Huygens relativity

Galileo still spoke of objects as if they have a "proper place" in Nature (*il suo luogo*). He also identified circular motion as the only 'natural' motion (*moto naturale*).

Huygens was the first to realize that mechanics may be built on a *principle of relativity*.

Because it appears to be impossible to read off the position $\vec{x}$ or the time t on any particle, one should consider $(\vec{x}, t)$ to be relative, and only *changes* thereof are observable.

Thus, the equation of motion of a particle is not an algebraic equation but (as Newton formulated it) a *differential equation*, in which

$$\vec{v} \equiv \frac{d\vec{x}}{dt} \tag{1}$$

But Huygens also noted that it appears to be impossible to read off the velocity $\vec{v}$ on any particle.

Thus, one should consider $\vec{v}$ to be relative, and only *changes* thereof are observable. Accordingly, one should consider the acceleration

$$\vec{a} \equiv \frac{d\vec{v}}{dt} = \frac{d^2\vec{x}}{dt^2} \tag{2}$$

But wait a minute– why then not continue, and go on with

$$\vec{b} \equiv \frac{d\vec{a}}{dt} = \frac{d^3\vec{x}}{dt^3} \tag{3}$$

and so forth?

Because it turns out to be possible to read off the acceleration $\vec{a}$ on a particle!

Accelerations are absolute, not relative. As is now known, this is due to the existence of an *absolute* velocity, the speed of light c.

The roots of classical mechanics, as enumerated above, show up everywhere in hydrodynamics, and even in the design of *numerical* hydro methods. Basically,

$$(t, \vec{x}, \vec{v}) \tag{4}$$

are the coordinates of any system of particles, and $\vec{a}$ is prescribed externally by the usual

$$\vec{F} = m\,\vec{a} \tag{5}$$

Particle averages

If we do not have a single particle, but consider the average of the motion of a large number of particles which are closely coupled by thermal collisions, we can no longer use the derivative d/dt. Because we have taken an average, *we must specify a place and a time* where our average is taken. Accordingly, we must expand the difference dQ of any quantity Q in the space spanned by Eq.(4):

$$dQ = \frac{\partial Q}{\partial t}\, dt + \frac{\partial Q}{\partial x_j}\, dx_j + \frac{\partial Q}{\partial v_j}\, dv_j \tag{6}$$

where we have used the Einstein convention for summation over repeated indices. Accordingly, any function f of the variables enumerated in Eq.(4) obeys the equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_j}\, \frac{dx_j}{dt} + \frac{\partial f}{\partial v_j}\, \frac{dv_j}{dt} \tag{7}$$

or, using Eqs.(1,2),

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + v_j\, \frac{\partial f}{\partial x_j} + a_j\, \frac{\partial f}{\partial v_j} \tag{8}$$

This “three-difference” form will occur again and again, not only in the derivation of the equations of motion of hydrodynamics, but even in the basic design of all numerical hydro methods.

First sketch of averaging

If we have no imposed external acceleration, $\vec{a} = 0$, and

$$\begin{aligned} \frac{d}{dt} &= \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y} + \frac{dy}{dt}\frac{\partial}{\partial z} \\ &= \frac{\partial}{\partial t} + v_j \, \frac{\partial}{\partial x_j} \end{aligned} \tag{9}$$

so that we expect the velocity to evolve according to

$$\frac{\partial v_i}{\partial t} + v_j \, \frac{\partial v_i}{\partial x_j} = 0 \tag{10}$$

Note that this equation is much more difficult to deal with than the classical Eq.(5), because it is

- a partial differential equation
- a spacetime equation
- nonlinear due to the factor v_j
- strongly three-dimensional due to $\partial v_i / \partial x_j$ We will find that these unpleasant properties hold for all the equations of hydrodynamics:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_i}{\partial x_i} = \frac{\partial \rho}{\partial t} + \mathrm{div}\, \rho \vec{v} = 0 \tag{11}$$

$$\frac{\partial v_i}{\partial t} + v_j \, \frac{\partial v_i}{\partial x_j} = -\frac{1}{\rho} \, \frac{\partial P}{\partial x_i} + \vec{a} \tag{12}$$

(and likewise for the energy equation, which we will see later).
If we split the velocity into a systematic part $\vec{w}$, the 'wind speed', and a random (thermal) part $\vec{u}$,

$$\vec{v} \equiv \vec{w} + \vec{u} \tag{13}$$

then we find the averages

$$\begin{aligned} < \vec{v} > &= \vec{w} \\ < v^2 > &= w^2 + \alpha T = w^2 + \beta \frac{P}{\rho} \end{aligned} \tag{14}$$

where T is the gas temperature, P is the pressure, and ρ is the mass density. The second equation can be written even more clearly as

$$\begin{aligned} \frac{1}{2} < v^2 > &= \frac{1}{2} w^2 + \frac{\gamma}{\gamma - 1} \frac{P}{\rho} \\ &= \frac{1}{2} w^2 + \frac{s^2}{\gamma - 1} \end{aligned} \tag{15}$$

in which s is the speed of sound:

$$\mathcal{R} T = \gamma \frac{P}{\rho} = s^2 \tag{16}$$

Application to HH34

$$\frac{T}{10,000\,\mathrm{K}} \approx \left(\frac{s}{10\,\mathrm{km/s}}\right)^2 \tag{17}$$

$$\sin\,\alpha = \frac{st}{wt} = \frac{s}{w} = \frac{1}{\mathcal{M}} \tag{18}$$

From now on, we will use $\vec{v}$ for the wind velocity

$$\begin{aligned} \textbf{mass} \quad & \rho \\ \textbf{momentum} \quad & \rho\,\vec{v} \\ \textbf{energy} \quad & \frac{1}{2}\rho v^2 + \rho\,\frac{s^2}{\gamma - 1} \end{aligned} \tag{19}$$

Distribution function f

$$\frac{df}{dt} = \text{local effects of collisions} \tag{20}$$

and if we average over a box that is much larger than the collision mean free path,

$$\frac{df}{dt} = 0 \tag{21}$$

$$df = dt\,\frac{\partial f}{\partial t} + dx_j\,\frac{\partial f}{\partial x_j} + dv_j\,\frac{\partial f}{\partial v_j} \tag{22}$$

Here we see the "three-difference" form in action; it will occur again and again,

dt	absolute time does not exist
dx	neither does absolute space
dv	nor does absolute velocity

and therefore

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + v_j\,\frac{\partial f}{\partial x_j} + a_j\,\frac{\partial f}{\partial v_j} \tag{23}$$

Probability function

$$< Q > \equiv \int Q\, f(t, \vec{x}, \vec{v})\ d^3v \tag{24}$$

Remember that, because f vanishes at the boundaries, expectation values of v-derivatives are zero, so that

$$\begin{aligned} \int v_i \,\frac{\partial f}{\partial v_j}\ d^3v &= \int \frac{\partial f v_i}{\partial v_j}\ d^3v - \int f\, \frac{\partial v_i}{\partial v_j} d^3v \\ &= 0 - \int f\, \delta_{ij}\ d^3v \end{aligned} \tag{25}$$

What is the equation for the expectation value of $Q = 1$? We have

$$\begin{aligned} \frac{df}{dt} &= \\ &\frac{\partial f}{\partial t} + v_j\, \frac{\partial f}{\partial x_j} + a_j\, \frac{\partial f}{\partial v_j} = \\ &\frac{\partial f}{\partial t} + v_j\, \frac{\partial f}{\partial x_j} + \frac{F_j}{m}\, \frac{\partial f}{\partial v_j} = 0 \end{aligned} \tag{26}$$

Integrating over v-space, the third term drops out because f vanishes at the boundaries, so that expectation values of v-derivatives are zero. The second term can be rewritten

$$v_j \frac{\partial f}{\partial x_j} = \frac{\partial f v_j}{\partial x_j} - f \frac{\partial v_j}{\partial x_j} \tag{27}$$

Upon integration, the second term of this vanishes also, for the same reason. Thus, what remains is

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_j} (n w_j) = 0 \tag{28}$$

in which n is the particle density

$$n \equiv \int f \; d^3 v \tag{29}$$

and $\vec{w}$ is the wind speed. If all the particles have the same mass,

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} = 0 \tag{30}$$

which is the equation of mass conservation, or *continuity equation.*