the Boltzmann equation, so that the number of hydrogen atoms in the upper level, n_1, to the lower level, n_0 is given by

$$\frac{n_1}{n_0} = \frac{g_1}{g_0} e^{-\Delta E/kT_S}, \tag{21}$$

[2] Note that the numbers in Komatsu et al for $z_{\rm ri}$ differs somewhat because they are not obtained using the simplified expression equation (19).

where $g_1 = 3$ and $g_0 = 1$. The spin temperature is particularly important in that it determines whether the 21 cm line will appear in emission or in absorption on the CMB. If the spin temperature is lower than the temperature of the CMB, the 21 cm line will appear in absorption against the CMB. If it is higher than the CMB temperature at that redshift, it will appear in emission.

The spin temperature is expected to be close to the kinetic temperature of matter, T_K in general. At high redshift this is expected to be lower than that of the CMB because the matter temperature declines as $T_K \propto (1+z)^2$ after $z \sim 100$. When energetic sources start to inject energy into the IGM during re-ionisation, the matter temperature will go up and will normally be considerably above the CMB temperature — leading to 21 cm in emission. Thus we expect that the 21 cm line will primarily be observed in emission when exploring the epoch of re-ionisation.

One important case where absorption would be important, is in the case of very high redshift radio galaxies. If there were luminous radio sources at $z > 6$, say, they would provide sensitive probes of the state of the IGM at high redshift.

Whatever the actual state, it is clear that probing the IGM at high redshift using the 21 cm line is of major interest, and a number of facilities will aim to detect the 21 cm line from the epoch of re-ionisation.

3 The thermal balance of the IGM

The thermal balance of the IGM is of major importance, in part because it allows us to gain some insight into the heating sources that inject energy into the IGM.

3.1 Photo-ionisation equilibrium

If we start out with considering a pure H gas, then in photoionisation equilibrium we have that

$$\frac{n_{\mathrm{HI}}}{t_{\mathrm{ion}}} = \frac{n_{\mathrm{HII}}}{t_{\mathrm{rec}}}, \tag{22}$$

where t_{ion} is the time-scale for ionisation and t_{rec} is the time-scale for recombinations. Thus the equation states that the number of ionisations should equal the number of recombinations. Adding He to this relationship is straightforward but considering H alone is simpler and sufficient for what we want here.

If the gas is highly ionised, as will be the case for the IGM in general, we have that $n_{\mathrm{HII}} \approx n_{\mathrm{H}}$ and hence

$$\frac{n_{\mathrm{HI}}}{n_{\mathrm{H}}} \approx \frac{t_{\mathrm{ion}}}{t_{\mathrm{rec}}}. \tag{23}$$

The ionisation time-scale depends on the presence of sources and the intensity of their radiation, while the recombination time-scale is set by atomic physics. Specifically the recombination time-scale is given by

$$t_{\mathrm{rec}} = \frac{1}{n_e \alpha_r}, \tag{24}$$

where $\alpha_r \approx 4 \times 10^{-13} T_4^{-0.7}$ cm^3/s, with the temperature written as $T = T_4 \times 10^4$K.

If the gas is fully ionised, then we have shown previously that we can write

$$n_e = \frac{\rho_b}{m_P}\left(X + \frac{2(1-X)}{4}\right) = n\left(\frac{1+X}{2}\right), \tag{25}$$

where X is the hydrogen mass fraction. This then gives a recombination time-scale of

$$t_{\rm rec} \approx 5.5 \times 10^9 \text{ yr} \left(\frac{n_{\rm H}}{\bar{n}_{\rm H}}\right)^{-1} \left(\frac{1+z}{4}\right)^{-3} T_4^{0.7} \left(\frac{1+X}{1.75}\right)^{-1} \left(\frac{\Omega_b h^2}{0.023}\right)^{-1}, \tag{26}$$

where I have used that $n/\bar{n} = n_{\rm H}/\bar{n}_{\rm H}$.

This is the relevant time-scale only as long as it is comparable to or shorter than the Hubble time at redshift z. This is given by

$$t_H \approx 1.5 \times 10^9 h^{-1} \text{ yr} \left(\frac{\Omega_m}{0.3}\right)^{-1/2} \left(\frac{1+z}{4}\right)^{-3/2}, \tag{27}$$

so we find that the ratio of the two can be written

$$\frac{t_{\rm rec}}{t_H} \approx 3.6 h T_4^{0.7} \left(\frac{n_{\rm H}}{\bar{n}_{\rm H}}\right)^{-1} \left(\frac{\Omega_m}{0.3}\right)^{1/2} \left(\frac{1+z}{4}\right)^{-3/2} \left(\frac{\Omega_b}{0.023}\right)^{-1} \left(\frac{1+X}{1.75}\right)^{-1}. \tag{28}$$

Since this is > 1 it is in fact questionable to make the assumption of steady state, and indeed it is not to be expected that the Universe as a whole will be in photoionisation equilibrium. However in small enough patches it will be acceptable and it is good enough for getting the approximate scalings right.

Observationally it is found that the Gunn-Peterson optical depth at $z \sim 3$ is $\tau_{\rm GP} \approx 0.3$ (e.g. Songaila 2004, ApJ, 127, 2598). Putting this into equation (17), we get that the typical neutral fraction at this redshift is $n_{\rm HI}/n_{\rm H} \approx 2.3 \times 10^{-6}$. If we also assume that the temperature is $T_4 \sim 2$, we get that

$$t_{\rm rec} \sim 9.3 \times 10^9 \text{ yr}, \tag{29}$$

and hence that $t_{\rm ion}$ can be estimated as

$$t_{\rm ion} = \frac{n_{\rm HI}}{n_{\rm H}} t_{\rm rec} \approx 2 \times 10^4 \text{ yr}, \tag{30}$$

and it is common to define the ionisation rate, $\Gamma \stackrel{\rm def}{=} t_{\rm ion}^{-1}$, and we find that $\Gamma \approx 10^{-12}$ s^{-1}, which leads us to define $\Gamma_{-12} = 10^{12}\Gamma$.

Using this definition we can now write the neutral fraction as

$$\frac{n_{\rm HI}}{n_{\rm H}} = \frac{t_{\rm ion}}{t_{\rm rec}} = 5.8 \times 10^{-6} \frac{n_{\rm H}}{\bar{n}_{\rm H}} \left(\frac{1+X}{1.75}\right) \left(\frac{\Omega_b h^2}{0.023}\right) \left(\frac{1+z}{4}\right)^3 T_4^{-0.7} \Gamma_{-12}^{-1}, \tag{31}$$

which we can use to write that the Gunn-Peterson optical depth is

$$\tau_{\rm GP} \approx 1 \left(\frac{n_{\rm H}}{\bar{n}_{\rm H}}\right)^2 \left(\frac{\Omega_b h^2}{0.023}\right)^2 \left(\frac{\Omega_m h^2}{0.147}\right)^{-1/2} \left(\frac{1+z}{4}\right)^{9/2} \left(\frac{X}{0.75}\right) \left(\frac{1+X}{1.75}\right) \Gamma_{-12}^{-1} T_4^{-0.7}. \quad (32)$$

What is important to note here, is that while Ly-α absorption is less suited to very high neutral fractions, it is a very sensitive probe of mild overdensities and it can therefore be used to build up a map of the gas distribution. This is particularly powerful around $z = 3$, becuase we see from equation (32) that as $z \gg 3$, the absorption will increase significantly, while at $z \ll 3$ there is little absorption and the probe is less efficient.

4 The energy balance of the IGM

The temperature of the IGM is a crucial variable, for instance for predicting the properties of Ly-α-absorbers. In the last lecture I argued that the temperature in the gas responsible for the Ly-α-forest ought to be around 10^4K, but it is unlikely that at gas at different densities have the same temperature so we really would like to know what the T-ρ relation looks like.

One way to do this is to carry out a numerical simulation and Figure 2 shows the result of one recent run. However it is essential to understand where different features in this diagram comes from, in part to understand what aspects are more robust than others. That is the topic of this part of the lecture.

The equations for the energy balance of the IGM can be derived from the first law of thermodynamics

$$dE = dQ - pdV. \quad (33)$$

If we write $E = uV$, where $u = \rho k_B T/(\gamma - 1)\mu m_P$ is the internal energy of the gas, we have

$$d(uV) = \frac{1}{\gamma - 1}\frac{k}{m_P}\left[d(\rho V)\frac{T}{\mu} + \frac{\rho V}{\mu}dT - \rho V T\frac{d\mu}{\mu^2}\right]. \quad (34)$$

The first term in the parenthesis, $d(\rho V)$ is zero as long as mass is conserved so we ignore this in the following. The right hand side of equation (33) will balance the energy input and we can generally write the full equation as

$$\frac{1}{\gamma - 1}\frac{k}{m_P}\left[\frac{\rho V}{\mu}dT - \rho V T\frac{d\mu}{\mu^2}\right] = (\Lambda_h - \Lambda_c)\, V dt - \frac{\rho k_B T}{\mu m_P}dV, \quad (35)$$

where Λ_h is the radiative heating per volume and Λ_c is the radiative cooling term per volume. Dividing this equation by $uVdt$, we get

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{\mu}\frac{d\mu}{dt} + \frac{\Lambda_h - \Lambda_c}{u} - (\gamma - 1)\frac{1}{V}\frac{dV}{dt}. \quad (36)$$

The last term in on the right hand side represents adiabatic cooling due to the expansion of the Universe. The middle term provides the balance between radiative heating and

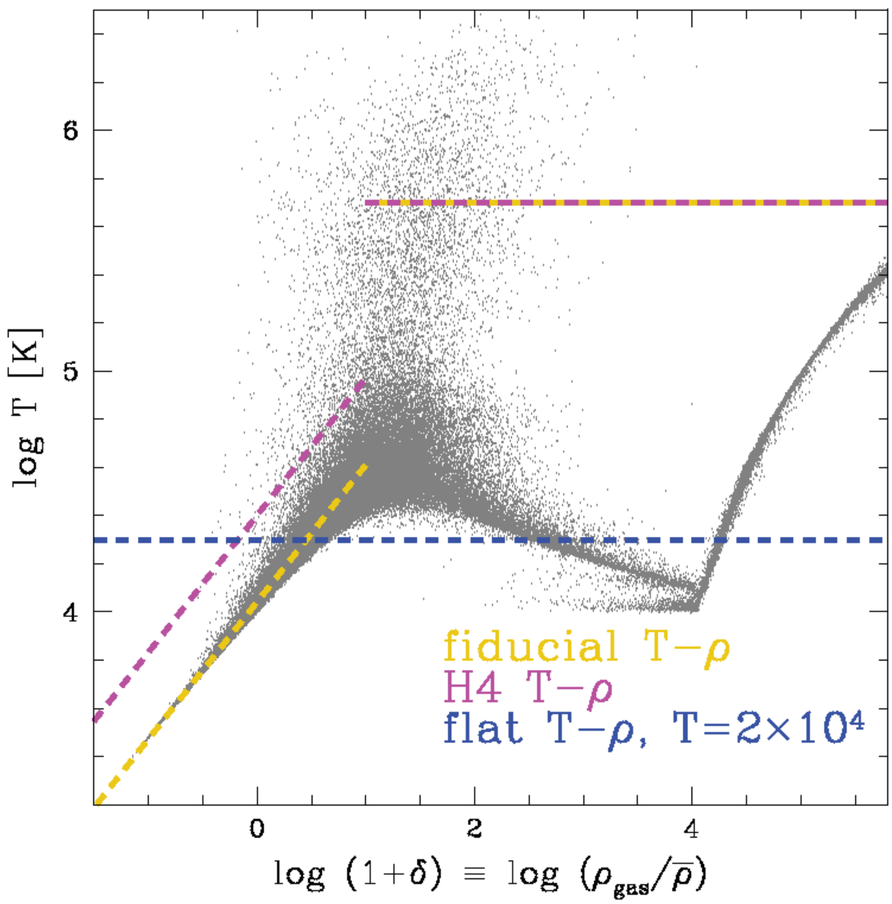

Figure 2: A density-temperature diagram from a recent simulation by Peeples et al (2010, MNRAS, 404, 1281). The various coloured lines show analytical relations used in the paper, what is relevant for us is the distribution of points in the diagram which shows the location of 1% of the particles in the simulation at $z = 3$.

cooling. The first term on the right hand side encapsulate the energy change associated with a change in particle number.

As long as mass is conserved we have

$$\frac{dV}{V} = -\frac{d\rho}{\rho}, \tag{37}$$

and since $\rho = \bar{\rho}(1+\delta)$, we have

$$\frac{d\rho}{\rho} = \frac{d\bar{\rho}}{\bar{\rho}} + \frac{d\delta}{1+\delta} = -3\frac{da}{a} + \frac{\delta}{1+\delta}\frac{da}{a}, \tag{38}$$

where the the second equality follows because $\bar{\rho} \propto a^{-3}$ and $\delta \propto a$ (which is a good approximation at high redshift as long as $\delta \ll 1$). The $\delta/(1+\delta)$ term is going to be very small as long as $\delta \ll 1$ or we consider a fixed δ. In that case we find that

$$-\frac{1}{V}\frac{dV}{dt} \approx -3H(t), \tag{39}$$

and insertion into equation (36) gives us

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{\mu}\frac{d\mu}{dt} + \frac{\Lambda_h - \Lambda_c}{u} - 3(\gamma-1)H(t). \tag{40}$$

In Mo, van den Bosch & White, this equation is given as a derivative in $\ln(1+z)$ which can be easily achieved by using $d/dt = -H(t)d/d\ln(1+z)$.

We can now use equation (40) to look at how the temperature changes with time. If we first consider a virialized halo where $u/\Lambda_c = t_{\rm cool} < t_H \sim 1/H(t)$ and where the matter is fully ionized so that $d\mu/dt \approx 0$, we have

$$\frac{d\ln T}{dt} = -\frac{\Lambda_c - \Lambda_h}{u} = -\frac{1}{t_{\rm cool}}, \tag{41}$$

where we have extended our definition of the cooling time to also account for the heating term. This then can be formally integrated to yield

$$T = T_i e^{-(t-t_i)/t_{\rm cool}}, \tag{42}$$

where subscript i corresponds to the initial values.

After reionisation we expect the temperature of the gas to a few times 10^4K and again we expect the neutral fraction to be very low, so again we can ignore the derivative of μ. However the density in this case is going to be low so the cooling will primarily be inverse Compton cooling as discussed in earlier lectures. So $\Lambda_c \approx \Lambda_{\rm ic}$. This allows us to write

$$\frac{1}{T}\frac{dT}{dt} = \frac{\Lambda_h - \Lambda_{ic}}{u} - 3(\gamma-1)H(t) \tag{43}$$

At sufficiently high redshift inverse Compton cooling is efficient in the sense that the cooling time is shorter than the Hubble time. In this case we get an evolution similar to equation (42). If we are at lower redshift where inverse Compton cooling is inefficient the main cooling source will be adiabatic expansion and much slower.

Turning now to the sources of heating. The main heating source we will consider here is photoionization heating. It is then convenient to relate the rate of photoionization to the ionization flux. This is generally given by

$$\Gamma_{\rm pi} = \int_{\nu_{\rm pi}}^{\infty} \frac{4\pi J(\nu)\sigma_{\rm pi}}{h\nu} d\nu, \tag{44}$$

where the term in the integrand gives the number of photons per second per Hertz. $\nu_{\rm pi}$ is the frequency corresponding to the ionization energy $\nu_{\rm LyC} = 13.6{\rm eV}/h$ for hydrogen; $\sigma_{\rm pi}$ is the photoionization cross section, and for hydrogen this is given by

$$\sigma_{\rm LyC} \approx 6.3 \times 10^{-18} \left(\frac{\nu}{\nu_{\rm LyC}}\right)^{-3} {\rm cm}^2, \tag{45}$$

where LyC denotes the Lyman continuum which corresponds to an energy of 13.6 eV, a wavelength of $\lambda_{\rm LyC} = 912$Å and a frequency of $\nu_{\rm LyC} \approx 3.29 \times 10^{15}$Hz.

It is useful to see how this integral behaves if the ionizing radiation is given by a power-law

$$J(\nu) = \left(\frac{\nu_{\rm LyC}}{\nu}\right)^{\beta} J_{-21}\ {\rm erg\ cm^{-2}\ Hz^{-1}\ sr^{-1}}, \tag{46}$$

which upon insertion in equation (44) gives us

$$\Gamma_{-12} = \frac{12}{\beta+3} J_{-21}. \tag{47}$$

In particular we note that the ionization rate is proportional to the strength of the ionizing radiation.

We can then easily calculate the energy input from photoionization by writing (now specializing to hydrogen):

$$\Lambda_{h,H} = n_{\rm HI} \int_{\nu_{\rm LyC}}^{\infty} \frac{4\pi J(\nu)\sigma_{\rm LyC}(\nu)}{h\nu} h(\nu - \nu_{\rm LyC}) d\nu, \tag{48}$$

where the integrand in equation (44) has been augmented by the energy per photon. We also see that this expression is proportional to the density of hydrogen, and not to the density squared as is true for most cooling processes.

Again, if we have photoionization equilibrium, we have that the number of ionizations should balance the number of recombinations, or in other words

$$n_{\rm HI}\Gamma = \alpha_r n_e n_{\rm HII}. \tag{49}$$

If our gas is nearly fully ionized, then n_e and $n_{\rm HII}$ are both $\propto \rho$. Thus we have that

$$n_{\rm HI} \propto \rho^2 T^{-0.7} \Gamma^{-1}, \tag{50}$$

and since $J(\nu) \propto \Gamma$, we have

$$\Lambda_{h,H} \propto \rho^2 T^{-0.7}. \tag{51}$$

With this background we can now in principle solve equation (40). The simplest case is to ask what one would expect in thermodynamical equilibrium. In this case, $dT/dt = 0$, regardless of the density. If we assume fully ionized gas, then we can observe from equation (40) that $H(t)$ is independent of ρ and $\Lambda_{ic}/u \propto n_e/n$ is also independent of density for a fully ionized gas, this leaves the heating term.

To be able to ensure that $dT/dt = 0$ at one density implies that it holds at all densities, as is required for thermodynamical equilibrium, we need that

$$\frac{\Lambda_{h,H}}{u} \propto \frac{\rho^2 T^{-0.7}}{\rho T} \propto \rho T^{-1.7} \tag{52}$$

is independent of density. This is satisfied if $T \propto \rho^{0.59}$. We therefore expect that in thermodynamic equilibrium that there is a relationship between temperature and density like that.

4.1 Summarising the thermal state of the IGM

If we now return to the theoretical calculation shown in Figure 2 we can try to make some sense of it all. As we will see below, at re-ionization the input of energy will lead to approximately the same energy per baryon and hence an approximately constant temperature at various densities with $T \sim 10^4$K. Subsequently the gas will slowly cool due to adiabatic expansion and a thermodynamic equilibrium is set up so that $T \propto \rho^{0.59}$.

Winds blowing into the IGM and other processes of shock formation will heat the gas around forming galaxies, when the density is sufficiently high this gas will cool down again but at low density the temperature will remain high which you can se around $1 < \delta < 2$.

At slightly higher densities, the gas will start forming galaxies and start collapsing and the temperature will decrease because the higher densities lead to a stronger cooling. Then at very high densities where virialised halos have formed, the gas will start getting hotter as the density increases because the gas will be heated to the virial temperature.