Lecture 3

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1 Introduction

Last week we looked at the growth of perturbations in linear theory focusing on the Newtonian regime. We also looked at the important concept of horizon crossing. Here we will continue this and see how this introduces a characteristic scale on the distribution of galaxies in the sky. But first we will complete the discussion of what slows down/affects perturbations. Last week we examined the effect of pressure — this introduced the Jeans mass and showed that perturbations below a particular size, the Jeans length, have an oscillatory solution.

2 Slowing down growth of perturbations by expansion

The growth of perturbations is slowed down because of the Hubble expansion which acts as a drag force against the collapse. Intuitively, when the expansion is faster than the collapse you will not have collapse. Since the expansion time is $t_{\mathrm{expansion}} \sim H(t)^{-1} \sim \rho_{\mathrm{dominant}}^{-1/2}$ and the collapse time for matter is $t_{\mathrm{dynamic}} \sim \rho_{\mathrm{matter}}^{-1/2}$ you see that if the density of the dominant ingredient is much higher than that of the matter you get no growth. This is in general true in the radiation dominated epoch.

To do a more careful analysis, we will assume that the relativistic background is smooth. And we introduce a variable, $y = \rho_{\mathrm{NR}}/\rho_{\mathrm{R}}$, which is the ratio of the matter density that of the radiation and can also be written $y = a/a_{\mathrm{eq}}$. Introducing this into the perturbation equation **??** and using the Friedman equation (ignoring the cosmological constant and curvature terms), one gets (after some straightforward but tedious manipulations):

$$2y(1+y)\frac{d^2\delta}{dy^2} + (2+3y)\frac{d\delta}{dy} = 3\delta. \tag{1}$$

This equation is solved by $\delta \propto 1 + 3y/2 = 1 + \frac{3}{2}\frac{a}{a_{\mathrm{eq}}}$. From this we can immediately see that growth during the radiation dominated regime is very slow at maximum a factor of 5/2.

In a more accurate analysis (see e.g. MvdBW section 4.2) appropriate for the make-up of a Universe like ours, perturbations in the radiation when sub-horizon scale oscillate, and baryons couple to these as we will discuss below. Perturbations in the cold dark matter do however grow, albeit only logarithmically. For our purposes it is sufficiently accurate to say that sub-horizon perturbations in the matter do not grow, or grow very little, during the radiation dominated epoch.

3 Free streaming

Consider a scenario where the Universe has a significant contribution from a particle that interacts very weakly with photons. You can think of this as neutrinos if you wish to be concrete as that is the most commonly suggested candidate for such a particle.

In general, if particles in a perturbation can stream out of the perturbation faster than it can collapse, the perturbation will be washed out. This is clearly a scale-dependent question for a given velocity. We can easily estimate the scale below which this is important: If we take a region with proper size, l, and particles with velocity, v, the criterion of interest is when the crossing time, l/v is shorter than the dynamical time, $t_{\rm dyn}$:

$$\frac{l}{v} < t_{\rm dyn} \sim \frac{1}{\sqrt{G\rho}} \Rightarrow \lambda_{\rm FS} \sim \frac{v}{\sqrt{G\rho}} \sim \frac{v}{H(t)} \sim v t_H, \tag{2}$$

where $\lambda_{\rm FS}$ is called the free-streaming length and $H(t)$ is the Hubble parameter as usual. The second-to-last equality follows from the Friedmann equation which says $H(t)^2 = 8\pi G\rho/3$. When particles go non-relativistic we have that $v \propto a^{-1}$, because of the expansion of the Universe, which means that the importance of this process quickly declines. Thus we care mostly about the period while the particles are relativistic.

We can make this quantitative as follows (and you will do this more carefully in the problem class). First we consider a short period of time, dt, in which a particle crosses a physical distance of $a\,dr$. Integrating this from $t = 0$ to t we get that the co-moving free-streaming length is:

$$r_{\rm FS} = \int_0^t dr = \int_0^t \frac{v(\tau)}{a(\tau)}\, d\tau. \tag{3}$$

To carry out this integral it is convenient to split it into the period while the particle is relativistic, up until $t = t_{\rm NR}$ and one period while it is traveling non-relativistically at which point the velocity declines by Universe expansion $v \propto a^{-1}$. If we further assume that the particle becomes non-relativistic during the radiation dominated epoch, we can write

$$r_{\rm FS} = \int_0^{t_{\rm NR}} \frac{c\,d\tau}{a(\tau)} + \int_{t_{\rm NR}}^{t_{\rm eq}} \frac{c a_{\rm NR}}{a(\tau)^2}\, d\tau, \tag{4}$$

where we have used $v \propto a^{-1}$ to write the velocity after the particle has gone non-relativistic as $v = (a_{\rm NR}/a)c$ and we focused on the radiation dominated period here. Since we are only concerned with the radiation dominated epoch here we can take $a \propto t^{1/2}$. If we insert this into the integral above, we can carry out the integration and get:

$$r_{\rm FS} = \frac{2ct_{\rm NR}}{a_{\rm NR}} \left[1 + \ln \frac{a_{\rm eq}}{a_{\rm NR}}\right], \tag{5}$$

where we can ignore the logarithmic factor for light dark matter particles that stay relativistic during most of the radiation dominated epoch. In the problem set you will extend this calculation somewhat.

You can show (see problem set 2), that for a massive neutrino with mass, m_ν, the typical co-moving free-streaming length is:

$$r_{\rm FS} \approx 30.5 \left(\frac{m_\nu}{30 \text{ eV}}\right)^{-1} \text{ Mpc}, \tag{6}$$

and this corresponds to a free streaming mass in neutrinos of

$$M_{\rm FS} = \frac{\pi}{6}\rho r_{\rm FS}^3 \approx 1.3 \times 10^{15} \left(\frac{m_\nu}{30\text{eV}}\right)^{-2} M_\odot. \tag{7}$$

While we have made some approximations here, the result captures the essential aspects. It is important to note that a heavier particle goes non-relativistic earlier and thus the free-streaming mass decreases.

4 Silk damping

Literature: MvdBW section 4.1.6 and 4.2.5, C&L 11.6, Longair 12.5

Before decoupling, photons do not free stream because they couple tightly to baryons through Thompson scattering off free electrons. However they do carry out a random walk due to the collisions. Early on the mean free path is minuscule, but as the time approaches the time of decoupling the mean free path of the photons becomes noticeably non-zero: they will slowly diffuse out of perturbations, dragging baryons with them and wiping out all baryonic perturbations below some size, known as the Silk damping scale.

The mean free path, $l_{\rm mfp}$, for Thompson scattering is given as

$$l_{\rm mfp} = \frac{1}{n_e \sigma_T}, \tag{8}$$

where n_e is the number density of electrons and $\sigma_T = 6.65 \times 10^{-25}\text{cm}^2$ is the Thompson cross-section.

If we count the motion of a photon in steps between individual scattering events, In a time dt, a photon will take $N = cdt/l_{\rm mfp}$ steps. From kinetic theory[1] we have that the distance travelled is

$$\lambda_{\rm Diffusion}^2 = \frac{N}{3} l_{\rm mfp}^2. \tag{9}$$

From our expression for N above we can then integrate this over time and move to co-moving coordinates. This gives an expression for the comoving Silk damping scale:

$$r_S^2 = \int_0^{t_{\rm dec}} \frac{c\, l_{\rm mfp}}{3a^2}\, dt. \tag{10}$$

[1]If you want to see the details of this you can do worse than consult Chandrasekhar's exposition in Rev. Mod. Phys. 1943, Vol 15(1).

To get a rough estimate of this we first note that since n_e is a density we have that $l_{\rm mfp} \propto 1/n_e \propto a^3$ as long as there is no destruction of particles. We can then write:

$$l_{\rm mfp}(t) = l_{\rm mfp}(t_{\rm dec}) \frac{a(t)^3}{a_{\rm dec}^3}, \tag{11}$$

which we can insert into the integral and convert it to an integral over a:

$$r_S^2 = \frac{c\, l_{\rm mfp}(t_{\rm dec})}{3a_{\rm dec}} \int_0^{t_{\rm dec}} a dt. \tag{12}$$

To carry out the integral we need to know the variation of $a(t)$ in the relevant period. It turns out that the main contribution to the Silk damping takes place fairly close to decoupling, and in that case we have that $a(t)/a(t_{\rm dec}) = (t/t_{\rm dec})^{2/3}$ because the Universe is matter dominated. We can therefore carry out the integral and we get

$$r_S^2 \approx \frac{1}{5} c\, l_{\rm mfp}(t_{\rm dec}) t_{\rm dec} \frac{1}{a_{\rm dec^3}^2}. \tag{13}$$

To evaluate this quantity, we write the ionization fraction as $X_e = n_e/n_H$. If we write the hydrogen density as

$$n_{\rm H,dec} = \frac{X \Omega_b \rho_{\rm crit,0} (1+z_{\rm dec})^3}{m_H}, \tag{14}$$

where a subscript dec refers to the value at the time of matter-radiation equality and X is the hydrogen mass fraction. To get a rough estimate of the time of decoupling we can make the approximation of a matter-dominated Universe and write

$$t_{\rm dec} \approx \frac{2}{3H(t_{\rm dec})} \approx \frac{2}{3H_0 \Omega_m^{1/2} (1+z_{\rm dec})^{3/2}}, \tag{15}$$

and we get

$$l_{\rm mfp}(t_{\rm dec}) \approx \frac{m_H}{X_e \sigma_T} \frac{1}{X \Omega_b \rho_{\rm crit,0} (1+z_{\rm dec})^3} \tag{16}$$

Putting this all together we find

$$r_S^2 = \frac{2}{15} \frac{c m_H}{\sigma_T} (X\, X_e\, \rho_{\rm crit,0}^{-1})^{-1} (1+z_{\rm dec})^{-5/2} (h^3\, \Omega_b\, \Omega_m^{1/2}) \tag{17}$$

and putting in numbers we have

$$r_S \sim 28\, {\rm Mpc}, \tag{18}$$

which corresponds to a Silk mass

$$M_S \sim 3 \times 10^{14} M_\odot, \tag{19}$$

for $h = 0.704$, $X_e = 0.1$, $X = 0.75$, $z_{\rm dec} = 1088.2$, $\Omega_b = 0.0456$, and $\Omega_m = 0.227$. Thus baryonic perturbations on scales smaller than this will have been erased by the time of decoupling. This placed a very strong constraint on models for galaxy formation in a purely baryonic Universe but since we now know dark matter is important we need to consider weakly interactive non-relativistic matter.

5 Baryon Acoustic Oscillations

Before turning to this, however, let us look at a problem of great importance in astronomy/cosmology today which has a link to the physics just discussed, namely Baryon Acoustic Oscillations (BAOs). These are structures in the overall galaxy distribution that we can measure and which provide good constraints on cosmological parameters.

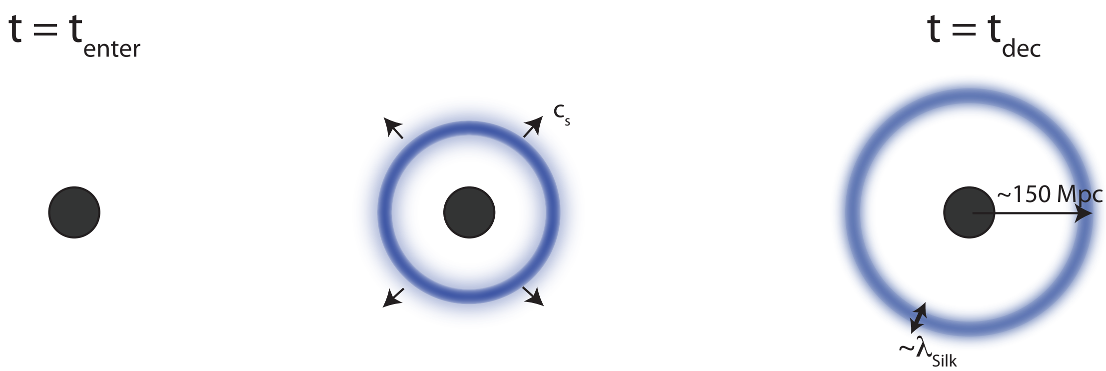

Figure 1: An illustration of the creation of circular patterns in the sky from photon+baryon pressure wave travel up until the time of de-coupling.

The story begins with a perturbation just entering the horizon, which is at that time filled with photons, baryons and dark matter. It then evolves as follows (see Figure 1):

- Since the pressure is high within the perturbation the photon+baryon fluid will expand away from the center of the perturbation, while the dark matter (which is pressure-less) will remain.
- The photons+baryons travel as a pressure wave with a sound speed of $c_s \sim c/\sqrt{3}$ until they decouple.
- At de-coupling the photons travel freely while the baryons will have been dragged out into a spherical shell around the original perturbation with diameter close to the sound-speed horizon at decoupling.
- The shell will of course travel somewhat further due to momentum and the radius of the shell is found to be $\sim$ 150Mpc.
- Because of the non-perfect coupling between photons and baryons as decoupling approaches, the mean free path of the photons increases and this blurs the shell of baryonic matter on a scale equal to the Silk scale ($\sim$ 10Mpc).

This process is now actively used observationally because it leads to a pattern in the galaxy distribution with a characteristic scale in the correlation function of $\sim$ 150Mpc.

This pattern in the correlation function leads to a set of wiggles, sinusoidal waves, in the power spectrum which has now been seen in several surveys. These structures effectively provide a standard measuring rod that can be used to probe cosmological parameters quite effectively.

A more in-depth discussion of this with nice illustrations is provided on Martin White's BAO page: http://astro.berkeley.edu/~mwhite/bao/.

6 Coupled perturbations

So free streaming of relativistic weakly interacting particles, neutrinos, removes structure at scales $M < M_{\rm FS} \approx 1.3 \times 10^{15} \left(\frac{m_\nu}{30{\rm eV}}\right)^{-2} M_\odot$. Silk damping removes *baryonic* structure on scales $M < M_{\rm Silk} \sim 10^{15} M_\odot$. Finally the rapid expansion of the Universe during the radiation dominated epoch nearly stops structure growth during this period. And observations of the microwave background tell us that large-scale perturbations were very small at the time of last scattering.

One might worry how we can have any structures at all today given this starting situation. Here cold (ie. not relativistic), non-interactive matter comes to the rescue. Since dark matter does not suffer from Silk damping because it does not interact (significantly) with radiation, the perturbations in the dark matter is free to grow slowly during the radiation dominated epoch and is ready to collapse more rapidly at $t_{\rm eq}$.

To understand this, it is now useful to introduce the perturbation equation for multiple fluid components. If we define

$$L = \frac{\partial^2}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial}{\partial t}, \tag{20}$$

then we can write the perturbation equation for component X in Fourier space as

$$L\delta_k^X = 4\pi G\bar{\rho}\delta_k^{\rm tot} - \frac{c_s^2 k^2}{a^2}\delta_k^X. \tag{21}$$

There are as many of these expressions as there are relevant components. In this expression, the first term on the right hand side couple these different components because they all contribute to the overall density. The second term on the right is only present for components that feel pressure forces.

You can then write this out fully (and in the problem set you will do). If we consider the Universe after decoupling, we can focus on dark matter and baryons only. In that case we get:

$$L\delta_k^B = 4\pi G\bar{\rho}\left(\Omega_B\delta_k^B + \Omega_{\rm DM}\delta_k^{\rm DM}\right) - \frac{k^2 c_s^2}{a^2}\delta_k^B \tag{22}$$

$$L\delta_k^{\rm DM} = 4\pi G\bar{\rho}\left(\Omega_B\delta_k^B + \Omega_{\rm DM}\delta_k^{\rm DM}\right), \tag{23}$$

where we have assumed the dark matter to be pressure-free and where B stands for baryons. If we assume that $\Omega_{\rm DM} \gg \Omega_B$, we can ignore the contribution of the baryons to the gravitational potential.

In the latter case we can make use of the (co-moving) Jeans wave number which is

$$k_J = \left(\frac{4\pi G\rho}{c_s}\right)^{1/2} a \tag{24}$$

and write the equation for the baryons as

$$\ddot{\delta}_k^B + 2\frac{\dot{a}}{a}\dot{\delta}_k^B = 4\pi G\bar{\rho}\left(\delta_k^{\rm DM} - \frac{k^2}{k_J^2}\delta_k^B\right). \tag{25}$$

After baryons decouple from photons, they will feel the gravitational pull of the dark matter perturbations and will soon follow these. Thus it is reasonable to try a solution that has the form $\delta_k^B = w_k \delta_k^{\rm DM}$. If you do that, you will find that this requires a solution where

$$w_k = \frac{1}{1 + k^2/k_J^2}, \tag{26}$$

or that

$$\delta_k^B(t) = \frac{\delta_k^{\rm DM}(t)}{1 + k^2/k_J^2}, \tag{27}$$

where we can alternatively write (why?):

$$k_J^2 = \frac{3a^2 H(t)^2}{2c_s^2}. \tag{28}$$

On large scales we see that $\delta^B \to \delta^{\rm DM}$ which implies that the baryonic matter falls into the perturbations created by the dark matter. At small scales the pressure in the baryons cannot be neglected. The overall form of the solution in the general case show similar behaviour.

7 Summarising the growth of perturbations of different scale

Figure 2 summarises the growth of perturbations visually. In addition, if free-streaming is important it will also impose a scale below which we have no perturbations. If we start with a perturbation at an early time, t_i, that is $\delta(t_i)$, we can summarise its subsequent growth as:

$$\delta_r(t) = \begin{cases} 0 & r < r_{\rm FS} \\ \delta(t_i)\left(\frac{a_{\rm eq}}{a_i}\right)^2\left(\frac{a}{a_{\rm eq}}\right)\left(\frac{r}{r_{\rm eq}}\right)^2 & r_{\rm FS} < r < r_{\rm eq} \\ \delta(t_i)\left(\frac{a_{\rm eq}}{a_i}\right)^2\left(\frac{a}{a_{\rm eq}}\right) & r > r_{\rm eq} \end{cases} \tag{29}$$

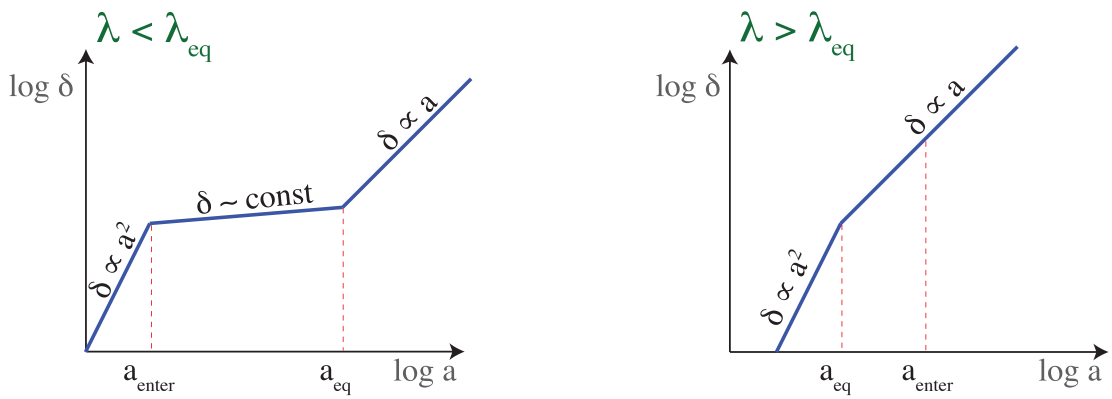

Figure 2: Summarising the growth of perturbations that enter the horizon in the radiation dominated epoch (left) and in the matter dominated epoch (right).

Where I have ignored the growth of sub-horizon scale perturbations in the radiation dominated epoch and I have observed that since $a_{\rm ent} r \sim ct_{\rm ent}$ and $a \propto t^{1/2}$ in the radiation dominated epoch, we can write $a_{\rm ent}/a_{\rm eq} = r/r_{\rm eq}$.

This scale-dependence in the growth, means that the horizon at the time of matter-radiation equality is imprinted into the large-scale structure of the Universe and we will see later how this impacts galaxy formation.

8 Statistical properties

The overdensity $\delta(\vec{x}, t)$ contains a wealth of information — to much indeed to be easy to handle. To make progress it is often necessary to summarise this information into statistical properties. In general $\delta(\vec{x})$ will be a random field given by some probability distribution function, f. To characterise δ we then calculate moments of this distribution. The first moment is

$$\langle \delta \rangle = \int \delta(\vec{x}) f(\vec{x}) d^3\vec{x} = 0 \tag{30}$$

which follows from the definition of $\delta(\vec{x})$. The second moment is generally given by

$$\langle \delta(\vec{x}_1)\delta(\vec{x}_2) \rangle = \int \delta(\vec{x}_1)\delta(\vec{x}_2) f(\vec{x}_1) f(\vec{x}_2) d^3\vec{x}_1 d^3\vec{x}_2. \tag{31}$$

For general PDFs this is not a trivial integral to do. It is simplified somewhat by the realisation that in an isotropic Universe it can only depend on $|\vec{x}_1 - \vec{x}_2| = r$.

Standard inflation models predict that initial density perturbations are distributed as a Gaussian model. In this case, we know that the first two moments are sufficient to characterise the entire distribution. The assumption of primordial Gaussianity is just that, an assumption, but it does also seem to match the observations quite well. Finding signs of primordial non-Gaussianity is a very active field of research. In the case of a Gaussian distribution we then have

$$P(\delta)d\delta = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\delta^2}{2\sigma^2}} d\delta, \tag{32}$$

where σ characterises the width of the distribution (which we will return to shortly), and P is the probability distribution of δ — not to be confused with the power spectrum introduced later. Note that this distribution has the same general functional form both in real space and Fourier space because the Fourier transform of a Gaussian is a Gaussian.

But note that even if the density field is expected to be Gaussian at an early time, it will not remain so forever. This is because by definition, $\delta > -1$ which means that as soon as δ grows appreciably it will no longer give rise to a symmetric distribution and hence gravitational collapse leads to non-Gaussian structures. The width of the distribution will of course also change because, as we will see, or as you can infer from the second moment expression above and some knowledge of Gaussian distributions, $\sigma_k^2 \sim \delta^2$. Thus, when $\delta \sim 1$ the density distribution will have significant change of $\delta < -1$ if it were Gaussian so the conclusion is that the distribution is now non-Gaussian. But this is distinct from primordial non-Gaussianity which would be visible in the linear regime.

Ignoring primordial non-Gaussianity, in the linear regime where $\delta \ll 1$, we can focus on Gaussian distributions. It is then useful to consider what this means for the individual Fourier modes. These can in general be written

$$\delta_k = \delta_R + i\delta_I, \tag{33}$$

where both δ_R and δ_I are Gaussian distributed variables with a mean of zero and the same variance, $\langle|\hat{\delta}(k)|^2\rangle/2$ (because of symmetry and independence). Since δ_R and δ_I are statistically independent, we can write their joint probability distribution as

$$\begin{aligned} P(\delta_R, \delta_I) &= \frac{1}{\sqrt{2\pi\frac{1}{2}\langle|\hat{\delta}(k)|^2\rangle}} e^{-\frac{\delta_R^2}{2\langle|\hat{\delta}(k)|^2\rangle/2}} \times \\ &\quad \frac{1}{\sqrt{2\pi\frac{1}{2}\langle|\hat{\delta}(k)|^2\rangle}} e^{-\frac{\delta_I^2}{2\langle|\hat{\delta}(k)|^2\rangle/2}} d\delta_R d\delta_I \\ &= \frac{1}{\pi P(k)} e^{-(\delta_R^2+\delta_I^2)/P(k)} d\delta_I d\delta_R \end{aligned} \tag{34}$$

which we can recast in a common form by moving to polar coordinates. If we write $\delta_R = A\cos\phi$ and $\delta_I = A\sin\phi$ we have a Jacobian $= A$ and the probability distribution transfers as

$$P(\delta_R, \delta_I) \to P(A, \phi) = \frac{1}{\pi P(k)} e^{-A^2/P(k)} A dA\, d\phi. \tag{35}$$

which shows that the amplitudes of the perturbations are Rayleigh distributed and the phases are uniformly random. This can be used to draw initial perturbations that satisfies Gaussianity.

9 The correlation function

It is useful to connect the concepts we discuss here a bit more closely to observations. In other words, how do you calculate the power spectrum for instance? To fix our discussion, let us say that the number density of galaxies is $n(\vec{x})$, which is connected to the underlying density distribution by $n(\vec{x}) = b\rho(\vec{x})$ where b is a parameter that encapsulates the degree to which galaxies trace the true underlying density distribution. In reality this parameter depends on various properties of the galaxies.

What do we mean when we say something is distributed randomly? In general we then mean that their distribution follows a Poisson process/a Poisson distribution. The chance in this case of find one object in a small volume, dV_1 and another one in a volume dV_2 is then given as

$$P(\vec{x}_1, \vec{x}_2) \equiv P_{1,2} = \bar{n}dV_1\bar{n}dV_2, \quad (36)$$

where $\bar{n}$ is the average number of galaxies.

In general we would have

$$P_{1,2} = b^2 dV_1 dV_2 \langle \rho(\vec{x}_1)\rho(\vec{x}_2)\rangle, \quad (37)$$

which we can change by inserting $\rho = \rho_b(1+\delta)$ and recalling that $\langle\delta\rangle = 0$. We then get

$$\begin{aligned} P_{1,2} &= b^2 dV_1 dV_2 \langle(\rho_b(1+\delta_1)\rho_b(1+\delta_2)\rangle & (38)\\ &= \bar{n}^2 dV_1 dV_2 (1 + \langle\delta_1\delta_2\rangle) & (39)\\ &= \bar{n}^2 dV_1 dV_2 (1+\xi(r)), & (40)\end{aligned}$$

where the last line defines the correlation function, $\xi(r)$, and where we have defined $r = |\vec{x}_1 - \vec{x}_2|$ which follows from isotropy as mentioned above.

$\xi(r)$ measures the excess or deficit in probability of finding a galaxy at a distance r from another galaxies with respect to a Poisson process. This is immediately obvious by comparing equation 36 and equation 40.

We can write this in Fourier space through

$$\begin{aligned} \xi(|\vec{x}_1 - \vec{x}_2|) &= \langle\delta(\vec{x}_1)\delta(\vec{x}_2)\rangle & (41)\\ &= \int \frac{d^3\vec{k}_1\, d^3\vec{k}_2}{(2\pi)^6} \left\langle \hat{\delta}(\vec{k}_1)\hat{\delta}(-\vec{k}_2)\right\rangle e^{-(i\vec{k}_1\cdot\vec{x}_1 - i\vec{k}_2\cdot\vec{x}_2)}, & (42)\end{aligned}$$

which is just the Fourier transform as usual and where we made a change of variable $\vec{k}_2 \to -\vec{k}_2$. The product inside the integral,

$$\hat{\delta}(\vec{k}_1)\hat{\delta}(-\vec{k}_2) = \hat{\delta}(\vec{k}_1)\hat{\delta}^*(\vec{k}_2), \quad (43)$$

because $\delta(\vec{x})$ is a real valued function. We then define the power spectrum, $P(k)$, through

$$\langle\hat{\delta}(\vec{k}_1)\hat{\delta}^*(\vec{k}_2)\rangle = (2\pi)^3 P(\vec{k}_1)\delta_{\rm Dirac}(\vec{k}_1 - \vec{k}_2), \quad (44)$$

where the Dirac delta function is obvious in the linear regime because there modes on different scales develop independently (as can be seen from the perturbation equation directly).

Integrating once we are then left with

$$\xi(|\vec{x}_1 - \vec{x}_2|) = \xi(r) = \int \frac{d^3\vec{k}}{(2\pi)^3} P(k) e^{-i\vec{k}\cdot\vec{r}}, \tag{45}$$

which shows that ξ is the Fourier transform of the power spectrum and vice-versa.

Since ξ is a real function we have that

$$e^{-i\vec{k}\cdot\vec{r}} \to \cos(kr\cos\theta), \tag{46}$$

and so if we change to spherical coordinates in equation 45 we get

$$\xi(r) = \frac{1}{2\pi^2} \int k^2 P(k) \frac{\sin kr}{kr} dk = \frac{1}{2\pi^2} \int k^3 P(k) \frac{\sin kr}{kr} d\ln k, \tag{47}$$

where the latter equality emphasises the importance of $k^3 P(k)$ for determining what scales are important. It is common to encapsulate this by defining

$$\Delta^2(k) = \frac{1}{2\pi^2} k^3 P(k), \tag{48}$$

which by insertion gives

$$\xi(r) = \int \Delta^2(k) \frac{\sin kr}{kr} d\ln k. \tag{49}$$

We can also note that $\xi(0) = \langle \delta(\vec{x})^2 \rangle = \sigma^2$, which is the variance of the field is given by:

$$\xi(0) = \sigma^2 = \int \Delta^2(k) d\ln k, \tag{50}$$

which is a useful equation for later and which also provides the σ for the Gaussian distribution above!

Finally, it is worth pointing out that we introduced $\delta(\vec{x})$ as a random field earlier. This means that $\langle \cdot \rangle$ denotes an *ensemble* average. I have however converted these quietly to *spatial* averages above. This makes an assumption of ergodicity. The ergodic hypothesis can be formulated as: *Ensemble averages equal spatial averages taken over one realisation of the random field.* This appears to be a reasonable assumption for the Universe but it is of little use when your sampling box is comparable to the size of the Universe (on really large scales) because then you have essentially no spatial averages you can make. This introduces a fundamental limitation in the accuracy of results on really large scales and is known as cosmic variance.