1a) We are not "special": the Universe is isotropic and homogeneous
isotropic: there is no prefered direction
homogeneous: there are no variations in its properties

b) fluctuations that enter the horizon before t_{eq} cannot grow due to the Meszaros effect, which causes a break as larger scales that enter after t_{eq} preserve the shape of the primordial power spectrum; the location of the break corresponds to horizon at t_{eq}

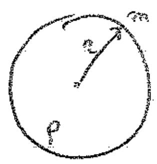
c) To reach a low ionization fraction there should be few ionizing photons, whose energy distribution is described by the Planck curve; more photons delay the moment of recombination, as there are more high energy photons

d) Anisotropies in the radiation field due to temperature quadrupoles lead to anisotropic scattering, which leads to a net polarization signal

e) The observed value of the cosmological constant is 60-120 orders of magnitude smaller than one would expect from naive calculations of the expected contribution of quantum fluctuations to the energy density of the vacuum. Somehow these contributions do not cancel perfectly.

f) It solves the flatness problem ($\Omega_{tot}=1$) and the horizon problem (the CMB is smooth) and reduces the relics of phase transitions in the early universe (monopole problem)

2a)



force on the test mass m: $F = \frac{GMm}{r^2} = \frac{4\pi G}{3}\rho m r$ (as $M = \frac{4\pi}{3}\rho r^3$)

The test mass has potential energy $V = -\frac{GMm}{r} = -\frac{4\pi G\rho}{3} m r^2$

The velocity is $\dot{r}$ and thus the kinetic energy is $T = \frac{1}{2} m \dot{r}^2$

$\Rightarrow$ Total energy $U = T + V = \underbrace{\text{constant}}_{\text{energy conservation}} = \frac{1}{2} m \dot{r}^2 - \frac{4\pi}{3} G\rho m r^2$

$\vec{r} = a(t)\vec{x}$

physical coordinate — comoving coordinate $\Rightarrow \dot{x} = 0$ by definition

$$U = \frac{1}{2} m \dot{a}^2 x^2 - \frac{4\pi G}{3}\rho a^2 x^2 m = \text{constant} \Rightarrow \frac{2U}{x^2 m a^2} = \left(\frac{\dot{a}}{a}\right)^2 - \frac{4\pi G}{3}\rho$$

$$\Leftrightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{4\pi G}{3}\rho + \underbrace{\frac{2U}{x^2 m}}_{\text{constant}}\frac{1}{a^2} \qquad \frac{2U}{x^2 m} \equiv -k = \text{constant}$$

$$\Leftrightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{4\pi G}{3}\rho - \frac{k}{a^2}$$

b) $k \propto U$ which is the total energy; in GR k is interpreted as the curvature of the space-time of the Universe

c) For the critical density $k = 0 \Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{4\pi G}{3}\rho_{crit}$

$$\Rightarrow \rho_{crit} = \frac{3H^2}{4\pi G} \quad \text{and} \quad \rho_{crit}(\text{now}) = \frac{3H_0^2}{4\pi G}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2\Big[\underbrace{\frac{\rho}{\rho_{crit}}}_{=\Omega_m} - \underbrace{\frac{k}{H^2 a^2}}_{=\Omega_k}\Big] = H^2\left[\Omega_m + \Omega_k\right] \qquad \text{where } \Omega_m \equiv \frac{\rho}{\rho_{crit}} \quad \Omega_k = \frac{-k}{a^2 H^2}$$

$\Omega_m + \Omega_k = 1$ by definition $\Rightarrow \Omega_k = (1 - \Omega_m) \Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\frac{\Omega_{m,0}}{a^3} + \frac{1-\Omega_{m,0}}{a^2}\right]$

d) expansion stops when $\frac{\dot{a}}{a} = 0 \Rightarrow 0 = \underbrace{\frac{\Omega_{m,0}}{a^3}}_{\text{always} > 0} + \frac{1-\Omega_{m,0}}{a^2}$

$\Rightarrow$ expansion can only stop if $\frac{1-\Omega_{m,0}}{a^2} < 0 \Leftrightarrow \Omega_{m,0} > 1$

$$0 = \frac{\Omega_{m,0}}{a_{max}^3} + \frac{(1-\Omega_{m,0})}{a_{max}^2} \Leftrightarrow \frac{\Omega_{m,0}}{a_{max}} + 1 - \Omega_{m,0} = 0 \Leftrightarrow \frac{\Omega_{m,0}}{a_{max}} = \Omega_{m,0} - 1 \Leftrightarrow a_{max} = \frac{\Omega_{m,0}}{\Omega_{m,0} - 1}$$

e) One way to write the Friedmann equation $H^2 = H^2\Omega_m - \frac{k}{a^2}$

$\Rightarrow (\Omega_m - 1) = \frac{k}{a^2H^2}$ but also $\Omega_0 - 1 = \frac{k}{H_0^2} \Rightarrow k = H_0^2(\Omega_{m,0} - 1)$

$\Rightarrow \frac{\Omega_{m,0}}{a^3} - 1 = \frac{H_0^2}{H^2}\,\frac{\Omega_{m,0} - 1}{a^2}$

but the Friedmann eqn when matter dominates (e.g. early on) $\Rightarrow H^2 = H_0^2\frac{\Omega_{m,0}}{a^3} \Rightarrow \left(\frac{H_0}{H}\right)^2 = \frac{a^3}{\Omega_{m,0}}$

$\Rightarrow \Omega_m - 1 = \frac{\Omega_{m,0} - 1}{\Omega_{m,0}}\, a \quad \Rightarrow |1 - \Omega_m| \propto a(t)$

If $\Omega_{m,0} = 1$ then $k = 0$ and Ω_m remains constant $= 1$, but a value of $\Omega_m \sim 0.3$ at the present implies that the deviation from ~~[illegible]~~ unity must have been very small when $a \sim 0$ $\Rightarrow$ implies fine tuning

~~f) A period of accelerated expansion leads to a shrinking horizon, such that the observable universe is causally connected (horizon problem). It also ensures that at the end of inflation $k \sim 0$ (flatness problem). GUT models predict the creation of defects such as monopoles and inflation reduces their density to negligible values (monopole problem)~~

3a) The neutrons and protons are in ~~thermal~~ equilibrium and then the number densities are given by $n(T) = g\left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2} e^{-mc^2/k_BT}$

At decoupling $\left(\frac{n_n}{n_p}\right) = \underbrace{\left(\frac{m_n}{m_p}\right)^{3/2}}_{\approx 1} e^{-\Delta mc^2/k_BT_{dec}}$

$\Delta mc^2 = 1.29\,\mathrm{MeV}$

$k_BT_{dec} = 1\,\mathrm{MeV}$

$\Rightarrow \frac{n_n}{n_p} = e^{-1.29} = 0.275$

b) The helium fraction is $Y = \frac{4n_{He}}{n_H + 4n_{He}}$ if all the neutrons end up in ^4He

$\Rightarrow n_{He} = \frac{n_n}{2} \qquad n_H = n_p - n_n \quad \Rightarrow Y = \frac{4n_n/2}{n_p - n_n + \frac{4n_n}{2}} = \frac{2n_n}{n_p + n_n} = \frac{2(n_n/n_p)}{1 + (n_n/n_p)}$

$\Rightarrow Y = 0.431$

c) The binding energy per nucleon is high for ^4He whereas there are no stable nuclei with $A = 5$ $\rightarrow$ very difficult to make heavier elements; synthesis of nuclei with $A > 7$ is hindered by absence of stable nuclei with $A = 8$

4 Entropy is conserved during e^+e^- annihilation

$$S = \frac{\rho c^2 + p}{T} \qquad \text{for relativistic particles } S = \frac{4}{3}\frac{\rho c^2}{T}$$

The energy density $\rho c^2 = \frac{1}{2} g^* \sigma_r T^4 \qquad S \propto g^* T^3$

where $g_* = \sum g_B + \frac{7}{8}\sum g_F$ is the effective degrees of freedom

$$g^*(\text{before})T^3(\text{before}) = g^*(\text{after})T^3(\text{after}) \Leftrightarrow \left(\frac{T(\text{before})}{T(\text{after})}\right)^3 = \frac{g^*(\text{after})}{g^*(\text{before})}$$

$$g^*(\text{before}) = \underset{\text{photon}}{2} + \frac{7}{8} \times \underset{e^+e^-}{2} \times \underset{\text{spin states}}{2} = \frac{11}{2} \qquad g^*(\text{after}) = 2$$

$$\Rightarrow \frac{T(\text{before})}{T(\text{after})} = \frac{2}{11/2} = \frac{4}{11} \quad \Rightarrow T_{\text{after}} = \left(\frac{11}{4}\right)^{1/3} T_{\text{before}} \Rightarrow T_{\nu,0} = 1.95\,K$$

5 a) $$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) = -\frac{4\pi G}{3}\rho(1+3w)$$

$$a(t) = \left(\frac{t}{t_0}\right)^n \Rightarrow \dot{a} = \left(\frac{1}{t_0}\right)^n n t^{n-1} = n\, a(t)\, t^{-1}$$

$$\ddot{a} = n(n-1)\left(\frac{1}{t_0}\right)^n t^{n-2} = a(t)\, n(n-1)\, t^{-2}$$

$$\Rightarrow \frac{n(n-1)}{t^2} = -\frac{4\pi G}{3}\rho(1+3w) \Rightarrow \rho = \frac{-3n(n-1)}{4\pi G(1+3w)}\frac{1}{t^2}$$

b) We also have the Friedmann equation $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{n^2}{t^2} = \frac{8\pi G}{3}\rho \Rightarrow \rho = \frac{3n^2}{8\pi G}\frac{1}{t^2}$$

$$\Rightarrow \frac{3n^2}{8\pi G} = \frac{-3n(n-1)}{4\pi G(1+3w)} \Leftrightarrow \frac{n^2}{2} = \frac{-n(n-1)}{1+3w} \Leftrightarrow n = \frac{-2(n-1)}{1+3w}$$

$$\Rightarrow (1+3w)n + 2n = 2 \Rightarrow 3(1+w)n = 2 \Leftrightarrow n = \frac{2}{3(1+w)}$$

c) $$\left(\frac{\dot{a}}{a}\right)_{t=t_0} = H_0 = \frac{n}{t_0} \Rightarrow t_0 = \frac{2}{3H_0}\frac{1}{1+w}$$

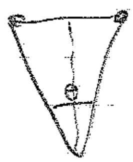
6 a) Consider the coordinate distance travelled by a photon from the Big Bang to time t

For photons $ds^2 = 0 = c^2 dt^2 - a^2(t)\,dr^2$

$$\Rightarrow \frac{dr}{dt} = \frac{c}{a(t)} \qquad \text{for flat matter dominated Universe } a(t) = \left(\frac{t}{t_0}\right)^{2/3}$$

$$\Rightarrow \frac{dr}{dt} = c\,t_0^{2/3}\,t^{-2/3} \qquad \Rightarrow \int_0^r dr' = c\,t_0^{2/3}\int_0^t t'^{-2/3}\,dt' \qquad \Rightarrow r = 3c\,t_0^{2/3}\,t^{1/3}$$

$$\left(\frac{t}{t_0}\right)^{1/3} = a^{1/2} = \frac{1}{\sqrt{1+z}} \qquad \Rightarrow r = \frac{3ct_0}{\sqrt{1+z}} = \frac{2c}{H_0\sqrt{1+z}} \equiv D_H(z)$$

b)  $D_H(0) - D_H(z)$ the comoving distance between the two points is therefore $2\sin(\theta/2)\,[D_H(0) - D_H(z)]$

The maximum distance for causal contact: $2D_H(z)$ $\Rightarrow 2D_H(z) = \frac{4c}{H_0\sqrt{1+z}} < 2\sin(\theta/2)\left[\frac{2c}{H_0} - \frac{2c}{H_0\sqrt{1+z}}\right]$

$$\Leftrightarrow \sin(\theta/2)\left[1 - \frac{1}{\sqrt{1+z}}\right] < \frac{1}{\sqrt{1+z}} \Rightarrow \sin(\theta/2) < \frac{1}{\sqrt{1+z}-1} \simeq 0.0311 \text{ for } z = 1100$$

$$\Rightarrow \theta_{max} \sim 3.6^\circ$$