Lecture 7

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1 Recap

In the last lecture we discussed the extended Press-Schechter formalism and argued that by looking at the density field filtered at different scales and see how the density contrast varies as the smoothing scale changes, we could count the number of virialised halos in a rigorous way.

In addition we saw that if we look at the field with different thresholds, we could use the time dependence of $\delta_c(t)$ to look at the build up of halos with time.

We closed by an introduction to the virial theorem and virial relations and we complete this today.

2 Virial relations

2.1 The virial radius

The spherical collapse model has led us to write the criterion for collapse at redshift z as

$$\rho(z) \gtrsim \Delta_c \bar{\rho}(z). \tag{1}$$

If we denote the radius of the virialised region as r_{vir}, and recall that the critical density is $\rho_{\text{crit}} = 3H(z)^2/8\pi G$, we can write this criterion as

$$\rho(z) \quad \frac{1}{\bar{\rho}(z)} \quad \gtrsim \Delta_c \tag{2}$$

$$\frac{3M}{4\pi r_{\text{vir}}^3} \quad \frac{8\pi G}{\Omega_m 3H(z)^2} \gtrsim \Delta_c \tag{3}$$

During the epoch of matter domination, we can in general write $H^2(z) \approx H_0^2(1+z)^3\Omega_m$. In that case we get an expression for the virial radius

$$r_{\text{vir}} = \left(\frac{2G}{H_0^2 \Delta_c}\right)^{1/3} M^{1/3}(1+z)^{-1}\Omega_m^{-2/3}. \tag{4}$$

To get an idea for the order of magnitude we will here and in the following give a quantitative estimate of this referred to a halo with mass $M = 10^{12}h^{-1}M_\odot$ at $z = 0$, similar to the Milky Way; and to one at $z = 10$ with mass $M = 10^8 h^{-1}M_\odot$. Inserting values we have

$$r_{\rm vir} \approx 378 \left(\frac{M}{10^{12}h^{-1}M_\odot}\right)^{1/3} \left(\frac{\Omega_m}{0.3}\right)^{-2/3} (1+z)^{-1}h^{-1}{\rm kpc} \tag{5}$$

for the MW-like structure, while in the other case we have

$$r_{\rm vir} \approx 1.6 \left(\frac{M}{10^{8}h^{-1}M_\odot}\right)^{1/3} \left(\frac{\Omega_m}{0.3}\right)^{-2/3} \left(\frac{1+z}{11}\right)^{-1} h^{-1}{\rm kpc} \tag{6}$$

The sun is at $\sim$ 8kpc and the MW halo is $\sim 10^{12}M_\odot$ so we can conclude that the virial radius of the dark matter halo is considerably larger than that of the baryonic component. This is reasonable because baryons can release energy through radiation and hence can collapse further.

I have tacitly assumed that Δ_c is the collapse threshold derived from the spherical collapse model to get the numerical expressions above. Is this right? Well, equation 2 can be taken to define a characteristic size for the halo independently of the spherical collapse model. It is important when you come across virial radii in papers that you check what definition was used because there is no universally agreed upon definition.

2.2 The virial temperature

If we write down the virial theorem in Equation (**??**) ignoring external pressure and the d^2I/dt^2 term, we have

$$\frac{3M_{\rm gas}k_BT_{\rm vir}}{\mu m_P} - f_S\frac{GM_{\rm gas}M}{r_{\rm vir}} = 0, \tag{7}$$

which defines the virial temperature $T_{\rm vir}$. For a uniform sphere we have

$$T_{\rm vir} = \frac{1}{5}\frac{GM\mu m_P}{r_{\rm vir}k_B}. \tag{8}$$

It is also here common to introduce the virial velocity and in terms of this we have

$$T_{\rm vir} = \frac{1}{5}\frac{\mu m_P}{k_B}V_c^2. \tag{9}$$

If we insert $r_{\rm vir}$ from equation (4) we get

$$T_{\rm vir} = \frac{(GM)^{2/3}\mu m_P}{2^{1/3}5k_B}\Omega_m^{2/3}(1+z)H_0^{2/3}\Delta_c^{1/3}, \tag{10}$$

which gives

$$T_{\rm vir} = 1.6\times 10^5 \left(\frac{M}{10^{12}h^{-1}M_\odot}\right)^{2/3} \left(\frac{\Omega_m}{0.3}\right)^{2/3} \left(\frac{\mu}{0.59}\right)(1+z)K, \tag{11}$$

and

$$T_{\rm vir} = 3.9 \times 10^3 \left(\frac{M}{10^8 h^{-1} M_\odot}\right)^{2/3} \left(\frac{\Omega_m}{0.3}\right)^{2/3} \left(\frac{\mu}{0.59}\right) \left(\frac{1+z}{11}\right) K, \tag{12}$$

Thus we see that unless significant cooling has taken place, our galaxy is embedded in a hot halo. We will see next time how cooling influences this.

While the derivation of the virial temperature (spherical collapse, virial equilibrium, no cooling) are all very simplistic, the virial theorem remains a useful concept. Upon mergers of dark matter halos we expect in general that the gas within them gets heated to the virial temperature of the new halo, and subsequent cooling is required for star formation to start.

2.3 The circular velocity

We have made use of the circular velocity repeatedly above. This is a handy way to specify the properties of a halo because it encapsulates both the radius and mass, and for a virialised structure we can write

$$V_c = \left(\frac{GM}{r_{\rm vir}}\right)^{1/2} \tag{13}$$

$$= \left(\frac{G^2 H_0^2}{2}\right)^{1/6} \Delta_c^{1/6} \Omega_m^{1/3} M^{1/3} (1+z)^{1/2}. \tag{14}$$

which shows that the circular velocity increases with mass, and also with formation redshift. Thus for the same mass and rotational velocity, the objects are more compact at high redshift.

Quantitatively this results for our examples in:

$$V_c = 107 \left(\frac{M}{10^{12} h^{-1} M_\odot}\right)^{1/3} (1+z)^{1/2} \left(\frac{\Omega_m}{0.3}\right)^{1/3} \text{km/s} \tag{15}$$

$$= 16 \left(\frac{M}{10^8 h^{-1} M_\odot}\right)^{1/3} \left(\frac{1+z}{11}\right)^{1/2} \left(\frac{\Omega_m}{0.3}\right)^{1/3} \text{km/s} \tag{16}$$

The Milky Way has a circular velocity of $V_c \approx 220$km/s which is somewhat higher than what you expect from the above scalings. A possible interpretation is that the halo of the Milky Way virialised at $z \sim 3$, but recall that this is a very simplistic model.

3 The structure of dark matter halos

See section 5.2.1 in MvdBW for a discussion of the self-similar spherical collapse model. Section 7.5 in the same book discusses the internal structure of dark matter halos.

It is possible to show, but cumbersome, that in a spherical collapse model with self-similar solutions the density profile of the resulting dark matter halo can be expected to be a power-law, and one can argue from this that typical dark matter halos should have density profiles that are close to isothermal,

$$\rho_{isothermal} \propto r^{-2}. \tag{17}$$

This is a handy profile from an analytic point of view and it is useful to tabulate some classical relationships for this profile.

$$\rho(r) = \frac{2k_B T}{\mu m_P} \frac{1}{4\pi G r^2} \tag{18}$$

$$M(r) = \frac{2k_B T}{\mu m_P} \frac{r}{G}, \tag{19}$$

but it is also common to see these relations phrased in terms of the circular velocity, $V_c = \sqrt{GM/r}$, which gives:

$$V_c^2 = \frac{2k_B T}{\mu m_P} \tag{20}$$

$$\rho(r) = \frac{V_c^2}{4\pi G r^2} \tag{21}$$

$$M(r) = \frac{V_c^2 r}{G}. \tag{22}$$

Since this profile does not have a convergent mass, it is common to truncate this at the virial radius, creating the truncated isothermal sphere.

Simulations of dark matter halos do however show that their profiles are not isothermal spheres. Notably, however, simulations *do* show that dark matter profiles have a universal shape. This was noticed by Navarro, Frenk & White (1996) and their fit to this profile, the Navarro-Frenk-White (NFW) profile has become the most widely used density profile for dark matter halos.

This profile has the shape

$$\rho(r) = \bar{\rho} \frac{\delta_{\rm char}}{(r/r_s)(1 + r/r_s)^2}, \tag{23}$$

with mass

$$M(< r) = 4\pi \bar{\rho} \delta_{\rm char} r_s^3 \left[\ln(1 + cx) - \frac{cx}{1 + cx} \right], \tag{24}$$

where $c = r_{\rm vir}/r_s$, $x = r/r_{\rm vir}$ and if you combine this with the criterion for top-hat collapse which we reviewed above, you can show that

$$\delta_{\rm char} = \frac{\Delta_c}{3} \frac{c^3}{\ln(1 + c) - c/(1 + c)}. \tag{25}$$

The parameter c is known as the concentration parameter and from equation (25) one can see that M and c is sufficient to specify the shape of the halo. It is known that c is a function of halo mass and formation time and fits have been provided in the literature for this dependence.

The fact that mass profiles to first order depend only on the halo mass is an important observation, and one that we will make use of later.

4 A story of villages

Imagine you live in medieval, feudal Europe. Most of the population lives in small villages with fields around, and are therefore spatially separated from other villages. This is evidently a very clustered distribution, so as a naturally inquisitive child you want to create a model for the correlation function of people in your part of the world.

You could of course go around and keep track of every person and calculate a complicated correlation function, but you can also be more clever: Consider first small separations - if you are in your village chances are quite high that you will be close to someone else in your village, but very unlikely that you are close to someone from another village. I have here made the assumption that medieval villages were compact, which is a reasonable assumption. Thus you have one component of your correlation function; On small scales you are dominated by local correlations and this is approximately independent of the other villages around.

What about the large-scale behaviour? When you ask about the chance to have another person 100 km away, your village should not matter. What matter here is the distribution of the villages overall - the village-village correlation function. Thus our simple model for the correlation of people would be:

$$\xi_{\mathrm{pp}}(r) = \xi_{\mathrm{pp}}^{\mathrm{1village}}(r) + \xi_{\mathrm{pp}}^{\mathrm{village-villlage}}(r). \tag{26}$$

Where the first term goes to zero at large distances because the people living in your village are not very adventurous, while we expect that the second term goes to zero at small distances because other villages are like yours.

If we want to develop this theory further we will see that the devil is in the details, for instance we need to take into account the fact that more populous villages are larger but as long as their size only depends on the number of people living there we can handle that. More complicated is the fact that geography of your world is complex. If only the world was homogeneous and isotropic!

5 The halo model of large-scale structure

An comprehensive overview of the halo model is given in the review by Cooray & Sheth (2002, Phys. Rep. 371, 1–129) but it is quite a bit to read. The theory is also summarised in section 7.5 in MvdBW and a pedagogical introduction to the model and its ingredients is provided in van den Bosch et al (2013, MNRAS 430, 725–746).

The Universe is to first approximation not so different. We have argued that the typical end-product of gravitational collapse is a halo. The halo model makes the simplifying assumption that all dark matter resides in virialised dark matter halos with well-defined density profiles.

Last lecture we saw that the density profile of dark matter halos (in simulations) is a function of halo mass and concentration/formation time, but as the concentration is to first order a function of mass. In that case we can write the density distribution of a halo as

$$\rho_h(\vec{x}) = M u(\vec{x}|M), \tag{27}$$

where $u(\vec{x}|M)$ is a normalised halo profile so that

$$\int d^3\vec{x}\, u(\vec{x}|M) = 1. \tag{28}$$

Now imagine dividing the Universe into small volumes, ΔV_i, so that there is a maximum of 1 halo centre in each cell. We can then write the number per cell as

$$\mathcal{N}_i = \begin{cases} 0 & \text{No halo centre in } \Delta V_i \\ 1 & \text{One halo centre in } \Delta V_i \end{cases} \tag{29}$$

We assume we can always do this by making ΔV_i small enough.

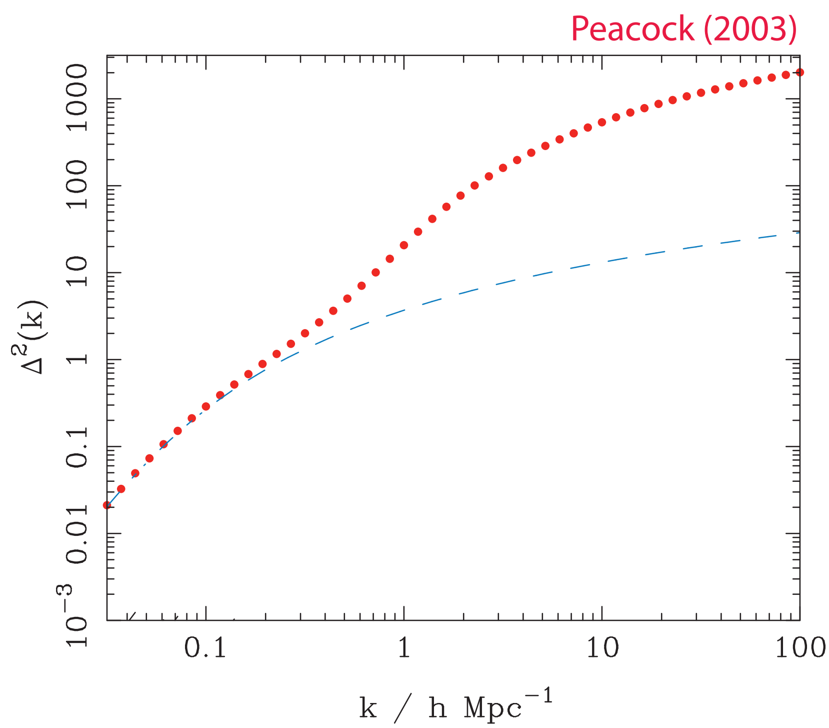

Figure 1: The linear power spectrum is plotted as the dashed line, while the non-linear power-spectrum, plotted as red circles, differs very strongly. How can we reconcile the two? Figure based on Peacock (2003).

Our goal now is now to use this simple framework to understand the non-linear evolution of the Universe. That we need this is clear from Figure 5. This shows an approximate total non-linear power spectrum as the red solid dots, while the blue dashed line shows the *linear* power-spectrum. The two obviously differ quite substantially on small scales and there is a pronounced kink in the power spectrum at scales of a few h^{-1} Mpc. We now want to understand what the physics behind this shape is.

To make the step we will first work with the correlation function, which is the Fourier transform of the power-spectrum, and we will start with the correlation function of the density field, so $\xi = \langle \rho(\vec{x})\rho(\vec{y})\rangle$. Thus we need an expression for $\rho(\vec{x})$. We can get this formally by summing over all volumes ΔV_i:

$$\rho(\vec{x}) = \sum_i \rho_{h,i}(\vec{x})\mathcal{N}_i = \sum_i M_i u(\vec{x} - \vec{x}_i|M_i)\mathcal{N}_i. \tag{30}$$

This expresses the assumption that all dark matter is bound up in virialised halos. We will now define the ensemble average, ie. the average over many independent realisations of the Universe, to be an average over mass and if we use $\langle \cdot \rangle$ to denote this average we have

$$\langle \mathcal{N}_i M_i u(\vec{x} - \vec{x}_i|M_i)\rangle = \int dM\ n(M)M\Delta V_i u(\vec{x} - \vec{x}_i|M). \tag{31}$$

This allows us to calculate the average over the density:

$$\langle \rho(\vec{x})\rangle = \int dM\ n(M)M \sum_i \Delta V_i u(\vec{x} - \vec{x}_i|M), \tag{32}$$

where the term in blue will turn into an integral over space which gives us

$$\langle \rho(\vec{x})\rangle = \int dM\ n(M)M \int d^3\vec{x}' u(\vec{x'} - \vec{x}|M) \tag{33}$$

$$= \int dM\ n(M)M \tag{34}$$

$$= \bar{\rho}. \tag{35}$$

Thus the average value is indeed the mean density of the Universe, which we naturally would expect. The second equality above comes from the normalisation requirement, equation 28.

5.1 The correlation function

We start by calculating the density field cross-correlation function and will continue to the overdensity field cross-correlation function further below. The cross correlation function of the density field is given by

$$\langle \rho(\vec{x})\rho(\vec{y})\rangle = \sum_{i,j} \langle \mathcal{N}_i M_i \mathcal{N}_j M_j u(\vec{x} - \vec{x}_i|M_i)u(\vec{y} - \vec{x}_j|M_j)\rangle. \tag{36}$$

We can simplify this somewhat by thinking back to our overall model, or back to the villages. We expect that physically there should be a difference between correlations *within* a halo, ie. with $i = j$, and correlations outside the halo where $i \neq j$.

We will therefore separate these two terms. The first, the $i = j$ term, we will refer to as the **one-halo** term, and the $i \neq j$ term will be referred to as the **two-halo** term. Since the one-halo term is simpler, let us start with that.

We can now make use of the fact that $\mathcal{N}^2 = \mathcal{N}$ since the occupation number only can be 1 or 0. This allows us to write the $i = j$ term as

$$\xi_\rho^{1h}(\vec{x}, \vec{y}) = \sum_i \langle \mathcal{N}_i M_i^2 u(\vec{x} - \vec{x}_i | M_i) u(\vec{y} - \vec{x}_i | M_i) \rangle \tag{37}$$

$$= \int dM n(M) M^2 \sum_i \Delta V_i u(\vec{x} - \vec{x}_i | M_i) u(\vec{y} - \vec{x}_i | M_i), \tag{38}$$

$$= \int dM n(M) M^2 \int d^3\vec{x'} u(\vec{x} - \vec{x'} | M) u(\vec{y} - \vec{x'} | M) \tag{39}$$

$$\tag{40}$$

As we will see below, this term is easier to handle in Fourier space but for now we will turn to the 2-halo term.

The $i \neq j$ case is slightly more complex, and we have

$$\xi_\rho^{2h}(\vec{x}, \vec{y}) = \sum_{i \neq j} \langle \mathcal{N}_i \mathcal{N}_j M_i M_j u(\vec{x} - \vec{x}_i | M_i) u(\vec{y} - \vec{x}_j | M_j) \rangle \tag{41}$$

$$= \int dM_1 n(M_1) M_1 \int dM_2 n(M_2) M_2 \sum_{i \neq j} \Delta V_i \Delta V_j u(\vec{x} - \vec{x}_i | M_i) u(\vec{y} - \vec{x}_j | M_j). \tag{42}$$

The integral here is similar to the one above, but we have a more complicated sum. We will transform the sums to spatial integrals, as we have done with the blue terms above, but we also need to account for the fact that the halos are correlated, thus the two volumes are not independent. We can account for this by introducing the halo-halo correlation function, ξ_{hh}. Note that this is different from the density-density correlation function that we are actually interested in here, but as we will see below it is related quite closely to the overdensity correlation function — quite naturally because we assume that all matter is in virialised halos.

Introducing ξ_{hh} we get

$$\xi_\rho^{2h}(\vec{x}, \vec{y}) = \int dM_1 n(M_1) M_1 \int dM_2 n(M_2) M_2$$

$$\int d^3\vec{x'} \int d^3\vec{y'} \left[1 + \xi_h h(\vec{x'} - \vec{y'} | M_1, M_2)\right] u(\vec{x} - \vec{x'} | M_1) u(\vec{y} - \vec{y'} | M_2) \tag{43}$$

$$= \bar{\rho}^2 + \int dM_1 n(M_1) M_1 \int dM_2 n(M_2) M_2$$

$$\int d^3\vec{x'} \int d^3\vec{y'} \xi_h h(\vec{x'} - \vec{y'} | M_1, M_2) u(\vec{x} - \vec{x'} | M_1) u(\vec{y} - \vec{y'} | M_2) \tag{44}$$

5.2 The correlation function of the overdensity field

In previous lectures we have focused on the correlation function of the overdensity field and this is what we normally want to consider. This is

$$\xi(r) = \langle \delta(\vec{x})\delta(\vec{y})\rangle = \left\langle \left(\frac{\rho(\vec{x}}{\bar{\rho}} - 1\right)\left(\frac{\rho(\vec{y}}{\bar{\rho}} - 1\right)\right\rangle \tag{45}$$

from the definition of δ. We can expand this to give

$$\xi(r) = \frac{1}{\bar{\rho}^2}\langle \rho(\vec{x})\rho(\vec{y})\rangle - \frac{2}{\bar{\rho}}\langle \rho(\vec{x})\rangle + 1 \tag{46}$$

$$= \frac{1}{\bar{\rho}^2}\langle \rho(\vec{x})\rho(\vec{y})\rangle - 1 \tag{47}$$

$$= \xi^{1h}(r) + \xi^{2h}(r), \tag{48}$$

where $r = |\vec{x} - \vec{y}|$ and the 1-halo and 2-halo terms come from equation 39 and 44 above.

This is a very useful and noteworthy result because it shows how you can separate the local, non-linear effects from the large-scale linear effect. In reality there is of course a transition region but to first order the idea of separating effects as coming from the same halo and from different halos is very useful.

Since the Fourier transform is linear, and the power spectrum is the Fourier transform of the correlation function, we also have that

$$P(k) = P^{1h}(k) + P^{2h}(k), \tag{49}$$

where for illustration we can easily get the 1-halo term:

$$P^{1h}(k) = \frac{1}{\bar{\rho}^2}\int dM M^2 n(M) \left|\hat{u}(k|M)\right|^2, \tag{50}$$

which follows straight from equation 39 with $\hat{u}$ being the Fourier transform of the halo profile function.

The 1-halo term consists of terms that we know how to calculate, so let us instead turn to the 2-halo term to try to get a better understanding of this. For this we can go to very large-scales. If the scales are large enough, then the detailed shape of the halo is irrelevant and much smaller than the scale of interest. We can then replace $u(\vec{x})$ by the Dirac delta function $\delta^D(\vec{x})$. This allows us to carry out the spatial integrals in equation 44 easily and we get

$$\xi^{2h}(r) = \frac{1}{\bar{\rho}^2}\int dM_1 M_1 n(M_1)\int dM_2 M_2 n(M_2)\xi_{hh}(r|M_1, M_2), \tag{51}$$

so we are left with having to figure out what ξ_{hh} is.

We will not discuss this in detail here, a discussion can be found in section 7.4 of MvdBW if you are interested, or in the Cooray & Sheth review paper mentioned at the start. The basic idea is that if halos are associated with high peaks in the density field, then they are likely to be more clustered. Thus the overdensity of halos might not follow precisely the overdensity of matter.

It is possible to get an expression for this using the extended Press-Schechter formalism, or by expanding the overdensity of halos as a Taylor series in the matter overdensity, δ_m:

$$\delta_h(\vec{x}; M, z) = \sum_0^\infty \frac{b_{h,n}(M, z)}{n!} \delta_m^n(\vec{x}, z). \tag{52}$$

On large scales, which is what we consider here, $\delta \ll 1$, so we can ignore all but the $n = 1$ term (the $n = 0$ term is usually ignored because it would only contribute to $k = 0$ in the power spectrum). In that case we have

$$\delta_h(\vec{x}; M, z) \approx b_h(M, z)\delta_m(\vec{x}, z), \tag{53}$$

where I have written b_h for $b_{h,1}$. There are various expressions for b_h in the literature, the simplest is probably

$$b_h(M, z) = 1 + \frac{\nu^2 - 1}{\delta_c(z)}, \tag{54}$$

where $\nu = \delta_c(t)/\sigma(M)$. This has the characteristic feature that halos with mass, $M = M_*$, where M_* corresponds to the mass where $\nu = 1$, are unbiased. More massive halos are positively biased (more strongly clustered) than the matter, and less massive halos are less clustered. In general that formula is a bit approximate and the exact fitting formula does depend on how you define masses of halos, thus I will not include details here but the discussion in Tinker et al (2010, ApJ, 724) is a good place to learn more about this.

Armed with a bias factor we can then write

$$\begin{aligned} \xi_{hh}(r|M_1, M_2, z) &= \langle \delta_h(\vec{x_1}, M_1, z)\delta_h(\vec{x_2}, M_2, z)\rangle & (55)\\ &\approx b_h(M_1, z)b_h(M_2, z)\xi^{\text{lin}}_{mm}(r, z), & (56) \end{aligned}$$

where $\xi^{\text{lin}}_{mm}(r, z)$ is the linear matter power spectrum — which we have looked at extensively in previous lectures, and where $r = |\vec{x_1} - \vec{x_2}|$.

We can now go back to equation 51 and write it

$$\begin{aligned} \xi^{2h}(r) &= \frac{1}{\bar{\rho}^2} \int dM_1 M_1 n(M_1) \int dM_2 M_2 n(M_2) \xi_{hh}(r|M_1, M_2) & (57)\\ &\approx \xi^{\text{lin}}_{mm}(r, z) \left[\frac{1}{\bar{\rho}} \int b_h(M) M n(M) dM\right]^2 & (58)\\ &\approx \xi^{\text{lin}}_{mm}(r, z). & (59) \end{aligned}$$

The last equality follows approximately because matter should be unbiased with regards to itself, that is M, and in the halo model all matter is in halos so on large enough scales at least, the integral should converge to $\bar{\rho}$ and hence the expression in parenthesis approach 1.

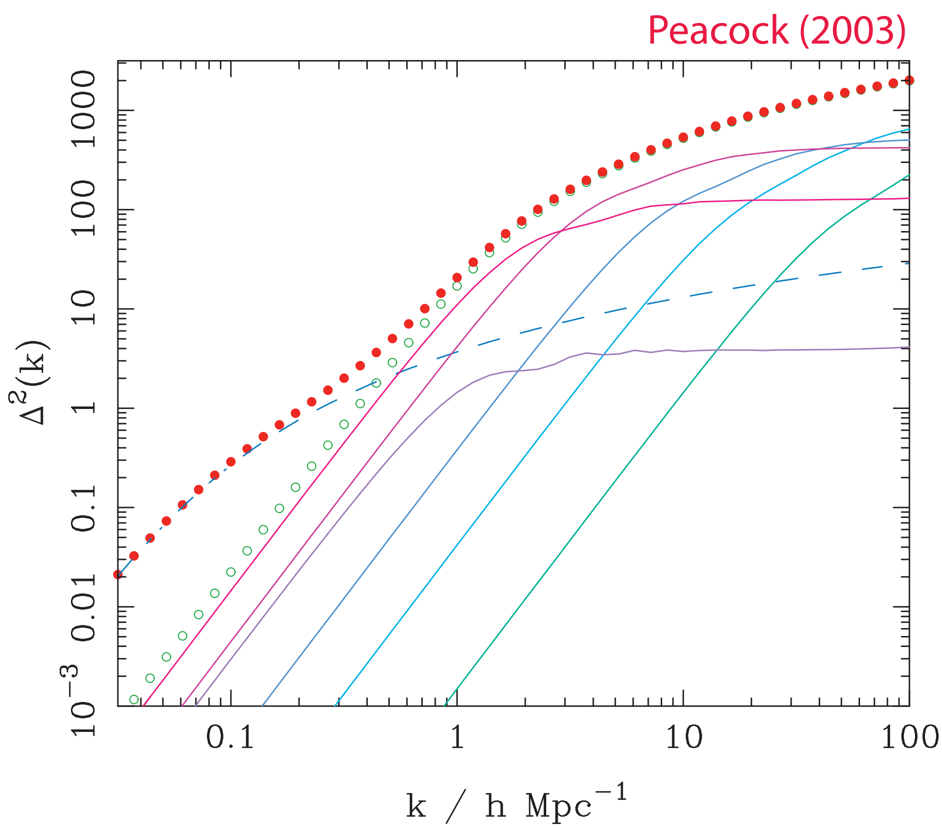

Figure 2: The dashed blue line shows the linear power spectrum, plotted as $\Delta^2(k)$, while the open circles show the contribution of the 1-halo term summed over all masses. The red filled dots show the sum of the 1-halo and 2-halo terms and the individual curves show contributions to the 1-halo term in decades in mass, starting with 10^{10}–$10^{11} M_\odot$ on the right.

The functioning of this model is illustrated in Figure 5.2, taken from Peacock (2003), which shows the linear power spectrum as the dashed blue line and the sum of the 1-halo and 2-halo terms as the solid red circles. The open circles show the 1-halo term and the individual curves show the contributions to this term in bins of a factor of 10 in mass. The figure shows quite clearly how the excess power, or increased correlation function if you wish, at small scales is due to the 1-halo term. This is a non-linear effect and essential for understanding observational data.

6 Galaxies in the halo model

We would now need to include galaxies into this model. The way this is done is very similar to what we did for the halo term above. The challenging aspect is that we need some sort of prescription for the number of galaxies of luminosity L that reside in a halo of mass M. That this is possible to should be fairly clear. One could for instance argue that galaxies that cluster in a particular way should be associated to dark matter halos that cluster in the same way.

We can express this by writing for halos $\xi_{hh}(r|M) = b_h^2(M)\xi_{mm}^{\rm lin}(r)$, while for galaxies we can write an analogous galaxy-galaxy correlation function $\xi_{gg}(r|L) = b_g^2(L)\xi_{mm}^{\rm lin}(r)$. If we have a good model for the halo bias, and observations of the galaxy-galaxy correlation function, we can then get an indirect constraint on the mass through

$$b_h^2(M) = \frac{\xi_{gg}(r|L)}{\xi_{mm}^{\rm lin}(r)}, \tag{60}$$

which we can then solve for M. That will give us the average mass of a halo that houses galaxies of luminosity L_*.

In practice a convenient way to encapsulate this is through the conditional luminosity function (CLF), $\phi(L|M)$. This is defined through

$$\phi(L, z) = \int \phi(L|M)n(M, z)dM, \tag{61}$$

where L is the luminosity of the galaxy and M is the mass of the halo, $n(M)$ is the mass function of halos as used before. The CLF must be estimated from data, an influential effort to do this is the group catalogue from the Sloan Digital Sky Survey assembled by Yang et al (2007, ApJ, 671, 153).

Given the conditional luminosity function you can easily calculate a number of useful quantities. For instance, the average number of galaxies with luminosity $L \in [L_1, L_2]$ in a halo of mass M can be written

$$\langle N_g|M\rangle(L_1, L_2) = \int_{L_1}^{L_2} \phi(L|M)dL, \tag{62}$$

and from this you can then get the average number density of galaxies in this luminosity range

$$\bar{n}_g(z; L_1, L_2) = \int \langle N_g|M\rangle(L_1, L_2)n(M, z)dM, \tag{63}$$

where $n(M,z)$ is the halo mass function as before.

6.1 Centrals and satellites

An important result from both observations and semi-analytic models of galaxy formation is that central galaxies in halos evolve differently from satellites. This means that it is beneficial to split the CLF into a contribution from centrals and one from satellites, so we write

$$\phi(L|M) = \phi_c(L|M) + \phi_s(L|M), \tag{64}$$

with $\phi_c(L|M)$ being the CLF for centrals and $\phi_s(L|M)$ that of satellites. The CLF for centrals is often modelled as a log-normal distribution

$$\phi_c(L|M)dL = \frac{\log e}{\sqrt{2\pi}\sigma_c} e^{-\frac{(\log L - \log L_c)^2}{2\sigma_c^2}} \frac{dL}{L}, \tag{65}$$

and the CLF for satellites as a modified Schechter function

$$\phi_s(L|M)dL = \phi_s^* \left(\frac{L}{L_s^*}\right)^{\alpha_s+1} e^{-(L/L_s^*)^2} \frac{dL}{L}. \tag{66}$$

It is modified because the drop-off is like e^{-x^2} rather than e^{-x}.

We now write the overdensity of galaxies as

$$\delta_g(\vec{x}, z) = \frac{n_g(\vec{x}, z) - \bar{n}_g(z)}{\bar{n}_g(z)}, \tag{67}$$

where n denotes number density. Introducing the central, satellite split we get

$$\delta_g(\vec{x}, z) = f_c(z)\delta_c(\vec{x}, z) + f_s(z)\delta_s(\vec{x}, 1), \tag{68}$$

where f_c and f_s are the central and satellite fractions respectively. Following the approach we had for the halos earlier we can expand this as

$$n_c(\vec{x}, z) = \sum_i \mathcal{N}_{h,i}\mathcal{N}_{c,i}\delta^D(\vec{x} - \vec{x_i}), \tag{69}$$

where δ^D is the Dirac delta function and it signifies that the centrals are assumed to be at the centre of each halo, while $\mathcal{N}_{c,i}$ is the central galaxy occupation number for each halo and as there can only be one central this can only be 0 or 1.

The satellite term is similar

$$n_s(\vec{x}, z) = \sum_i \mathcal{N}_{h,i}\mathcal{N}_{s,i}u_s(\vec{x} - \vec{x_i}|M_i, z), \tag{70}$$

where $\mathcal{N}_{s,i}$ is the satellite occupation number in volume i, and this can be any number.

We then form the correlation function $\xi_{gg} = \langle \delta_g(\vec{x})\delta_g(\vec{y})\rangle$, and it follows directly from the same approach we used for the correlation function earlier that this will be a sum of different terms. Phrased in terms of the power-spectrum we get

$$P_{gg}(k,z) = f_c^2(z)P_{cc}(k,z) + 2f_c(z)f_s(z)P_{cs}(k,z) + f_s^2(z)P_{ss}(k,z), \tag{71}$$

where cc denotes central-central, ss satellite-satellite and cs central-satellite.

Armed with these equations and the CLF it is then possible to predict the observed correlation function of galaxies, and this simple model works very well for that.