We need to describe how density evolves; so far we looked at pressureless matter, but this is not sufficient in general.

Use the first law of thermodynamics $dE + p\,dV = T\,dS$

expanding volume of unit comoving radius

The volume has a physical radius of a $\Rightarrow E = mc^2 = \frac{4\pi}{3} a^3 \rho c^2$

$$\Rightarrow \frac{dE}{dt} = 4\pi a^2 \rho c^2 \frac{da}{dt} + \frac{4\pi}{3} a^3 \frac{d\rho}{dt} c^2$$

$$\frac{dV}{dt} = 4\pi a^2 \frac{da}{dt}$$

According to the cosmological principle there can be no flow of heat $\Rightarrow dS = 0$
(i.e. reversible adiabatic expansion)

$$\Rightarrow \frac{dE}{dt} + p\frac{dV}{dt} = 0 \qquad \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) = 0$$

fluid equation

dilution in density due to increase in volume

loss of energy because pressure has done work as volume increased

has gone into potential energy

Note there are no pressure forces, as these require gradients.

We need an equation of state $P \equiv P(\rho)$ this function depends on material

Baryonic matter has eqn of state $P = w\varepsilon$ (energy density) with $w \simeq 0$:

low density, non-relativistic massive particles $P = \frac{\rho}{\mu} kT$ (mean mass)

$\varepsilon \simeq \rho c^2$ (only rest mass important) $\Rightarrow P \simeq \frac{kT}{\mu c^2} \varepsilon$

For a non-relativistic gas $3kT = \mu \langle v^2 \rangle \Rightarrow P_{non\text{-}rel} = w\, \varepsilon_{non\text{-}rel}$ where $w \simeq \frac{\langle v^2 \rangle}{3c^2} \ll 1$

Photons and highly relativistic particles have $w = \frac{1}{3}$ (as $\langle v^2 \rangle \sim c^2$)

w cannot take on arbitrary values: variations in P will travel at sound speed $c_s^2 = c^2 \left(\frac{dP}{d\varepsilon}\right)$
A substance with $w > 0$ $c_s = \sqrt{w}\, c \Rightarrow w \leq 1$ (as $c_s \leq c$)

$w < -1/3$ is interesting because $\ddot{a} > 0$; this would be "dark energy"
This can be seen from the "acceleration equation"

We can use the fluid equation and Friedmann equation to derive a third (but not independent) equation describing the acceleration of the scale factor

$$\frac{d}{dt}\left[\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}\right] \Rightarrow 2\left(\frac{\dot a}{a}\right)\left(\frac{\ddot a}{a} - \frac{\dot a^2}{a^2}\right) = 2\left(\frac{\dot a}{a}\right)\left(\frac{a\ddot a - \dot a^2}{a^2}\right) = \frac{8\pi G}{3}\dot\rho + \frac{2kc^2\dot a}{a^3}$$

Use the fluid equation $\dot\rho + 3\frac{\dot a}{a}\left(\rho + \frac{P}{c^2}\right)$ $\Rightarrow$ to substitute $\dot\rho$ and cancelling $\frac{2\dot a}{a}$ in each term

$$\Rightarrow \frac{\ddot a}{a} - \left(\frac{\dot a}{a}\right)^2 = -4\pi G\left(\rho + \frac{P}{c^2}\right) + \frac{kc^2}{a^2}$$

is given by Friedmann equation

$$\Rightarrow \frac{\ddot a}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$ in terms of energy density $\frac{\ddot a}{a} = \frac{-4\pi G}{3c^2}(\varepsilon + 3P)$

if a component has any pressure, this increases the "gravitational force" and further <u>decreases</u> the rate of expansion

Note that the acceleration equation does not depend on k

The Universe contains different components with different equations of ~~state~~; for sure non-relativistic matter and radiation, but there could be others...

$$\varepsilon = \sum_w \varepsilon_w \leftarrow \text{different components (only } w \text{ matters)} \qquad \Rightarrow P = \sum_w P_w = \sum_w w\varepsilon$$

As long as there is no interaction between components the fluid equation holds separately for each component

$$\dot\varepsilon_w + 3\frac{\dot a}{a}(\varepsilon_w + P_w) = 0 \iff \dot\varepsilon_w + 3\frac{\dot a}{a}(1+w)\cdot\varepsilon_w = 0$$

$$\iff \frac{d\varepsilon_w}{\varepsilon_w} = -3(1+w)\frac{da}{a}$$

If we assume w = constant $\varepsilon_w(a) = \varepsilon_{w,0}\, a^{-3(1+w)}$ (as $a=1$ at present)

- For radiation $w = 1/3 \Rightarrow a(t) = \left(\frac{t}{t_0}\right)^{1/2}$ and $\rho(t) = \frac{\rho_0 t_0^2}{t^2} \propto \underbrace{\frac{1}{a^3}}_{\text{volume}} \cdot \underbrace{\frac{1}{a}}_{\text{redshift}}$

Notice that the expansion is slower when radiation dominates due to deceleration caused by pressure ($a \propto t^{2/3}$ for matter)

- Λ-dominated $\dot a^2 = \frac{8\pi G\rho_\Lambda}{3c^2}a^2 \iff \dot a = H_0 a$ with $H_0 = \left(\frac{8\pi G\rho_\Lambda}{3c^2}\right)^{1/2}$

$$\Rightarrow a(t) = e^{H_0(t-t_0)}$$

Pressureless matter $\rho(t) = \rho_0/a^3$; for relativistic matter $\rho(t) = \rho_0/a^4$
and for vacuum energy $\rho(t) = \Lambda c^2/3 = \text{const}$.

For each component we can define $\Omega = \frac{\rho(t)}{\rho_{crit}}$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right) = H_0^2 \left(\frac{\Omega_{matter}}{a^3} + \frac{\Omega_{rad}}{a^4} + \frac{\Omega_\Lambda}{a^0} + \frac{\Omega_{curvature}}{a^2} \right)$$

By definition Ω's sum up to 1 (as $a = 1$ at present, and $\dot{a} = H_0$)

$\Rightarrow$ The evolution of the expansion depends on the current composition of the Universe and how these components get diluted during expansion

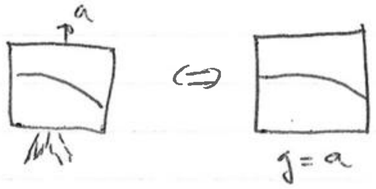
$\downarrow$

Measuring Ω's central quest of cosmology!

Now we need to connect our results to GR before we continue to examining the implications for observations.

We derived $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$ where k was interpreted as some "initial" amount of kinetic energy; the correct interpretation requires GR

Einstein started with the equivalence principle (gravitational mass = inertial mass)



gravity curves space!

John Wheeler:

Newton: mass tells gravity how to exert a force ($F = -\frac{GMm}{r^2}$)
force tells mass to accelerate ($F = ma$)

Einstein: mass-energy tells space how to curve
curved space-time tells ~~mass~~ mass-energy how to move

$\downarrow$

gives a natural explanation of the equivalence principle

In GR the curvature of space-time determines how objects move: they move along geodesics ($\equiv$ shortest paths) in a curved space-time

Space-time is described by a metric $g_{\mu\nu}$ which gives the distance ds between events $\vec{x} = (t, x, y, z)$ and $\vec{x} + d\vec{x} = (t + dt, x + dx, y + dy, z + dz)$