Pressureless matter $\rho(t) = \rho_0/a^3$; for relativistic matter $\rho(t) = \rho_0/a^4$
and for vacuum energy $\rho(t) = \Lambda c^2/3 =$ const.

For each component we can define $\Omega = \frac{\rho(t)}{\rho_{crit}}$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right) = H_0^2 \left(\frac{\Omega_{matter}}{a^3} + \frac{\Omega_{rad}}{a^4} + \frac{\Omega_\Lambda}{a^0} + \frac{\Omega_{curvature}}{a^2} \right)$$

By definition Ω's sum up to 1 (as $a=1$ at present, and $\dot{a} = H_0$)

$\Rightarrow$ The evolution of the expansion depends on the current composition of the Universe and how these components get diluted during expansion

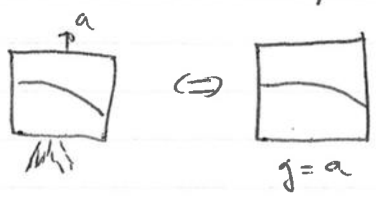
↓

Measuring Ω's central quest of cosmology!

Now we need to connect our results to GR before we continue to examining the implications for observations.

We derived $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$ where k was interpreted as some "initial" amount of kinetic energy; the correct interpretation requires GR

Einstein started with the equivalence principle (gravitational mass = inertial mass)



gravity curves space!

John Wheeler: Newton: mass tells gravity how to exert a force $\left(F = -\frac{GMm}{r^2}\right)$
force tells mass to accelerate $(F = ma)$

Einstein: mass-energy tells space how to curve
curved space-time tells mass-energy how to move
↓
gives a natural explanation of the equivalence principle

In GR the curvature of space-time determines how objects move: they move along geodesics ($\equiv$ shortest paths) in a curved space-time

Space-time is described by a metric $g_{\mu\nu}$ which gives the distance ds between events $\vec{x} = (t, x, y, z)$ and $\vec{x} + d\vec{x} = (t+dt, x+dx, y+dy, z+dz)$

Consider a 2-d surface with a coordinate system. The geometrical properties of the surface can be obtained by considering the distance between a pair of infinitessimal close points

$$dl^2 = \sum_{i,j=1}^{2} g_{ij}(\vec{x})\, dx^i\, dx^j$$

metric

Cartesian coordinates $(x,y) \Rightarrow g_{ij} = \delta_{ij}$ and $dl^2 = \underbrace{dx^2 + dy^2}_{\text{Cartesian}} = \underbrace{dr^2 + r^2 d\theta^2}_{\text{Polar}}$ Pythagoras

If we consider a sphere of radius R:

$$dl^2 = \frac{R^2 - y^2}{R^2 - x^2 - y^2} dx^2 + \frac{R^2 - x^2}{R^2 - x^2 - y^2} dy^2 + \frac{2xy}{R^2 - x^2 - y^2} dx\, dy$$

which is evident if we use spherical coordinates $dl^2 = R^2(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2)$ as $x = R \sin\vartheta \cos\varphi$, $y = R \sin\vartheta \sin\varphi$

Hence the metric does not only depend on the properties of the surface but also on the choice of coordinate system

Space time is described by the distance between two events

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\mu\nu}\, dx^\mu dx^\nu = c^2 dt^2 dx^i dx^j = c^2 dt^2 - \delta_{ij} dx^i dx^j$$

where $(x^0, x^1, x^2, x^3) = (ct, x, y, z)$ and $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ is the Minkowski metric that describes space-time in ~~gravity~~ special relativity (when there is no gravity)

For a general space-time $ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$

Since ds is invariant under coordinate transformation $x \to x'$ the metric must transform as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x)$$

The inverse four-metric $g^{\mu\nu}$ is the inverse of $g_{\mu\nu} \Rightarrow g_{\mu\alpha} g^{\alpha\nu} = \delta_\mu^\nu$

According to the principle of general relativity, all reference frames are equivalent and a physical law should have the same form under a general coordinate transformation (the general covariance)

In GR the curvature of space time is important: particles move such that

$$\delta \int_{path} ds = 0 \qquad \text{the integral is stationary}$$

In the reference frame comoving with the particle (where according to the principle of equivalence the space-time must be locally Minkowski with metric $\eta_{\mu\nu}$), the motion of the particle is given by

$$\frac{d^2\xi^\mu}{ds^2} = 0$$

where $\frac{ds}{c}$ is the proper time measured in the free fall frame
ξ^μ : space time coordinates of the particle

For a general reference frame with coordinates x^μ related to ξ^ν by $x^\mu(\xi)$, the metric is related to $\eta_{\mu\nu}$ by

$$g_{\mu\nu} = \eta_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial x^\mu} \frac{\partial \xi^\beta}{\partial x^\nu}$$

In the x-frame the equation of motion becomes $\frac{d^2x^\mu}{ds^2} = -\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}$

where $\Gamma^\mu{}_{\alpha\beta} = \frac{\partial x^\mu}{\partial \xi^\nu} \frac{\partial^2 \xi^\nu}{\partial x^\alpha \partial x^\beta}$ is the Christoffel symbol or affine connection

$$= \frac{1}{2} g^{\mu\sigma}\left(\partial_\beta g_{\sigma\alpha} + \partial_\alpha g_{\sigma\beta} - \partial_\sigma g_{\alpha\beta}\right) \quad \text{is determined by the metric}$$

Thus in the x-frame there is a force exerting on the free-fall particle: gravity!
In the perspective of GR it is because the particle is moving in a curved space

For a particle of rest mass m we can define the four-momentum as $p^\mu = m U^\mu$ with $U^\mu = c\frac{dx^\mu}{ds}$

In Newtonian and special relativity the conservation of mass, energy and momentum + equivalence of mass & energy leads to

$$\frac{\partial T_{\mu\nu}}{\partial x^\mu} = 0 \qquad \text{where } T_{\mu\nu} \text{ is the energy momentum tensor}$$

$T_{\mu\nu}$ describes the matter distribution; for a perfect fluid (no viscosity, heat flow or stresses) with pressure P and energy density ρ it is

$$T_{\mu\nu} = (P + \rho c^2) U_\mu U_\nu - P g_{\mu\nu} \qquad \text{as } U_\mu = g_{\mu\nu} U^\nu = g_{\mu\nu}\frac{dx^\nu}{ds},$$

$x^\nu(s)$ is worldline of a fluid element

The form of the space-time metric in the Newtonian limit of gravity will tell us how Newtonian gravity is interpreted in ~~the~~ terms of geometric quantities

Consider a reference frame O' which is in free fall in a Newtonian potential Φ which vanishes at large distances. Then the metric is Minkowski: $ds^2 = c^2 dt'^2 - dx'^2$

Now consider another reference frame O relative to which O' has the free-fall velocity given by $v^2 = -2\Phi$ in the x-direction. According to the Lorentz transformation we have

$$dt' = \left(1 + \frac{2\Phi}{c^2}\right)^{1/2} dt \qquad dx' = \left(1 - \frac{2\Phi}{c^2}\right)^{1/2} dx$$

Then the metric in terms of coordinates in the O system can be written as

$$ds^2 = c^2\left(1 + 2\frac{\Phi}{c^2}\right)dt^2 - \left(1 - \frac{2\Phi}{c^2}\right)(dx^2 + dy^2 + dz^2)$$

which is the metric in the Newtonian limit

⇓

Can be used to study gravitational lensing as $\Phi \ll c^2$ typically

In the Newtonian limit $T_{00} = \rho c^2$ and $g_{00} = \left(1 + 2\frac{\Phi}{c^2}\right)$

The Poisson equation ($\nabla^2\Phi = 4\pi G\rho$) becomes $\nabla^2 g_{00} = 8\pi G T_{00}/c^4$

This is a relation between the energy-~~density~~ momentum tensor and the derivatives of the metric. In general the field equation must be a covariant extension of this relation ⇒ The right hand side should be replaced by $8\pi G T_{\mu\nu}/c^4$ and the left hand side by a 4×4 tensor constructed from the metric and its derivatives

↓

to reduce to the Poisson equation ~~this~~ tensor can only contain the first two derivatives of $g_{\mu\nu}$ and have covariance

Einstein proposed

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

Ricci tensor

Ricci scalar measure of curvature

This tensor is the unique choice (with a very tedious proof)

$$\Rightarrow G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \qquad \text{Einstein field equations}$$

It is possible to write a modified set of field equations that are consistent with the conservation laws

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

cosmological constant

This modification allows for the construction of a static universe

What is the physical meaning of Λ? For this question it is useful to move the Λ term to the right hand side

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda g_{\mu\nu} \qquad T^{vac}_{\mu\nu} = \frac{c^4 \Lambda}{8\pi G} g_{\mu\nu} \; : \; G_{\mu\nu} = \frac{8\pi G}{c^4}(T_{\mu\nu} + T^{vac}_{\mu\nu})$$

If we recall that $T^{\mu\nu} = (\rho + \frac{P}{c^2}) U^\mu U^\nu - P g^{\mu\nu}$ then the Λ term can be included as an ideal fluid with

$$\rho = -\frac{P}{c^2} = \frac{c^2 \Lambda}{8\pi G}$$

Can the vacuum have energy? We do not understand the properties of the vacuum and this may well be critical for understanding the current accelerated expansion as well as the very beginning (inflation).

Note that in the case of vacuum energy $\rho c^2 + 3p < 0$

Pressure as a source of gravity: Newtonian gravity is modified in the case of a relativistic fluid (i.e. where we cannot assume $P \ll \rho c^2$). This leads to a modified Poisson equation

$$\nabla^2 \Phi = 4\pi G \left(\rho + \frac{3P}{c^2}\right)$$

For a gas of particles moving with the same speed u the effective gravitational mass density is $\rho(1 + u^2/c^2)$ $\Rightarrow$ a radiation dominated fluid generates an attraction that is twice as strong as one would expect from Newtonian arguments $\Rightarrow$ relevant for gravitational lensing.

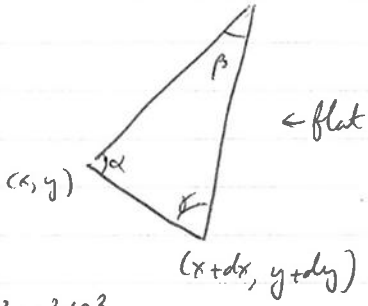
In GR the metric is key: but which one describes the Universe and obeys the cosmological principle?

↓

curvature has to be the same every where!

Start with 2-d spaces

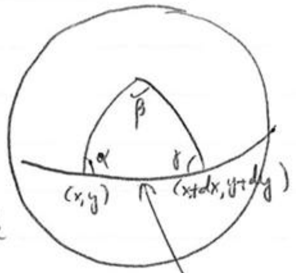
$\alpha+\beta+\gamma=\pi$ (in radians)



← flat 2d surface

We know that $ds^2=dx^2+dy^2=dr^2+r^2d\theta^2$

Now a curved space: the surface of a sphere



ds geodesic is part of a great circle (a circle with a centre that corresponds to the centre of the sphere)

area of the triangle

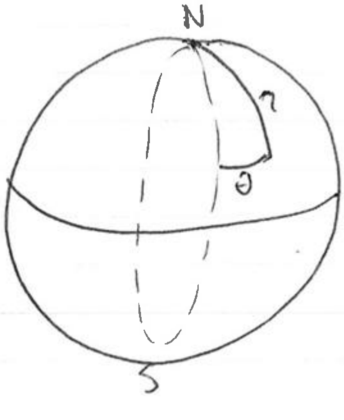
We now have $\alpha+\beta+\gamma=\pi+\frac{A}{R^2}$

radius of the sphere

if $\alpha+\beta+\gamma>\pi$: positively curved.

In the case of a sphere the curvature is homogeneous and isotropic

Polar coordinates $ds^2=dr^2+R^2\sin^2\left(\frac{r}{R}\right)d\theta^2$



Note that the surface has a finite area $4\pi R^2$ and a maximum separation πR

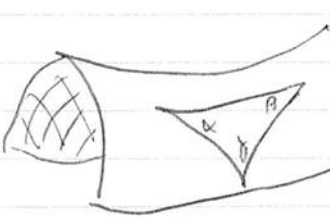
↓

quite different from flat space

Similarly we can define a negatively curved space

$$\alpha+\beta+\gamma = \pi - \frac{A}{R^2}$$

and $ds^2 = dr^2 + R^2 \sinh^2(r/R)\, d\theta^2$



infinite area, no upper limit on distance

Flat space in 3d: $ds^2 = dx^2+dy^2+dz^2$ or $ds^2 = dr^2 + r^2[d\theta^2 + \sin^2\theta\, d\varphi^2]$

positively curved: $ds^2 = dr^2 + R^2 \sin^2(r/R)[d\theta^2 + \sin^2\theta\, d\varphi^2]$

negatively curved: $ds^2 = dr^2 + R^2 \sinh(r/R)[d\theta^2 + \sin^2\theta\, d\varphi^2]$

All these metrics have constant curvature

$$ds^2 = dr^2 + S_k^2(r)\,[d\theta^2 + \sin^2\theta\, d\varphi^2]$$

$$\text{where } S_k(r) = \begin{cases} R\sin(r/R) & k=+1 \\ r & k=0 \\ R\sinh(r/R) & k=-1 \end{cases} \qquad \text{if } r \ll R \quad S_k \approx r$$

If we now change the coordinate system such that $r \rightarrow x \equiv S_k(r)$

$$\text{then } ds^2 = \frac{dx^2}{1-kx^2/R^2} + x^2[d\theta^2 + \sin^2\theta\, d\varphi^2]$$

or as written in the book $ds^2 = \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

This is the most general spatial metric with constant curvature; the only change we can make is to allow space to shrink or expand

$$\Rightarrow \text{Robertson-Walker metric: } ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

Any change to the time coordinate could be accounted for, so this is the most general form.

Plugging this metric into the Einstein equation yields 2 non-trivial equations

time-time: $\left(\frac{\dot a}{a}\right) + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho$ Where k is interpreted as the curvature of the Universe

2nd $\underbrace{2\frac{\ddot a}{a} + \left(\frac{\dot a}{a}\right) + \frac{kc^2}{a^2}}_{= \frac{8\pi G}{3}\rho \text{ according to Friedman eqn}} = -8\pi G \frac{P}{c^2}$ $\Rightarrow \frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$ acceleration equation