ORIGINS AND EVOLUTION OF THE UNIVERSE - SAMPLE EXAM

Read all questions carefully. Write your name and student number on the first page and number all pages.

The Friedmann Robertson-Walker metric is given by

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + (\sin\theta)^2 d\phi^2)\right].$$

The particle number density for a non-relativistic gas in thermal equilibrium at a temperature T is given by:

$$n(T) = g\left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2} e^{-mc^2/k_BT},$$

where k_B is Boltzmann's constant and g the statistical weight of the particle. For a relativistic gas in thermal equilibrium the number density is given by:

$$n(T) = 0.122 \times g \times F\left(\frac{k_BT}{\hbar c}\right)^3,$$

where $F = 1$ for bosons and $F = 3/4$ for fermions. The corresponding energy density of such a relativistic gas is given by:

$$\rho(T)c^2 = F' \times g \times \frac{\sigma_r T^4}{2},$$

where $\sigma_r = 7.565 \times 10^{-15}\mathrm{erg/cm^3/K^4}$ is the radiation density constant and $F' = 1$ for bosons and $F' = 7/8$ for fermions. Other useful numbers are 1 eV=1.16×10^4K and the hydrogen ionisation potential is 13.6 eV.

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1 [*40 points*] **Answer the following questions in a few sentences only, or with a diagram - no derivations**

(a) (*5 points*) State the cosmological principle and explain the terms used.
(b) (*5 points*) What is Silk damping?
(c) (*5 points*) How does the scale factor evolve with time in a radiation-dominated universe?
(d) (*5 points*) Why does the power spectrum of the temperature fluctuations in the CMB radiation show peaks?
(e) (*5 points*) What is the Kibble mechanism?
(f) (*5 points*) Since when is the temperature of the neutrino background different from the CMB radiation?
(g) (*5 points*) Reconcile the following statements:
- the Friedmann equations tell us that an empty universe has negative curvature.
- from General Relativity we know that curvature is cased by mass-energy.

(h) (*5 points*) Relate the present-day baryon-to-photon ratio to fundamental particle physics.

2 [*15 points*] Consider an isotropic and homogenous universe.

(a) (*5 points*) Show that the luminosity distance to a source at coordinate distance r whose light was emitted at time t_e and observed at time t_0 is $ra(t_0)^2/a(t_e)$.
(b) (*5 points*) Derive an integral that relates the luminosity distance only to redshift, for a curved isotropic and homogeneous universe with scale factor $a(t)$.
(c) (*5 points*) For a flat isotropic and homogeneous universe, calculate the luminosity distance to redshift $z = 1$ for matter domination and for dark energy domination. How can this be used as a cosmological test?

3 [*15 points*] Observations of the temperature fluctuations from the surface of last scattering provide important constraints on cosmological parameters.

(a) (*5 points*) What is the important physical change that the matter in the universe underwent at the time when the cosmic microwave background radiation last scattered?
(b) (*5 points*) Describe the dependence of this process on temperature by deriving the Saha equation.
(c) (*5 points*) If the baryon-to-photon ratio were 1, roughly at what temperature would last scattering occur? Why is the baryon-to-photon ratio important?

4 [*20 points*] Consider a spherical volume within a homogeneous expanding universe, containing only pressureless matter of density ρ. At the present time the density is ρ_0 and the scale factor $a = 1$.

(a) (*5 points*) Derive the acceleration equation for the expansion factor $a(t)$ from Newtonian physics.
(b) (*5 points*) Integrate the acceleration equation to yield the Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{C}{a^2}.$$

(c) (*5 points*) Define the critical density and rewrite the Friedmann equation in terms of the density parameter Ω_m.
(d) (*5 points*) Consider a universe that consists of only matter with a present density that is 5 times the critical density. How much larger can such a universe grow?

5 [*20 points*] The energy E in an expanding comoving volume $V(t)$ obeys $dE = -pdV$ for adiabatic changes, where p is the pressure.

(a) (*5 points*) Show that the energy density ρc^2 and the pressure obey the fluid equation:

$$\dot{\rho} + 3\left(\frac{\dot{a}}{a}\right)^2 (\rho + p/c^2) = 0.$$

Consider a homogeneous scalar field ϕ with a potential $V(\phi)$. Then its energy density is $\dot{\phi}^2/2 + V(\phi)$ and its pressure is $\dot{\phi}^2/2 - V(\phi)$.

(b) (*5 points*) When ϕ is small, what equation of state does the scalar field have? What is the effect of such an equation of state on the expansion of the universe?

(c) (*5 points*) Use the fluid equation to derive the equation for the evolution of the scalar field:

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) = 0.$$

(d) (*5 points*) Explain how a scalar field with a potential of the form $V(\phi) = (A^2 - \phi^2)^2$ can cause a finite period of inflation.