# Lecture 12

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## 1 Recap

Last lecture we discussed first the effect of supernova and AGN feedback, with a particular example being made for the energetics of feedback in changing the inner structure of dark matter halos. We then introduced the spin parameter, $\lambda$, whose form we justified by looking at the ratio of the spin of a structure to that if all constituents were on circular orbits. We commented that the spin parameter of dark matter halos was found to be quite low in general and that in order to reproduce the spin parameter seen in spirals in the present day Universe a significant spin-up must take place. However we also found that in the absence of dark matter this spin up would take too long, while including dark matter would make things match spirals in the present day Universe.

## 2 The origin of spin

*See Peacock 17.2, Mo, van den Bosch & White 7.5.4*

But how do you get this initial spin in the first place? The Universe as a whole is expected to be irrotational so one could imagine that dark matter halos do not spin. The generally accepted theory for the initial spin up of dark matter halos is that it is due to tidal forces between different collapsed structures.

A simple way to illustrate this (see Peacock 17.2 for more details) is to smooth the density field on scale $R$, with associated mass $M$. The typical size of perturbations will then be $R$. Let us further assume that the mass in each object is distributed as a dipole, with half the mass at one end and the other half the other end. In that case the tidal acceleration from the near side of the dipole on another structure can be written

$$a_{\rm tidal} \sim \frac{2GM\delta}{L^3}\Delta L, \qquad (1)$$

where $L = aR$ is the proper size of the system and $\Delta L = aR/2$ is the typical separation. $\delta$ is the overdensity of the structure — this is needed because only the mass corresponding to the density above the mean, $M\delta$, can contribute to the tidal field.

To get the torque, or time-derivative of the angular momentum, we need the lever arm, $aR/2$, and the mass $M/2$. Multiplying these together with $a_{\rm tidal}$ we get

$$\tau = \frac{dJ}{dt} \sim \frac{GM^2\delta}{4aR}. \tag{2}$$

For an Einstein-de Sitter universe, $\delta \propto a$, so in that case $\tau$ is constant and $J \propto t$. Thus the tidal field steadily spins up the halos. This spin-up continues until the structure collapses with a main contribution around turn-around when the objects are the largest.

We can do a slightly better job following White (1984). To start with we write the angular momentum as an integral over a Lagrangian volume (following the individual particles) so if the initial co-moving position is $\vec{q}$ we have

$$J(t) = \int_{V_L} [\vec{r}(\vec{q},t) - \langle \vec{r}(t)\rangle] \times \vec{v}(\vec{q},t)\rho_0 a^3 d^3\vec{q}, \tag{3}$$

where $\rho_0$ is the mean density and $a$ is the scale-factor as usual. It is now useful to re-write this using co-moving coordinates, in which case $\vec{r} = a\vec{x}$ and $\vec{v} = a\dot{\vec{x}}$. Inserting this in equation 3 we get (suppressing $t$ and $\vec{q}$):

$$J(t) = \rho_0 a^5 \int_{V_L} [\vec{x} - \langle \vec{x}\rangle] \times \dot{\vec{x}}\, d^3\vec{q}. \tag{4}$$

To make progress it is necessary to expand this to second order. Since $\dot{\vec{x}} = d\vec{q}/dt + \mathcal{O}(d^2\vec{q}/dt^2)$ it is already first order. We therefore expand $\vec{x} = \vec{q} + \mathcal{O}(d\vec{q}/dt)$ and get to second order

$$J(t) = \rho_0 a^5 \int_{V_L} [\vec{q} - \langle \vec{q}\rangle] \times \dot{\vec{x}}\, d^3\vec{q}. \tag{5}$$

To make further progress here, we need to have a short aside.

## 2.1 An aside: The Zel'dovich approximation

Previously in the course we focused on perturbation theory where we looked at expansion in density. If instead we consider all particles and do a perturbation in position we get the Zel'dovich approximation. This is a very useful approximation for numerical use.

It can be derived from the Euler equation

$$\dot{\vec{v}} + \frac{\dot{a}}{a}\vec{v} = -\frac{\nabla\phi}{a}. \tag{6}$$

For this we note that since the growth of perturbations, $\delta(t,\vec{x})$, can be separated into a time and spatial component, $\delta(t,\vec{x}) = D(t)\delta_i(\vec{x})$, we can write

$$\phi(\vec{x},a) = \frac{D(t)}{a}\phi_i(\vec{x}), \tag{7}$$

where subscript $i$ refers to initial values and $\nabla^2\phi_i = 4\pi G\bar{\rho}_m a^3 \delta_i(\vec{x})$. We can then integrate up the Euler equation (insert the result if you doubt it) and get

$$\vec{v} = -\frac{\nabla\phi_i}{a}\int \frac{D(t)}{a(t)}dt, \tag{8}$$

which can be rewritten (because $D(t)$ satisfies the growth equation of density perturbations) as

$$\vec{v} = -\frac{\dot{D}}{4\pi G\bar{\rho}_m a^2}\nabla\phi_i(\vec{x}). \tag{9}$$

If we integrate this up we get the following expression for the displacement:

$$\vec{r}(t,\vec{q}) = a(t)\left[\vec{q} - b(t)\nabla_{\vec{q}}\phi_i(\vec{q})\right], \tag{10}$$

which basically says that the new position is the starting position plus a term following the gravitational gradient.

## 2.2 Going back — using the Zel'dovich approximation

We now have an expression for $a\dot{\vec{x}} = \vec{v}$ from equation 9. Inserting this into equation 5 we get

$$J(t) = -\rho_m a^5 \frac{\dot{D}}{4\pi G\rho_m a^3}\int_{V_L}\left[\vec{q} - \langle\vec{q}\rangle\right]\times\nabla_{\vec{q}}\phi_i(\vec{q})\,d^3\vec{q}, \tag{11}$$

and converting this to a surface integral we have

$$J(t) \propto a^2\dot{D}\dot{b}\int_{V_L}\phi_i(\vec{q})\ \left[\vec{q} - \langle\vec{q}\rangle\right]\times d\vec{S}. \tag{12}$$

From this we can see that for a spherical volume $J(t)$ vanishes to first order because of symmetry, but in general it will be non-zero.

For the time-dependence, we see that is all encapsulated in $a^2\dot{D}$ which for an Einstein-de Sitter Universe is $\propto t$ which is the same scaling we got from the heuristic argument earlier.

To make further progress it is possible to expand $\phi$ as a Taylor series, see Mo, van den Bosch & White 7.5.4 for details. For an ellipsoidal structure it is possible to show that

$$J(t) = \frac{1}{5}\left(\frac{2G}{H(t)^2\Omega_m(t)}\right)^{2/3}\dot{D}(t)\delta_i M^{5/3}\mathcal{G}_{\{}, \tag{13}$$

where $\mathcal{G}_{\{}$ is a geometric factor. This scaling with $\delta$ and $M$ is the same as in equation 2 if we take the size to be proportional to the virial radius, $aR \propto r_{\rm vir} \propto M^{1/3}$.

# 3 Disk collapse in dark matter halos

*Mo, van den Bosch & White 11.2, which is based on Mo, Mao & White (1998, MNRAS, 295, 319).*

Above we ignored all details of dark matter halo structure. We will now turn to look at disk collapse in more realistic halos. To make things simple we will make two assumptions

- We assume that the halo density profile is isothermal.
- We ignore the self-gravity of the disk. In effect we then assume that the collapse of the disk has no effect on the dark matter halo at all.

We will see later that these are sufficient assumptions to give a qualitative description of disk formation.

Given those assumptions we can write the density distribution of the dark matter halo as

$$\rho(r) = \frac{V_c^2}{4\pi G r^2}, \tag{14}$$

and we also know from our study of spherical collapse that the average density within the virial radius is

$$\langle \rho(r < r_{\rm vir}) \rangle = \Delta_c \Omega_m \rho_{\rm crit}. \tag{15}$$

Combining this with the expression for $\rho_{\rm crit} = 3H(z)^2/8\pi G$, we have that the virial radius is

$$r_{\rm vir} = \sqrt{\frac{2}{\Omega_m \Delta_c}} \frac{V_{\rm vir}}{H(z)} \tag{16}$$

and the mass enclosed within it is

$$M_{\rm vir} = \sqrt{\frac{2}{\Omega_m \Delta_c}} \frac{V_{\rm vir}^3}{G\, H(z)}, \tag{17}$$

where $V_{\rm vir}$ is the circular velocity of halo.

If we write the disk mass as a fraction, $f_d$, of the halo mass[1], we have

$$M_d = f_d M_{\rm halo} = f_d M_{\rm vir}, \tag{18}$$

with $M_{\rm vir}$ given above. Putting in numbers we have

$$M_d \approx 1.3 \times 10^{11}\, M_\odot \left(\frac{f_d}{0.05}\right) \left(\frac{\Delta_c \Omega_m}{100}\right)^{-1/2} \left(\frac{H(z)}{H_0}\right)^{-1} \left(\frac{V_{\rm vir}}{200\,{\rm km/s}}\right)^3. \tag{19}$$

If we assume the disk to be exponential, we can write its surface density as

$$\Sigma(R) = \Sigma_0 e^{-R/R_d} \tag{20}$$

[1] Note that in the literature it is more common to use $m_d$ for the quantity I call $f_d$, but in the lecture I used $f_d$ for clarity on the board so I keep that notation here.

where $R_d$ is referred to as the disk scale-length. For these calculations we typically ignore the vertical structure of the disk and we will here ignore the bulge component of the disk. The mass of this disk is

$$M_d = 2\pi\Sigma_0 R_d^2. \tag{21}$$

The angular momentum of the disk, assuming particles are on circular orbits can be written as

$$J_d = \int \vec{r} V_c(\vec{r}) dM = 2\pi\Sigma_0 \int_0^\infty V_c(R) R^2 e^{-R/R_d} dR \approx 2 R_d M_d V_{\rm vir}, \tag{22}$$

where in the last equality we have made use of our assumption that there is no self-gravity in the disk. In that case the circular velocity profile is that of the halo and it is constant in the case of the isothermal sphere.

If we introduce a parameter $j_d = J_d/J$, where $J$ is the angular momentum of the halo, we can use the definition of $\lambda$ (equation (32)), to write

$$R_d = \frac{G M_{\rm vir}^{3/2}}{2|E|^{1/2} V_{\rm vir}} \left(\frac{j_d}{f_d}\right) \lambda. \tag{23}$$

For an isothermal sphere we have cannot actually define a gravitational binding energy because it would increase without bounds (out to a radius $R$ it is $W = R V_{\rm vir}^4/G$), so it is common to work with a truncated isothermal sphere, which we truncate at the virial radius. Making use of the virial theorem we find that $E = W + K = -M_{\rm vir} V_{\rm vir}^2/2$ for the truncated isothermal sphere. Inserting this into equation (60) and using that $R_{\rm vir} = G M_{\rm vir}/V_{\rm vir}^2$, we have

$$R_d = \frac{1}{\sqrt{2}} \left(\frac{j_d}{f_d}\right) \lambda R_{\rm vir}, \tag{24}$$

where the virial radius is given in equation (48) above, and quantitatively we have

$$R_d \approx 10 h^{-1}\,\mathrm{kpc} \left(\frac{j_d}{f_d}\right) \left(\frac{\lambda}{0.05}\right) \left(\frac{V_{\rm vir}}{200\,\mathrm{km/s}}\right) \left(\frac{\Omega_m \Delta_c}{100}\right)^{-1/2} \left(\frac{H(z)}{H_0}\right)^{-1}, \tag{25}$$

and using equation (21) we find the central mass density to be

$$\Sigma_0 \approx 207 h^{-1}\, M_\odot/\mathrm{pc}^2 \left(\frac{f_d}{0.05}\right) \left(\frac{j_d}{f_d}\right)^{-2} \left(\frac{\lambda}{0.05}\right)^{-2} \left(\frac{V_{\rm vir}}{200\,\mathrm{km/s}}\right) \left(\frac{\Omega_m \Delta_c}{100}\right)^{1/2} \left(\frac{H(z)}{H_0}\right). \tag{26}$$

If we again assume that the angular momentum per mass is conserved, we have that $j_d/f_d = 1$. If we insert values appropriate for the Milky Way we have $V_c \approx 220$km/s, $M_d \approx 5 \times 10^{10}\, M_\odot$ and $R_d \approx 3.5$kpc. If we assume that the flat part of the rotation curvey, $V_c$, is the halo virial velocity we find that $\lambda \approx 0.01$ and $f_d \approx 0.01$.

Is this reasonable? Firstly, from Figure 3 we see that a spin parameter of 0.01 is actually very unlikely — only about 1% of all halos have $\lambda < 0.01$. Secondly, the currently best estimates of the baryon and dark matter density gives that the fraction of baryons in the Universe is $\sim 20\%$. So if $f_d \approx 0.01$, we must conclude that only 5% of all baryons end up in the final disk. This is a very small number and it is hard to see how the process can be so inefficient.

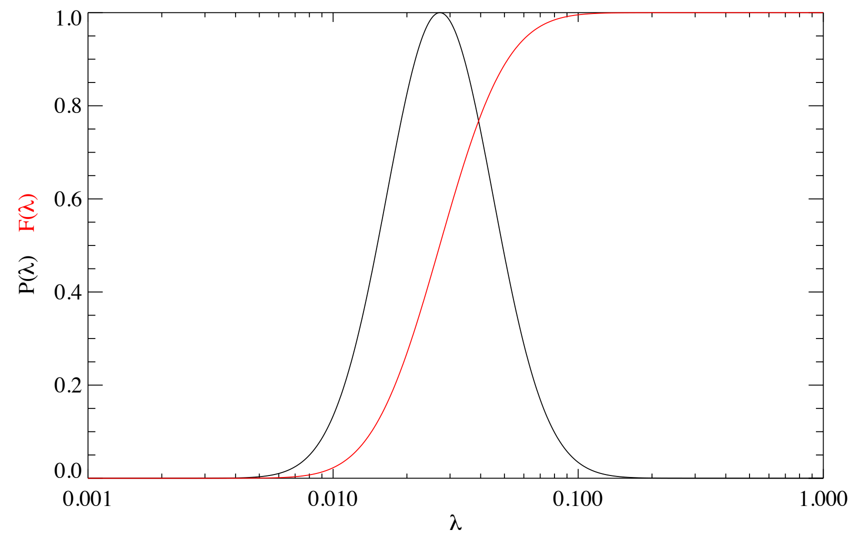


Figure 1: The black line shows the likelihood distribution of the spin parameter, $\lambda$, normalised to a peak of 1, for dark matter halos in the standard $\Lambda$CDM cosmology. The black line shows the cumulative likelihood for $\lambda$.

# 4 Fixing spiral formation

How can we fix this? One possibility is to reject the assumption that angular momentum per mass is conserved. If instead we assume that $j_d/f_d \approx 0.2$, we find a $\lambda \sim 0.05$, which is more reasonable. However we have also made two assumptions above, first by ignorning self-gravity and secondly by assuming that the halo is an isothermal sphere. Both of those are questionable assumptions.

## 4.1 Halo mass profile

In particular, if we use the Navarro-Frenk-White (NFW) profile we looked at in an earlier lecture,

$$\rho(r) = \rho_{\rm crit} \frac{\delta_{\rm char}}{(r/r_s)(1+r/r_s)^2}, \tag{27}$$

with mass

$$M(<r) = 4\pi \rho_{\rm crit} \Omega_m \delta_{\rm char} r_s^3 \left[ \ln(1+cx) - \frac{cx}{1+cx} \right], \tag{28}$$

where $c = r_{\rm vir}/r_s$, $x = r/r_{\rm vir}$ and

$$\delta_{\rm char} = \frac{\Delta_c}{3} \frac{c^3}{\ln(1+c) - c/(1+c)}. \tag{29}$$

The parameter $c$ is known as the concentration parameter and from equation (29) one can see that $r_s$ and $c$ is sufficient to specify the shape of the halo. It is known that $c$ is a function of halo mass and formation time and fits have been provided in the literature for this dependence.

### 4.2 Adiabatic contraction

*Mo, van den Bosch & White 11.1.3; The first careful study of adiabatic contraction was done in Blumenthal et al (1986; ApJ, 301, 27), see e.g. Gnedin et al (2004, ApJ, 616, 16) for a clear recent study.*

When a galaxy forms within a dark matter halo, its presence there will modify the mass distribution of the dark halo. If the collapse is very slow (ie. because gas is cooling out of the halo at a low rate), then adiabatic invariants will be approximately conserved. We ignored this effect also last week

The main adiabatic invariant for our case is the specific angular momentum, $rV_c(r)$.

$$r_i V_i(r_i) = r_f V_f(r_f), \tag{30}$$

where $r_i$ is the initial radius and $V(r)$ is the velocity profile. If we assume that everything is spherically symmetric, then $V(r) \propto \sqrt{M/r}$ means that we can write equation (30) as

$$r_i M_i(< r_i) = r_f M_f(< r_f), \tag{31}$$

where $M(< r)$ is the mass within radius $r$. The initial mass profile is set by the shape of the dark matter potential, and the final mass profile is a combination of the disk and the dark matter — note that we assume that the dark matter doesn't change, it is the total mass profile that will change in response to the collapsing baryons.

Initially the gas and the dark matter will be equally mixed and hence their mass distribution is the same, which allows us to write

$$r_i M_i(< r_i) = \frac{r_i}{1 - f_{\rm gas}} M_{\rm DM}(< r_i), \tag{32}$$

where $M_{\rm DM}$ is the dark matter mass and $f_{\rm gas}$ is the gas fraction in the halo.

After collapse, the mass, $M_f(< r_f)$, is composed of the dark matter plus any diffuse gas that follows the dark matter potential, and the galaxy, which means we can write

$$r_f M_f(< r_f) = r_f \left( \frac{1}{1 - f'_{\rm gas}} M_{\rm DM}(< r_i) + M_d(< r_f) \right). \tag{33}$$

by equation equation (32) and equation (33) we can solve for $r_f$ for any $r_i$ and thereby get the new halo shape. Note that I used $f'_{\rm gas}$ in equation (33) because this corresponds to the smoothly distributed gas (not the one in the disk). It is of course trivially calculated when you have $M_d$.

In general equation (33) will lead to $r_f < r_i$ when mass settles adiabatically in the central part of a halo. If instead there is mass flowing out of the system, or the collapse is far from adiabatic, expansion can occur. At the moment it is probably fair to say that it is uncertain what the true effect is.

The formalism above for adiabatic contraction assumes spherical symmetry and that particles move on circular orbits. Neither of these assumptions are correct in detail, indeed a disk galaxy is quite far from being spherically symmetric. Nonetheless, it appears that while the assumptions are flawed, the formalism gives results in fairly good agreement with numerical simulations and is therefore widely used, although the improved methodology of Gnedin et al (2004) is usually preferred which relaxes some of those assumptions.

# 5 More realistic models for spiral formation

**Note: this section was not covered in the lectures with the exception of the Kennicutt-Schmidt relation and is included here for completeness.**

*Mo, van den Bosch & White 11.2–11.5. Much of the discussion here follows closely Dutton & van den Bosch (2009, MNRAS, 396, 141)*

## 5.1 Why do spirals have exponential mass profiles?

Above we made progress on estimating disk sizes and mass densities by assuming that their mass profiles were exponential in shape. But we did not tackle the question of *why* the disks are exponential. Indeed there is currently no convincing theoretical framework for explaining why disks forming in dark matter haloes have approximately exponential mass distributions.

The two main ideas for the formation of an exponential surface mass density distribution can be summarised as

1. *The exponential shape is set by the initial angular momentum distribution of the dark matter halo.* This would have been a very tidy solution but it turns out be incorrect. Detailed simulations show that there is too much long angular momentum material in dark halos and this will fall into the central regions of the disk under the assumption of angular momentum conservation and this results in a galaxy that is much too peaked in the center to be consistent with a spiral galaxy shae.

   This central concentration of material could end up forming a bulge but that leaves the question of how to form a bulge-less spiral galaxy. Another possibility is that angular momentum is not conserved and that the low angular momentum material is somehow spun up although it is not clear what this effect is. Finally, it is possible that the low angular momentum material is more likely to be ejected from the disk due to feedback effects.

2. *The exponential shape is set by redistribution of angular momentum due to viscuous forces in the disk.* The gaseous content of a disk that rotates differentially will redistribute angular momentum due to the viscuous forces. This redistribution results quite naturally in exponential disks (Lin & Pringle 1987, ApJ, 320, L87). The

problem is that to achieve that you need to start with a disk that is less centrally concentrated than an exponential and as we saw in the previous point, this is not satistified when the disk forms in a realistic dark matter halo.

Both of these explanations probably contain important clues to how disk galaxies acquire their exponential mass distributions but a full theory is still somewhat lacking.

## 5.2 Adding mass to a disk

A dark matter halo is not an isolated entity that evolves in isolation from the rest of the Universe. It will accrete gas and dark matter from its surroundings throughout. Thus to create a reasonable model of spiral galaxy formation we need to incorporate this fact in our model for spiral formation.

To focus our mind, let us consider the evolution of mass in concentric circles, ie. we assume the spiral disk to be axisymmetric. This is a time-dependent quantity so we denote it $\Sigma(R, T)$. When material falls onto the disk we assume that it will conserve angular momentum and this determines where it will land in the disk. The specific angular momentum of a particle with angular momentum $J_i$ and mass $M_i$, $\ell \stackrel{\text{def}}{=} J_i/M_i$ then determines the radius at which the accreted mass will end up through

$$\ell = RV_c(R,t), \tag{34}$$

where $V_c(R,t)$ is the circular velocity profile of the disk galaxy. The distribution of $\ell$ would be denoted $P(\ell, t)$.

The accretion rate onto the disk we denote $\dot{M}_d(t)$ and this clearly will be related to the accretion rate onto the dark matter halo at some earlier time $\dot{M}_{\text{vir}}(t-\Delta t)$, where $\Delta t$ is set either by the longest time-scale of the cooling time and the gravtiational free-fall time. We can calculate both of these using the formalism in previous chapters and for each mass $\Delta M(R)$ added to the disk at radius $R$, we get an increase in $\Sigma(R,t)$ by

$$\Delta\Sigma(R,t) = \frac{\Delta M(R)}{2\pi R \Delta R}. \tag{35}$$

The material ending up at $R$ must satisfy equation (34), thus we can write $\Delta M(R)$ as

$$\Delta M(R) = \dot{M}_d(t) P(\ell, t)\Delta\ell. \tag{36}$$

From equation (34) we also have that

$$d\ell = V_c(R,t)dR + R\frac{dV_c(R,t)}{dR}dR, \tag{37}$$

so combining equations (35)–(37) we get

$$\dot{\Sigma}(R,t) = \frac{\dot{M}_d(t)}{2\pi R} P(\ell,t) R V_c(R,t)\left(1 + \frac{d\ln V_c(R,t)}{d\ln R}\right). \tag{38}$$

$P(\ell, t)$ must be taken from numerical simulations and we know how to calculate $V_c$ given a mass distribution. But we also need $\dot{M}_d(t)$. As mentioned above this must be related to $\dot{M}_{\rm vir}(t-\Delta t)$. It is possible to calculate $\dot{M}_{\rm vir}(t)$ from the extended Press-Schechter formalism discussed in earlier lectures and comparing this to numerical simulations. A good estimate is the relation given by Birnboim et al (2007, MNRAS, 380, 339) which we looked at earlier:

$$\frac{\dot{M}_{\rm vir}}{M_{\rm vir}} \approx 0.04 \left(\frac{M_{\rm vir}}{10^{12} M_\odot}\right)^{0.15} (1+z)^{2.25}\ {\rm Gyr}^{-1}, \tag{39}$$

which is valid for halo masses around $10^{12} M_\odot$.

## 5.3 Star formation in the disk

The preceding subsection provides us with a prescription to calculate $\dot{\Sigma}_{\rm gas}(R,t)$ for the forming spiral galaxy and from there we get $\Sigma_{\rm gas}(R,t)$.

Next, we want to form stars from the gas, and for that we have to make recourse to empirical estimates because we do not have a good a priori theory for star formation. The most common choice is to adopt the Schmidt-Kennicutt relation (see Kennicutt 1998, ApJ, 498, 541) which says that

$$\frac{\Sigma_{\rm SFR}}{M_\odot/{\rm pc}^2/{\rm Gyr}} \approx 0.25 \left(\frac{\Sigma_{\rm gas}}{1 M_\odot/{\rm pc}^2}\right)^{1.4}. \tag{40}$$

Using this prescription we can now estimate the amount of stars formed at each radius, $\Sigma_*(R,t)$.

However it is also clear that the criterion in equation (40) can only apply in situations where the disk is unstable to gravitational disturbances.

## 5.4 Stability of disks

The first question to ask is whether a small patch in a gaseous disk is unstable to gravitational collapse. If one carries out the perturbation analysis for small perturbations of a gaseous disk, one finds that the small patch in the disk is stable to small perturbations if

$$Q = \frac{c_s \kappa}{\pi G \Sigma_0} > 1, \tag{41}$$

where $\Sigma_0$ is the unperturbed gas density, $c_s$ the sound speed, and $\kappa$ the epicyclic frequency:

$$\kappa = \sqrt{2}\Omega(R) \left(1 + \frac{d \log V_c(R)}{d \log R}\right)^{1/2}, \tag{42}$$

where $\Omega(R)$ is the angular frequency at that radius, $\Omega(R) = V_c(R)/R$. Using this definition you can also write, $\kappa$:

$$\kappa = \sqrt{R \frac{d\Omega(R)^2}{dR} + 4\Omega(R)^2}. \tag{43}$$

The quantity $Q$ is known as the Toomre Q parameter, and there is an equivalent $Q_*$ parameter for stellar disks[2]. That stability criterion can be written

$$Q_* = \frac{\sigma_* \kappa}{3.36 G \Sigma_0} > 1, \tag{44}$$

for a locally stable disk. Here $\sigma_*$ is the velocity dispersion of the stars.

The physical reason why the Toomre parameter has the form it has, is that it represents a balance between thermal, rotational and gravitational forces. As a gaseous patch is compressed more and more, pressure forces will generally oppose the gravtiational collapse and this gives us a minimum size for the unstable patch (essentially the Jeans criterion), while on very large scales, rotation will oppose gravitational collapse, giving an upper limit to the possible patch size.

We can use this to estimate the stability criterion. If we just focus on the energy balance, we can write the thermal energy of the patch with size $L$ as

$$E_{\text{thermal}} \sim Mv^2 = \Sigma L^2 c_s^2, \tag{45}$$

the gravitational energy of the patch is

$$E_{\text{grav}} \sim \frac{GM^2}{L} \sim G\Sigma^2 L, \tag{46}$$

and finally the rotational energy is

$$E_{\text{rot}} \sim \Sigma L^2 (\Omega L)^2. \tag{47}$$

To get collapse we need

$$E_{\text{grav}} > E_{\text{thermal}} \quad \Rightarrow \quad L > \frac{c_s^2}{G\Sigma}, \tag{48}$$

which is essentially the Jeans criterion, and we need

$$E_{\text{grav}} > E_{\text{rot}} \quad \Rightarrow \quad L < \frac{G\Sigma}{\Omega^2}, \tag{49}$$

which we can combine and get

$$\frac{c_s^2}{G\Sigma} < L < \frac{G\Sigma}{\Omega^2} \tag{50}$$

$$\Downarrow$$

$$\frac{c_s \Omega}{\Sigma G} < 1 \tag{51}$$

as our criterion for local instability. A complete analysis gives the necessary constants and accounts for the differential rotation in the disk.

[2] This was in fact was what Toomre focused on in his 1964 paper (Toomre 1964, ApJ, 139, 1217)

The local stability of the disk is therefore determined by the value of the Toomre Q parameter. But we are also interested in the overall global stability of the disk. This is harder to investigate analytically so recourse is normally made to numerical simulations.

It turns out that what is important here is the relative importance of circular rotational motion relative to gravitational binding energy. If we define

**T** is the kinetic energy associated to ordered motion. In tensor form we have

$$T_{ij} = \frac{1}{2}\int \rho \langle v_i \rangle \langle v_j \rangle d^3\vec{x}, \tag{52}$$

**Π** is the kinetic energy associated to unordered motion. It is given by

$$\Pi_{ij} = \int \rho \sigma_{ij}^2 d^3\vec{x}, \tag{53}$$

where $\sigma_{ij}$ is the velocity dispersion tensor.

**W** is the gravitational potential energy. In tensor form this is known as the Chandrasekhar potential energy tensor and is defined through

$$W_{ij} = -\int \rho x_j \frac{\partial \Phi}{\partial x_i} d^3\vec{x}, \tag{54}$$

where $\Phi$ is the gravitational potential.

For completeness we have that the velocity averages, $\langle v_i \rangle$ are averages taken over the velocity distribution function, $f$:

$$\langle v_i \rangle = \frac{1}{n(\vec{x})}\int v_i f(\vec{x}, \vec{v}, t) d^3\vec{v}, \tag{55}$$

and

$$\langle v_i v_j \rangle = \langle v_i \rangle \langle v_j \rangle + \sigma_{ij}^2 = \frac{1}{n(\vec{x})}\int v_i v_j f(\vec{x}, \vec{v}, t) d^3\vec{v}, \tag{56}$$

which also defines $\sigma_{ij}$.

Using these quantities the standard criterion for global stability found by Ostriker & Peebles (1973, ApJ, 193, L1) is

$$\frac{T}{|W|} < 0.14 \quad \text{or} \quad \frac{\Pi}{T} > 5, \tag{57}$$

for global stability to axisymmetric perturbations. Thus you need to have some non-circular motion to have a stable disk. A simpler criterion that results in a similar quantity is

$$\frac{V_{\max}}{\sqrt{GM_d/R_d}} > \alpha, \tag{58}$$

for stability. Here $\alpha \approx 1.1$ for a purely stellar disk and $\alpha \approx 0.9$ for a purely gaseous disk (Efstathiou et al 1982, MNRAS, 199, 1069).

It is instructive to calculate this latter quantity for an isolated exponential disk without a dark halo. The circular velocity curve for an exponential disk can be calculated (see Binney & Tremaine 2008 for instance) to be

$$V_c^2(R) = 4\pi G \Sigma_0 R_d y^2 \left( I_0(y) K_0(y) - I_1(y) K_1(y) \right), \tag{59}$$

where $y = R/2R_d$, $I_n$ are modified Bessel functions of the first kind and $K_n$ modified Bessel functions of the second kind. This function has a maximum for $y = 1.075$, or $R = 2.15 R_d$. The value at maximum is

$$V_c(2.15 R_d) \approx 1.56 \sqrt{G R_d \Sigma_0}, \tag{60}$$

and since $M_d = 2\pi \Sigma_0 R_d^2$ we find that $\frac{V_{\rm max}}{\sqrt{G M_d / R_d}} \approx 0.62$ for an exponential disk. Thus an isolated exponential disk is globally unstable and will typically develop a bar.

## 5.5 Feedback from formed stars

We have now a prescription for where stars can form in our model galaxy and a fraction of these stars will have significant winds and some will explode as supernovae. The resulting energy/momentum input into the gas is what we denote stellar feedback.

One common prescriptions for this feedback is the energy driven feedback, for which the mass ejected from a radius $R$ is given by

$$\Delta M_{\rm ejected}(R, t) = \epsilon \frac{E_{\rm SN} \eta_{\rm SN}}{\frac{1}{2} V_{\rm esc}^2(R)} \Delta M_*(R, t), \tag{61}$$

where $\epsilon$ is an overall efficiency factor, $\eta_{\rm SN}$ is the number of supernovae per stellar mass formed, $E_{\rm SN}$ is the energy per supernova, and $V_{\rm esc}$ is the escape speed at radius $R$.

Another popular form of feedback is the momentum driven wind, for which it is not the energy ejected by a supernova that is the important ingredient in the feedback process, but rather the momentum. In this case the mass ejected is given by

$$\Delta M_{\rm ejected}(R, t) = \epsilon' \frac{p_{\rm SN} \eta_{\rm SN}}{\frac{1}{2} V_{\rm esc}(R)} \Delta M_*(R, t), \tag{62}$$

where $\epsilon'$ is another efficiency factor and $p_{\rm SN}$ is the momentum per supernova.

It is not clear exactly what feedback prescription most accurately match reality so it is common to keep both and compare their predictions. In addition one might also want to include a prescription for AGN feedback.

The process of stellar evolution is also responsible for enriching the gaseous of the galaxies. This leads to a gradual increase in their metallicity if not too much of the metals is ejected out of the galaxy in question. Since less massive galaxies have lower escape speeds, they will lose proportionally more of their metals than more massive galaxies. The resulting correlation between galaxy mass and metallicity is known as the mass-metallicity relation and can be compared to observational data to place constraints on model parameters.