Probability theory

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Levend Heelal, 23 mei 2005

Contents

1. Calculating probabilities

P stands for Probability!

Example

Throw with 8 dice

$$P(\text{all 8 outcomes are 6}) = \left(\frac{1}{6}\right)^8$$

$P(A) =$ the probability of the event A

Properties (axioms):

- for every event A, we have $P(A) \geq 0$
- $P(\text{impossible event}) = 0$
- if the events A and B cannot occur together (that is, A **and** B is impossible), then we have the **sum rule**

$$P(A \text{ or } B) = P(A) + P(B)$$

- $P(A \text{ does not occur}) = 1 - P(A)$.

Random selection

Definition

$$P(A) = \frac{\text{number of instances where } A \text{ occurs}}{\text{total number of instances}}.$$

Example. Throw die once.

$$P(\text{odd outcome}) = \frac{3}{6}.$$

Probabilities can often be best understood using **urn models**.

Example. Probability of mutation = 10^{-9} means: selecting the red ball in an urn of 999 999 999 green balls and one red ball.

Random selection commonly means that each possible outcome is *equally* likely.
We probabilists, call this the **uniform** distribution, to distinguish it from random distributions where each outcome is random but *not necessarily equally* likely. (In many discussions (e.g. evolution theory), the distinction is not made.)

Otherwise stated:
some things are more random than others.

2. Combinatorics

Law of multiplication

- number of license plates, consisting of 2 letters, 2 digits and 2 letters

$= 26 \times 26 \times 10 \times 10 \times 26 \times 26 = 44687600$

- number of telephone numbers with 10 digits

$= 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$

$= 10\ 000\ 000\ 000$

$= 10^{10}$

- number of telephone numbers consisting of 10 **different** digits

$= 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 1\ 088\ 640$

$= 10!$

- probability of a telephone number with 10 different digits

$= \frac{1\ 088\ 640}{10\ 000\ 000\ 000} = 0.001088640$

- number of books with 100 pages:

- 26 letters, comma, blank, period, question mark

$\rightarrow$ 30 possibilities

- 75 positions per line

- 40 lines per page

-100 pages

-TOTAL: $30^{75 \times 40 \times 100}$

$= 30^{300\ 000}$

$> 10^{145\ 000}$

binomial coefficient

The number of ways to select k items from n is equal to

$$\frac{k!(n-k)!}{n!} = \binom{n}{k}$$

So

$$\binom{5}{3} = 10$$

$$\binom{7}{3} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(3 \times 3 \times 1) \times (4 \times 3 \times 2 \times 1)}$$

$$= \frac{7 \times 6 \times 5}{3 \times 2} = 35$$

etc.

Example.

Urn with 6 red balls and 4 green balls.

Select at random 5 balls.

Probability of 3 red balls:

With replacement:

$$\binom{5}{3}(0.6)^3(0.2)^2 = 0.3456$$

Without replacement:

$$\frac{\binom{6}{3} \times \binom{4}{2}}{\binom{10}{5}}$$

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$$\frac{\binom{6}{3} \times \binom{4}{2}}{\binom{10}{5}} = \frac{20 \times 6}{252} = 0.4762$$

With replacement

→ **binomial** distribution

Without replacement

→ **hypergeometric** distribution

Pascal’s triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

$\binom{8}{3}$ 1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

etc.

3. Conditional probability

Definition **The probability of A given B is**

$$P(A|B) = \frac{P(\text{both } A \text{ and } B)}{P(B)}$$

Example.

$P(\text{medical sci.}|\text{university IV}) = 1/34$

$P(\text{university III}|\text{law}) = 7/34$

faculty \ university	I	II	III	IV	
natural sci.	16	3	2	13	34
social sci.	5	10	11	8	34
law	9	6	7	12	34
medical sci.	4	15	14	1	34
	34	34	34	34	136

Good news: with conditional probabilities we can take into account new information.

Example.

$P(\text{extraterrestrial life}) = 0.25$ (say)

$P(\text{extraterrestrial life}|little\ green\ man)$

> 0.25

Example.

$P(\text{birth before 40 weeks}) = 0.65$

$P(\text{birth between week 40 and week 41}) = 0.30$

$P(\text{birth between week 41 and week 42}) = 0.05$

$\Rightarrow$

$$P(\text{birth between week 40 and week 41}| \cdots$$
$$\cdots 40 \; weeks \; pregnant) = \frac{0.30}{0.35} = 0.8571$$

Bad news: conditional probabilities are conceptually hard.

Example: blood test for certain disease.
It is a reliable test:

$$P(\text{test gives correct result}) = 0.99$$

Suppose now that you are tested and the result is positive. (Bad news)

Are you in bad shape then?
What is the probability that you have the disease given that you test result is positive?

To calculate this probability, you have to know the incidence rate of the disease, i.e., the probability that someone has the disease.
For example, suppose

$$P(\text{disease}) = 0.001$$

Then

$$P(\text{disease}|\mathit{test\ positive}) = 0.112!$$

(Good news!)

This follows from

$$P(A|B) = \frac{P(\text{both } A \text{ and } B)}{P(B)}$$

so that by interchanging the role of A and B

$$P(B|A) = \frac{P(\text{both } A \text{ and } B)}{P(A)}.$$

In other words

$$P(B|A) = P(A|B)\frac{P(B)}{P(A)}.$$

This is called **Bayes rule**.

Example. Let

A= your test result is positive

B = you have the disease

We want to calculate

$$P(B|A) = P(A|B)\frac{P(B)}{P(A)}.$$

We know

$$P(A|B) = 0.99$$

and

$$P(B) = 0.001.$$

But how do we know

$$P(A)?$$

Theorem **Multiplication rule:**

$$P(A) = P(A|B)P(B)$$
$$+P(A|B\ does\ not\ occur)(1 - P(B)).$$

Example.

$$P(A) = P(\text{positive}|disease)P(\text{disease})$$
$$+P(\text{positive}|healthy)P(\text{healthy})$$
$$= 0.99 \times 0.001 + 0.01 \times 0.999 = 0.0089$$

So

$$P(B|A) = P(A|B)\frac{P(B)}{P(A)}$$
$$= 0.99 \times \frac{0.001}{0.0089} = 0.112.$$

4. Prior and posterior probabilities

Bayes rule:

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)},$$

or

$$P(\text{beliefs}|\text{data}) \sim P(\text{data}|\text{beliefs})P(\text{beliefs})$$

Let B be the event

H_0: a certain hypothesis holds true,

with alternative

H_1: the hypothesis does not hold true

Let A be data.

We call

$P(H_0)$= **prior**

and

$P(H_0|\mathrm{data})$ = **posterior**.

Note:

$P(H_1) = 1 - P(H_0)$

and similarly

$P(H_1|\mathrm{data}) = 1 - P(H_0|\mathrm{data})$

We call

$P(H_0)/P(H_1)$= **prior odds**

and

$P(H_0|\text{data})/P(H_1|\text{data})$ = **posterior odds**.

Furthermore

$P(\text{data}|H_0)/P(\text{data}|H_1)$= **likelihood ratio**.

THEOREM

posterior odds = **prior odds** $\times$ **likelihood ratio**

Example.

$P(\text{extraterrestrial life}) = 0.25$

Hence, prior odds: $\frac{0.25}{0.75}$ = 1 : 3

Suppose

$P(\text{little green man}|\text{extraterrestrial life}) = 0.10$

and

$P(\text{little green man}|\text{no extraterrestrial life}) = 0.05$

Then likelihood ratio = 2,

so the posterior odds equals

$2\times$ (1 : 3) = 2 : 3

5. Independence

Definition A and B are called **independent** if

$$P(A|B) = P(A),$$

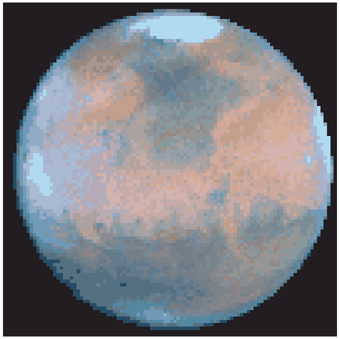
i.e., knowing whether or not B occurs does not change the probability of A.

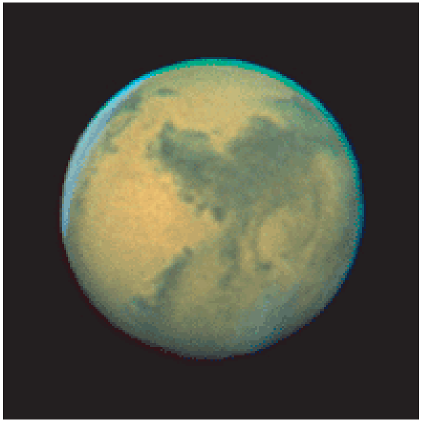
Thus, for independent events, we may multiply probabilities

$$P(\text{both } A \text{ and } B) = P(A) \times P(B).$$

Example.

MARS NEW PLANET





$$P(\text{life on Mars}) = 0.05$$
$$P(\text{life on new planet}) = 0.30$$

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$$P(\text{life on new planet}) = 0.30$$

Assume independence.

Then

$$P(\text{life on both Mars and new planet})$$

$$= 0.05 \times 0.30 = 0.015$$

$$P(\text{no life on both Mars and new planet })$$

$$= 0.95 \times 0.70 = 0.665$$

$$P(\text{ life on Mars and/or on new planet})$$

$$= 1 - 0.665 = 0.335$$

Example.

Let

n= number of planets similar to earth.

Say $n = 10^{15} = 1\,000\,000\,000\,000\,000$

A_i = life on planet i

$p = P(A_i)$

(assumed to be the same for all i)

According to Aczel:

$p = 5 \times 10^{-14} = 0.00000000000005$

Assume independence.

Exercise: Calculate the probability of life on one or more of these planets.

Solution

$$
\begin{aligned}
P(\text{life}) &= 1 - P(\text{no life on all } n \text{ planets}) \\
&= 1 - \left(P(\text{ no life on planet } i)\right)^n \\
&= 1 - \left(1 - P(\text{life on planet } i)\right)^n \\
&= 1 - \left(1 - P(A_i)\right)^n \\
&= 1 - \left(1 - p\right)^n \\
&= 1 - \left(1 - 0.0000000000000005\right)^{1\,000\,000\,000\,000\,000}
\end{aligned}
$$

Math:

$$1 - (1 - p)^n \geq 1 - \mathrm{e}^{-np}$$

so

$$P(\text{life}) \geq 1 - \mathrm{e}^{-5} = 0.9933.$$