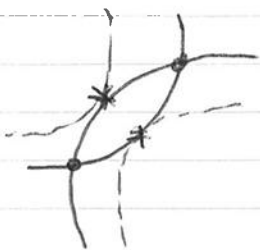
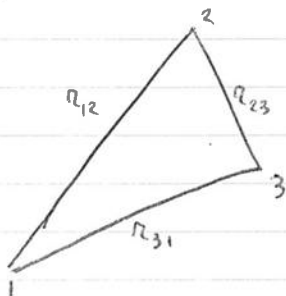


Cosmological Principle + Isotropy \rightarrow Homogeneity



"*" locations must be the same as "o" \rightarrow homogeneity

Homologous expansion



let the triangle expand such that $r_{12}(t) = a(t) r_{12}(t_0)$

$$r_{23}(t) = a(t) r_{23}(t_0)$$

$$r_{31}(t) = a(t) r_{31}(t_0)$$

an observer at location 1 will see galaxy 2 recede with a velocity $v_{12}(t) = \dot{a} r_{12}(t_0) = \left(\frac{\dot{a}}{a}\right) r_{12}(t)$

Similarly $v_{31}(t) = \left(\frac{\dot{a}}{a}\right) r_{31}(t)$

$$\Rightarrow v = \left(\frac{\dot{a}}{a}\right) r \equiv H r \quad \text{with } H \equiv \left(\frac{\dot{a}}{a}\right)$$

Accounting for peculiar motions:

$$v_{obs} = v_{pec} + H_0 r$$

$$\hat{H}_0 = \frac{v_{obs} - v_{pec}}{r}$$

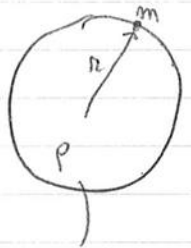
$$|\Delta H_0| = \frac{|v_{pec}|}{r}$$

$$\Rightarrow \frac{|\Delta H_0|}{H_0} = \frac{|v_{pec}|}{v} \Leftrightarrow \left| \frac{\Delta H_0}{H_0} \right| = \left| \frac{v_{pec}}{v} \right|$$

to get a 1% error in H_0 , the peculiar velocity needs to be < 1% of the Hubble velocity

$$H_0 = 70 \text{ km/s/Mpc} \Rightarrow 0.01 = \frac{600 \text{ km/s}}{70 \text{ km/s/Mpc } r_{min}} \Rightarrow r_{min} \geq 857 \text{ Mpc (or } \approx 70.2)$$

Derivation of the Friedmann equation



total mass $M = \frac{4\pi}{3} \rho r^3 \rightarrow F = \frac{GMm}{r^2} = \frac{4\pi G}{3} \rho m r$

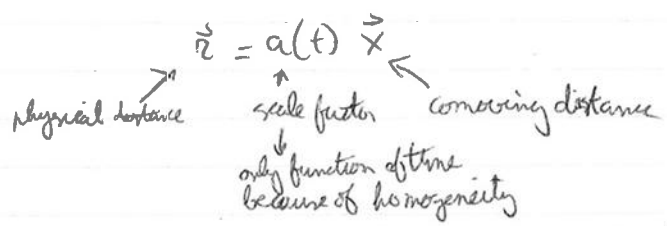
The particle has potential energy $V = -\frac{GMm}{r} = -\frac{4\pi G \rho}{3} m r^2$

The velocity of the particles is $\dot{r} \Rightarrow$ kinetic energy is $T = \frac{1}{2} m \dot{r}^2$

Energy conservation $U = T + V = \text{constant} = \frac{1}{2} m \dot{r}^2 - \frac{4\pi}{3} G \rho m r^2 \quad (*)$

this describes the evolution of the separation r between two particles

Homogeneity implies this holds for all particles \rightarrow we can change to comoving coordinates



Plug into (*) noting that $\dot{x} = 0$ by definition as objects are fixed in comoving coordinates

$U = \frac{1}{2} m \dot{a}^2 x^2 - \frac{4\pi}{3} G \rho a^2 x^2 m$ multiply by $\frac{2}{m a^2 x^2}$

$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{k c^2}{a^2}$ where $k = \frac{-2U}{m c^2 x^2}$ is time independent

$\leftarrow H^2$ \leftarrow has units $\frac{1}{\text{length}^2}$ \leftarrow fixed coordinates

$k=0 \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \frac{\rho_0}{a^3} \Leftrightarrow \ddot{a} = \sqrt{\frac{8\pi G \rho_0}{3 a}} \quad \dot{a} > 0 \text{ as } a \rightarrow \infty$

$k < 0 \quad \left(\frac{\dot{a}}{a}\right)^2 \rightarrow -\frac{k c^2}{a^2} \Rightarrow \ddot{a}^2 = |k| c^2 \quad \dot{a} > 0 \text{ as } a \rightarrow \infty$

$k > 0 \quad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho - \frac{k c^2}{a^2} \quad \ddot{a}^2 \text{ is initially positive but once } a_{\text{max}} = \sqrt{\frac{3 k c^2}{8\pi G \rho}}$

then $\ddot{a} = 0$ and expansion stops

\downarrow

what will happen?

$$\text{as } \ddot{a} = \sqrt{\frac{8\pi G \rho_0}{3a} - k c^2} \Rightarrow \ddot{a} = \frac{1}{2a} - \frac{8\pi G}{3} \frac{\rho_0}{a^2 \dot{a}}$$

$$\Leftrightarrow \ddot{a} = \frac{-4\pi G}{3} \rho_0 \frac{1}{a^2} < 0$$

\ddot{a} has to become negative and the sphere will contract

Note that $\ddot{a} < 0$ independent of $k \rightarrow a$ was smaller in the past and $\rightarrow 0$ as $t \rightarrow 0$

$$k=0: \dot{a}^2 = \frac{8\pi G \rho_0}{3} \frac{1}{a(t)}$$

We can solve this differential equation: (can be solved formally as this is a separable equation; but we will try a power law which is a common solution in cosmology)

$$a \propto t^q \Rightarrow \dot{a}^2 \propto t^{2q-2} \quad \text{but r.h.s } \frac{1}{a} \propto t^{-q}$$

$$\Rightarrow 2q-2 = -q \Leftrightarrow q = \frac{2}{3}$$

$$\Rightarrow a(t) = \left(\frac{t}{t_0}\right)^{2/3} \quad \rho(t) = \frac{\rho_0}{a^3} = \frac{\rho_0 t_0^2}{t^2}$$

↑
current age

The Universe expands forever and $H = \frac{\dot{a}}{a} = \frac{2}{3t} \rightarrow 0$ at $t \rightarrow \infty$

Note that $t_0 = \frac{2}{3} H_0^{-1}$

At present day $a=1 \Rightarrow H_0^2 = \frac{8\pi G \rho_0}{3}$ or $\rho_0 = \frac{3}{8\pi G} H_0^2 \equiv \rho_{crit}$

The critical density separates eternal expansion from recollapse; for $H_0 = 70 \text{ km/s}$ this corresponds to ~ 5 protons per m^3 or $2.78 \times 10^{11} h^2 \text{ M}_\odot \text{ Mpc}^{-3}$
 h_0 in units of 100 km/s/Mpc

With the definition of ρ_{crit} we can write

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[\frac{\rho(t)}{\rho_{crit}} - \frac{k}{H_0^2 a^2} \right] = H^2(a)$$

$\underbrace{\frac{k}{H_0^2 a^2}}_{\Omega_k/a^2} \leftarrow \text{curvature contribution}$

$\Omega_k(a)$ if we consider a particular ingredient

Now we need more than Friedmann eqn because we need to describe how the density evolves. We will also look into the GR interpretation of the Friedmann equation