Lecture 2

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1 Introduction

This lecture will focus on the growth of perturbations in general. This is important to understand for galaxy formation because in this game we start with a set of initial perturbations at some very early time. These perturbations then grow through time to provide the initial conditions for galaxy formation at at time after matter radiation decoupling. Note that I will attempt to denote co-moving sizes by r and proper sizes by λ or d in the following.

2 Size regimes and horizon crossing

The Hubble parameter provides us with a characteristic scale for the Universe, the Hubble radius, $d_H = c/H(t)$. We will call this loosely the "horizon" here although it is not formally identical to the proper horizon. The physical importance of the Hubble radius is that we have:

$$\begin{array}{lcl} \lambda \ll d_H & \Rightarrow & \text{Newtonian regime} \\ \lambda \gg d_H & \Rightarrow & \text{Curvature of space important, so GR is necessary.} \end{array}$$

Thus the scale defined by d_H is relevant to us. Furthermore in standard FRW cosmology all scales will at some point in the Universe have been larger than the horizon. This follows from the fact that the proper size of a region with size λ_0 today (co-moving or proper), has

$$\lambda(z) = \frac{\lambda_0}{1+z},$$

whereas the Hubble scale can be approximated as

$$\begin{array}{lcll} d_H(z) & \propto & (1+z)^{-3/2} & \text{Matter dominated} \\ d_H(z) & \propto & (1+z)^{-2} & \text{Radiation dominated,} \end{array}$$

and in either case $d_H(z_e) < \lambda(z_e)$ for $z > z_e$. This redshift, z_e, corresponds to the time a scale entered the horizon and we refer to it as the redshift of entering. At matter-radiation equality we have that the co-moving size of the horizon is given by:

$$r_{\rm eq} \sim 90\,{\rm Mpc} \qquad [14\,(\Omega_{\rm NR}h^2)^{-1}\,{\rm Mpc}]. \tag{1}$$

This is for three species of neutrinos and uses $d_H(t)$ as the horizon size. A more careful calculation (done in your problem set), gives that the comoving horizon is $\approx 16\,(\Omega_{\rm NR}h^2)^{-1}$ Mpc.

In general it is more convenient to refer to masses than to scales. Because the proper size scales as $\lambda \propto a$ and the density as $\rho \propto a^{-3}$, we find that the mass,

$$M \propto \rho\lambda^3 = \text{constant}.$$

Thus the mass enclosed within a proper size is a constant and provides a useful label to work with (we often evaluate this at $z = 0$ where proper size is equal to co-moving size and hence we often give densities relative to the critical density today). It is useful to look at some numerical values (the sizes here are diameters), setting $\rho = \rho_{\rm crit}$:

$$\begin{aligned} M(\text{within Mpc}) &\sim 1.45 \times 10^{11} h^2\, M_\odot \\ M(\text{within Mpc}) &\sim 1.45 \times 10^{14} h^2\, M_\odot. \end{aligned}$$

The former mass is of the order of a galaxy mass, while the latter is of the order of a cluster mass. Since both of these sizes are smaller than $r_{\rm eq}$, we conclude that all mass scales of astrophysical interest entered the horizon in the radiation dominated epoch.

When the scales are outside the horizon we really need a proper general relativistic treatment to get their evolution, but the smallness of perturbations on the Cosmic Microwave Background can be taken as evidence that the perturbations must have been small so GR in the linear regime is sufficient. Nevertheless, it is a somewhat tedious procedure to go through (see for instance section 4.2 in Mo, van den Bosch & White) and it is not essential for us. Instead we can use a simplified treatment to get the right scalings.

3 Superhorizon fluctuations — approximate treatment

Consider a spherical region embedded in a flat FRW Universe and take the surrounding Universe to have density $\rho_b(t)$ and scale factor $a_1(t)$. The spherical region is taken to have a slightly higher density, $\rho_b(t) + \Delta\rho(t)$ and scale-factor $a_2(t)$. From spherical symmetry the evolution inside the spherical region is unaffected by the external field.

In that case each of the two regions will evolve like a separate FRW Universe and we can write

$$H_1(t)^2 = \frac{8\pi G}{3}\rho_b(t) \tag{2}$$

$$H_2(t)^2 = \frac{8\pi G}{3}\left(\rho_b(t) + \Delta\rho(t)\right) - \frac{kc^2}{a_2^2}, \tag{3}$$

where $H_1 = \dot{a}_1/a_1$ and likewise for H_2. We then take the difference of these two equations:

$$H_2(t)^2 - H_1(t)^2 = \frac{8\pi G}{3}\Delta\rho - \frac{kc^2}{a_2^2}. \tag{4}$$

If we write $a_2 = a_1 + \Delta a$, which will be valid at an early time and expand in Taylor series in $\Delta a/a$, we find that

$$\frac{\Delta\rho}{\rho_b} = \delta = \frac{3kc^2}{8\pi G a_1^2 \rho_b}, \tag{5}$$

and this leads to two distinct scalings:

$$\begin{aligned} &\text{Radiation domination:} \qquad && \delta \propto \frac{1}{a^2}\frac{1}{a^{-4}} \propto a^2 \propto t \\ &\text{Matter domination:} \qquad && \delta \propto \frac{1}{a^2}\frac{1}{a^{-3}} \propto a \propto t^{2/3} \end{aligned}$$

which gives us the evolution behaviour of super-horizon scale perturbations. These scalings are also what is found using a full GR treatment.

The derivation above is actually even sloppier than just ignoring GR. I have ignored any discussion about what t means for instance. A slightly more careful derivation is the one given in the lecture where I used equation **??** and now made explicit that we need to compare these at the same time after the Big Bang. In that case you write out the time in the background

$$t_1(a_1) = \int_0^{a_1} \frac{da'}{a' H_1(a')}, \tag{6}$$

and the perturbed region

$$t_2(a_2) = \int_0^{a_2} \frac{da'}{a' H_2(a')} \tag{7}$$

Setting these $t_1 = t_2$ and expanding in a Taylor series using $a_2 = a_1 + \Delta a$ where $\Delta a \ll a_1$ gives

$$\int_0^{a_1} \frac{da'}{a' H_1(a')} =\approx \int_0^{a_1+\Delta a} \frac{da'}{a' H_1(a')} \left(1 + \frac{1}{2}\frac{\Delta K}{a'^2 H_1^2(a')}\right), \tag{8}$$

which can be split out (and assuming that $H_1(a)$ is constant over Δa) we get

$$\frac{\Delta a}{a_1} = -\frac{1}{2}\Delta K H_1(a_1) \int_0^{a_1} \frac{da'}{(a' H_1(a'))^3}, \tag{9}$$

which we can integrate up by using that $H^2(a) \propto a^{-3(1+w)}$ where $w = 1/3$ when radiation is the dominant component and $w = 0$ for matter domination. This gives exactly the same scalings as given above.

4 Subhorizon fluctuations

We will now focus on sub-horizon fluctuations. One can trace the fluctuations in various ways:

1. You can follow particle trajectories in detail. This is essentially the approach taken by numerical simulations with some simplifications.

2. The next best thing is to take a slightly coarser view and instead of individual particles evolve a distribution function, $f(\vec{x}, \vec{p}, t)$. This amounts to follow the Boltzmann equation in expanding coordinates and is an essential approach to do accurate calculations of the mixed photon and matter fluid before decoupling. In practice these calculations are done using large purpose-written software packages such as CMB-FAST[1] by U. Seljak and M. Zaldarriaga and CAMB[2] by A. Lewis & A. Challinor.

3. Finally one has the possibility of going to the fluid limit where physical variables are treated as smooth functions of space and time. In this case we have a well defined velocity at each point so an essential difference from the preceding approach is that in the fluid limit we have no dispersion in velocities at a given point.

We will assume that there is a smooth background Universe with density ρ_b and temperature T_b and we will consider perturbations of this field. As long as these perturbations are small, they do not influence the overall evolution of the Universe and we can use the scale factor of the background Universe. It is also handy to write the density, $\rho(\vec{x}, t)$ as

$$\rho(\vec{x}, t) = \rho_b(t)[1 + \delta(\vec{x}, t)] \quad \Leftrightarrow \quad \delta(\vec{x}, t) = \frac{\rho(\vec{x}, t) - \rho_b(t)}{\rho_b(t)}, \tag{10}$$

which also defines the overdensity, δ, which is the quantity we will focus our attention on.

In the fluid limit it is possible to write the fluid equations in an expanding Universe using co-moving coordinates as:

$$\begin{aligned}
\frac{\partial \delta}{\partial t} &+ \frac{1}{a}\nabla_{\vec{x}} \cdot (1+\delta)\vec{v} = 0 && (11)\\
\frac{\partial \vec{v}}{\partial t} &+ \left(\frac{\dot{a}}{a}\right)\vec{v} + \frac{1}{a}\left(\vec{v}\cdot\nabla_{\vec{x}}\right)\vec{v} = -\frac{\nabla_{\vec{x}}\Phi}{a} - \frac{\nabla_{\vec{x}}P}{a\rho_b(1+\delta)} && (12)\\
\nabla^2_{\vec{x}}\Phi &= 4\pi G\rho_b a^2\delta && (13)\\
\Phi &= \phi + \frac{1}{2}a\ddot{a}x^2, && (14)
\end{aligned}$$

where $\vec{v}$ is the peculiar velocity, P is the pressure, ϕ is the gravitational potential in non-comoving coordinates, and the gradients are taken along the co-moving coordinate $\vec{x}$. The notation for Φ is just a convenience of notation — it comes naturally from the change of

[1] http://lambda.gsfc.nasa.gov/toolbox/tb_cmbfast_ov.cfm

[2] http://lambda.gsfc.nasa.gov/toolbox/tb_camb_ov.cfm

variables in the Euler equation if you move all velocity independent terms to the right hand side.

We need to close this with an equation of state for the pressure. Generally we will take this to be barotropic equation of state, ie. $P = P(\rho)$. For the moment, we will however keep this more general and write $P = P(\rho, S)$, where S is the entropy of the gas. In this case we find

$$\frac{\nabla_{\vec{x}} P}{\rho_b} = c_s^2 \nabla_{\vec{x}} \delta + \frac{2}{3}(1+\delta) T_b \nabla_{\vec{x}} S, \tag{15}$$

where the sound speed is

$$c_s^2 = \left(\frac{\partial P}{\partial \rho}\right)_S . \tag{16}$$

The gradient in entropy gives rise to iso-curvature perturbations which can arise even in a Universe without curvature perturbations. The contribution of iso-curvature perturbations to the Universe is constrained to be small from observations of the CMB so in the following I will ignore it.

We can then linearise equations 11 to 13 and get:

$$\frac{\partial^2 \delta}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta}{\partial t} = 4\pi G \rho_b \delta + \frac{c_s^2}{a^2} \nabla_{\vec{x}}^2 \delta. \tag{17}$$

This is the equation that governs the evolution of density perturbations. It is however more tractable in Fourier space, so recalling that $\nabla_{\vec{x}} \to i\vec{k}$ and $\nabla_{\vec{x}}^2 \to -k^2$, we get:

$$\frac{\partial^2 \delta_k}{\partial t^2} + 2\frac{\dot{a}}{a}\frac{\partial \delta_k}{\partial t} = 4\pi G \rho_b \delta_k - \frac{c_s^2}{a^2} k^2 \delta_k, \tag{18}$$

where $k = 2\pi/\lambda$.

4.1 Jeans length and mass

If we ignore the expansion of the Universe in equation 18, we see that it turns into an equation of the form

$$\ddot{\delta}_k = \omega^2 \delta_k \tag{19}$$

with

$$\omega^2 = 4\pi G \rho_b - \frac{k^2 c_s^2}{a^2} \tag{20}$$

This gives as solution:

$$\delta_k(t) = A e^{\omega t} + B e^{-\omega t} \tag{21}$$

Thus, if $\omega^2 < 0$ the solution has oscillatory behaviour, whereas if $\omega^2 > 0$ the general solution is a superposition of a growing and a decreasing mode. In this case we would get exponential collapse.

This defines a typical length which is where $\omega = 0$, the co-moving Jeans length:

$$r_J = \frac{c_s}{a}\sqrt{\frac{\pi}{G\rho_b}}, \tag{22}$$

which is here written as the *co-moving* length. For an ideal gas the speed of sound is given by

$$c_s = \sqrt{\frac{5k_B T_b}{3m_p}}, \tag{23}$$

with m_p being the proton mass and 5/3 corresponding to the ratio of specific heats for a monatomic gas.

We can use this length to define a mass as before and we get the Jeans mass, M_J:

$$M_J = \frac{4\pi}{3}\left(\frac{r_J}{2}\right)^3 \rho_{c,0}\Omega_{B,0}h^2, \tag{24}$$

with the cosmological parameters being evaluated at the time today. Note that the density here is the *baryon* density because only the baryons feel the pressure forces (to first order at least). Quantitatively this gives

$$\begin{aligned} \text{Before recombination:} \quad M_J &\approx 1.2 \times 10^{16} \left(\Omega_{B,0}h^2\right)^{1/2} M_\odot \\ \text{After recombination:} \quad M_J &\approx 1.5 \times 10^{5} \left(\Omega_{B,0}h^2\right)^{1/2} M_\odot \end{aligned}$$

Why the large drop in Jeans mass? The reason is the huge change in sound speed — from near the speed of light to just a few thousands m/s after recombination. Related to this, there is also a big change in the pressure because the relevant number of particles (recall $P = nkT$ as the ideal gas law) drops from the number of photons, n_γ, to the number of baryons, n_B, after photons and baryons decouple. This corresponds to a drop in pressure (assuming no temperature change) of a factor of 10^8.

5 Solving the perturbation equation

Solving equation 18 in general requires numerical techniques, but a simple solution for the pressure-less case can be found. By noting that in that case the equation for δ is exactly the same as the equation satisfied by the Hubble parameter, $H(t)$. The simplest way to see this is perhaps by writing the time-time component of the Friedmann equation which is (e.g. MvdBW section 3.2.1)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + 3\frac{P}{c^2}\right) + \frac{\Lambda c^2}{3} \tag{25}$$

If you take the time derivative of this and note that Λ is constant you find an equation for $H(t)$ that is mathematically identical to equation 18. Thus one solution of equation 18 is $\delta_- \propto H(t)$. Since $H(t)$ decreases with time, this will be a decaying solution.

To get the physically interesting growing mode, we can make use of the fact that the Wronskian of the differential equation can be written as:

$$\delta_-\dot{\delta_+} - \delta_+\dot{\delta_-} = ka^{-2},$$

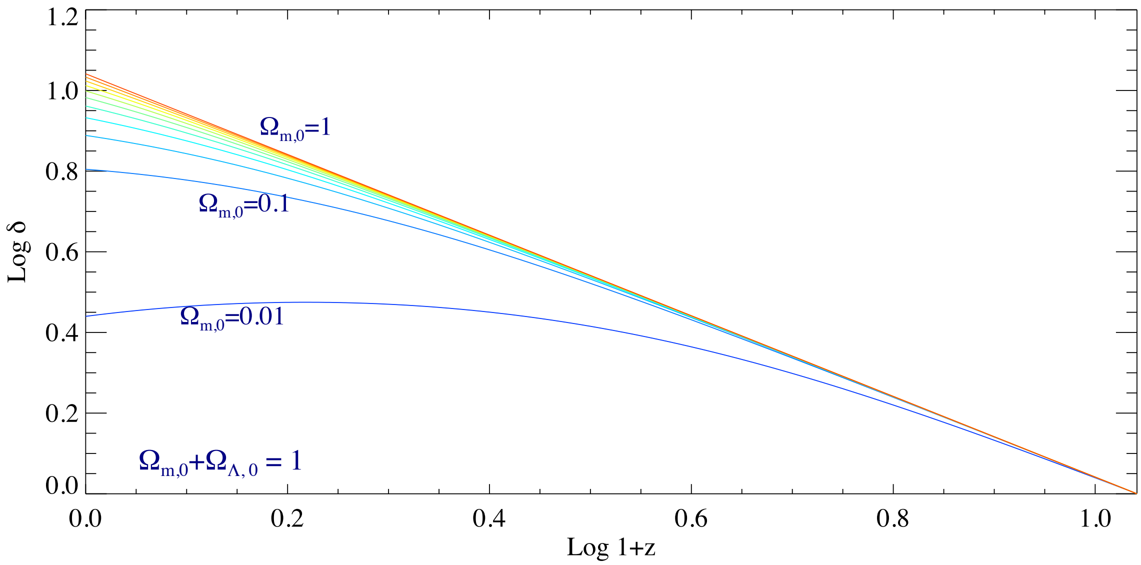

Figure 1: The growth of perturbations in flat Universes with varying matter content. They have all been normalised to the same value at z=10. The spacings between models is 0.1 in Ω_m, except that $\Omega_m = 0$ has been replaced by $\Omega_m = 0.01$.

and this is a first order linear equation which can be solved using an integrating factor, $u(t) = H(t)^{-1}$. We then obtain the full solution as

$$\delta_+(t) \propto H(t) \int_0^t \frac{d\tau}{a(\tau)^2 H(\tau)^2} dt \tag{26}$$

Numerical solutions to this equation for a flat Universe are shown in Figure 1. Several different density parameters are shown and as can be seen, lowering the matter content slows the growth of the perturbations because the Universe expands more rapidly.

6 What slows down perturbations?

6.1 The Hubble expansion

The growth of perturbations is slowed down because of the Hubble expansion which acts as a drag force against the collapse. Intuitively, when the expansion is faster than the collapse you will not have collapse. Since the expansion time is $t_{\text{expansion}} \sim H(t)^{-1} \sim \rho_{\text{dominant}}^{-1/2}$ and the collapse time for matter is $t_{\text{dynamic}} \sim \rho_{\text{matter}}^{-1/2}$ you see that if the density of the dominant ingredient is much higher than that of the matter you get no growth. This is in general true in the radiation dominated epoch.

To do a more careful analysis, the standard approach is to introduce a variable, $y = \rho_{\text{NR}}/\rho_{\text{R}}$, that is the ratio of the matter density that of the radiation. This is a convenient variable, and $y = a/a_{\rm eq}$ is another way to write it. Introducing this into the perturbation equation 17 and using the Friedman equation (ignoring the cosmological constant and curvature terms), one gets (after some straightforward but tedious manipulations):

$$2y(1+y)\frac{d^2\delta}{dy^2} + (2+3y)\frac{d\delta}{dy} = 3\delta. \tag{27}$$

This equation is solved by $\delta \propto 1 + 3y/2 = 1 + \frac{3}{2}\frac{a}{a_{\rm eq}}$. From this we can immediately see that growth during the radiation dominated regime is very slow at maximum a factor of 5/2.

Putting this all together gives us a picture of the growth of perturbations that can be summarised in the following table:

	radiation dominated	matter dominated
$a < a_{\rm enter}$	$\delta \propto a^2$	$\delta \propto a$
$a > a_{\rm enter}$	$\delta \sim$ near constant	$\delta \propto a$

Next time we will explore the consequences of this and discuss a number of other processes that affect the overall growth of the perturbations.