1a) Recall that the entropy is given by $S = \frac{(\rho c^2 + p)V}{T}$

Which means that the entropy density for relativistic particles is $s = \frac{4}{3}\frac{\rho c^2}{T}$

The energy density $\rho c^2 = \frac{1}{2} g^* \sigma_2 T^4 \quad \Rightarrow s \propto g^* T^3$

where $g^* = \sum_{\text{boson}} g_B + \frac{7}{8} \sum_{\text{fermion}} g_F$ is the effective number of degrees of freedom

The e^+e^- annihilations heat the photons but not the neutrinos, because they have already decoupled, as discussed in class
The entropy is conserved during the annihilations $\Longrightarrow$

$$g^*(\text{before})\, T^3(\text{before}) = g^*(\text{after})\, T^3(\text{after})$$

$$\Leftrightarrow \left(\frac{T(\text{before})}{T(\text{after})}\right)^3 = \frac{g^*(\text{after})}{g^*(\text{before})}$$

$$g^*(\text{before}) = \underset{\substack{\uparrow \\ \text{photons}}}{2} + \frac{7}{8} \times \underset{e^+, e^-}{2} \times \underset{\text{spin}}{2} = \frac{11}{2}$$

$$g^*(\text{after}) = 2 \qquad \Rightarrow \left(\frac{T(\text{before})}{T(\text{after})}\right)^3 = \frac{2}{11/2} = \frac{4}{11}$$

$$\Rightarrow T_{\text{after}} = \left(\frac{11}{4}\right)^{1/3} T_{\text{before}}$$

The e^+e^- annihilation heat the photons by a factor $\left(11/4\right)^{1/3} \sim 1.4$ but not the neutrinos
After this period the temperatures of both neutrinos and photon evolve $\propto \frac{1}{a}$ and the temperature ratio remains fixed

$$\Rightarrow T_{\nu,0} = \left(\frac{4}{11}\right)^{1/3} T_{\gamma,0}$$

The present day photon temperature $T_{\gamma,0} = 2.725\,K \Rightarrow T_{\nu,0} = 1.95\,K$

b) The neutrinos are relativistic fermions, so their energy density is

$$\rho_{\nu,0} c^2 = \frac{7}{8} g_\nu \frac{\sigma_2}{2} T_{\nu,0}^4$$

$g_\nu = 2 N_\nu$ where N_ν is the number of neutrino families.

Each family has ν and $\bar{\nu}$, each with one degree of freedom.

$$\Rightarrow \rho_{\nu,0} = \frac{7}{8} N_\nu \sigma_2 \frac{T_{\nu,0}^4}{c^2} \quad \text{and} \quad \Omega_{\nu,0} = \frac{7}{8} N_\nu \sigma_2 \frac{T_{\nu,0}^4}{\rho_{crit} c^2}$$

$$\left.\begin{array}{l} N_\nu = 3 \\ \sigma_2 = 7.565 \times 10^{-15} \text{ erg cm}^{-3} \text{K}^{-4} \\ T_{\nu,0} = 1.95 \text{K} \\ \rho_{crit} = 9.2 \times 10^{-30} \text{ g cm}^{-3} \\ c = 3 \times 10^{10} \text{ cm/s} \end{array}\right\} \Rightarrow \Omega_{\nu,0} = 3.47 \times 10^{-5}$$

$\Omega_{\nu,0}$ is comparable to the photon density $\Omega_{\gamma 0} = 5.1 \times 10^{-5}$

c) When the neutrinos decouple they are relativistic with a number density given by

$$n(T) = 0.1216 \begin{pmatrix} 3/4 \\ 1 \end{pmatrix} g \left(\frac{k_B T}{\hbar c}\right)^3$$

$\leftarrow$ fermion

$\uparrow$ boson

For photons $g = 2 \Rightarrow n_\nu = \frac{1}{2} \times \frac{3}{4}\, n_\gamma g_\nu = \frac{3}{8} n_\gamma g_\nu$

After decoupling n_ν decreases as $\frac{1}{a^3}$, but the photons are heated by the e^+e^- annihilation which changes n_γ: $n_\gamma \propto T_\gamma^3$

$$\Rightarrow \frac{n_\gamma(\text{before})}{n_\gamma(\text{after})} = \left(\frac{T_\gamma(\text{before})}{T_\gamma(\text{after})}\right)^3 = \frac{4}{11} \quad \text{(as we derived in part a)}$$

$$\Rightarrow n_\gamma(\text{before}) = \frac{4}{11} n_\gamma(\text{after})$$

d) The neutrinos are unaffected by the annihilation, so their number density is

$$n_\nu = \frac{3}{8} g_\nu n_\gamma(\text{before}) = \frac{3}{8} g_\nu \frac{4}{11} n_\gamma(\text{after}) = \frac{3}{22} g_\nu n_\gamma(\text{after})$$

One of the neutrino families has mass m_ν $\Rightarrow \rho_{\nu,0} = m_\nu n_{\nu,0} = \frac{3}{22} g_\nu m_\nu n_{\gamma,0}$

The present day photon number density is $n_\gamma = \dfrac{\rho_{\gamma,0} c^2}{2.7 k_B T}$ $\leftarrow T_{CMB} \sim 2.725\text{K}$

For a single neutrino species $g_\nu = 2$

$$\Rightarrow \rho_{\nu,0} = \frac{3}{11} \frac{m_\nu \rho_{\gamma,0} c^2}{2.7 k_B T_{\gamma,0}}$$

$$\left.\begin{aligned} m_\nu &= 0.04\,\text{eV}/c^2 = 7.12\times10^{-35}\,\text{g} \\ \Omega_{\gamma,0} &= 5.1\times10^{-5} \\ c &= 3\times10^{10}\,\text{cm/s} \\ k_B &= 1.38\times10^{-16}\,\text{erg K}^{-1} \\ T_{\gamma,0} &= 2.725 \end{aligned}\right\} \quad \Omega_{\nu,0} = 8.8\times10^{-4}$$

This density is more than an order of magnitude larger than the result obtained in part b)

2a) The neutron and proton are in thermal equilibrium and thus the number densities are given by

$$n(t) = g\left(\frac{m\,k_B T}{2\pi\hbar^2}\right)^{3/2} e^{-mc^2/k_BT}$$

At decoupling:

$$\Rightarrow \left(\frac{n_n}{n_p}\right)_{dec} = \underbrace{\left(\frac{m_n}{m_p}\right)^{3/2}}_{m_n\sim m_p} \frac{e^{-m_nc^2/kT_{dec}}}{e^{-m_pc^2/kT_{dec}}} \simeq e^{-\Delta mc^2/k_BT_{dec}}$$

$\Delta mc^2 = 1.29\,\text{MeV}$ and $k_BT_{dec} = 1\,\text{MeV}$ $\Rightarrow \left(\frac{n_n}{n_p}\right)_{dec} = e^{-1.29} = 0.275$

b) After decoupling the free neutrons will begin to decay with a half life $\tau_{1/2}$ until they form ^4He at $t=200$

$$\Rightarrow n_n(t) = n_{dec}\, e^{-t\ln(2)/\tau_{1/2}} \qquad \text{if we assume that } t_{dec}\simeq 1\text{s} \ll t$$

Neutron decay through $n \to p + e^- + \bar{\nu}_e$

$$\Rightarrow \text{proton number density is } n_p(t) = n_p(t_{dec}) + \left[n_n(t_{dec}) - n(t)\right]$$
$$= n_p(t_{dec}) + n_n(t_{dec})\left[1 - e^{-t\ln2/\tau_{1/2}}\right]$$

$\Rightarrow$ The neutron-proton ratio at time t is:

$$\left(\frac{n_n}{n_p}\right)_t = \frac{n_n(t_{dec})\, e^{-t\ln2/\tau_{1/2}}}{n_p(t_{dec}) + n_n(t_{dec})\left[1-e^{-t\ln2/\tau_{1/2}}\right]} = \frac{(n_n/n_p)_{dec}\, e^{-t\ln2/\tau_{1/2}}}{1+(n_n/n_p)_{dec}\left[1-e^{-t\ln2/\tau_{1/2}}\right]}$$

$(n_n/n_p)_{dec} = 0.275$, $t = 200$, $\tau_{1/2} = 620\,\text{s}$

$$\Rightarrow (n_n/n_p)_t = 0.208$$

The helium fraction is $Y = \frac{4 n_{He}}{n_H + 4 n_{He}}$

If we assume that all remaining neutrons end up in ^4He $\Rightarrow$

$$n_{He} = \frac{n_n}{2}, \quad n_H = n_p - n_n \quad \Leftrightarrow \quad Y = \frac{4 n_n/2}{n_p - n_n + 4n_n/2} = \frac{2n_n}{n_p + n_n} = \frac{2(n_n/n_p)}{1 + (n_n/n_p)}$$

$\Rightarrow Y = 0{,}344$

c) If the neutron half life is doubled : $\tau_{1/2} = 1240\,s$ $\Rightarrow \left(\frac{n_n}{n_p}\right)_{t=200} = 0.239$

$\Rightarrow Y = 0.387$ more neutrons are available increasing the yield of helium atoms

3 $m_e = 9.11 \times 10^{-28}\,g$
$c = 3 \times 10^{10}\,cm/s$
$k_B = 1.38 \times 10^{-16}\,erg\,K^{-1}$
$Q = 13.6\,eV = 2.18 \times 10^{-11}\,erg$

given $\eta = 2 \times 10^{-10}$ solving for T given that $x = 0.5$

$\downarrow$

$T_{rec} = 4120\,K$

$$T \propto \frac{1}{a} = 1+z \Rightarrow 1 + z_{rec} = \frac{T_{rec}}{T_0} = 4120/2.73 \Rightarrow z_{rec} = 1510$$

4a Friedmann equation $H^2 = \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho_{bg} + \frac{\Lambda}{3}$

Acceleration equation $\frac{\ddot a}{a} = -\frac{4\pi G}{3}\rho_{bg} + \frac{\Lambda}{3}$

as $H = \frac{\dot a}{a} \Rightarrow \dot a = aH \Rightarrow \ddot a = \dot a H + a\dot H \Rightarrow \frac{\ddot a}{a} = \frac{\dot a}{a} H + \dot H = H^2 + \dot H$

$\Rightarrow H^2 + \dot H = -\frac{4\pi G}{3}\rho_{bg} + \frac{\Lambda}{3}$ use Friedmann eqn for H^2

$$\Rightarrow \frac{8\pi G}{3}\rho_{bg} + \frac{\Lambda}{3} + \dot H = -\frac{4\pi G}{3}\rho_{bg} + \frac{\Lambda}{3} \Leftrightarrow \dot H = -4\pi G \rho_{bg}$$

also $H^2 + \dot H = \frac{\ddot a}{a} = \frac{\dot H}{3} + \frac{\Lambda}{3}$ $\Rightarrow \Lambda = 3H^2 + 2\dot H$

4b) $$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} + (c_s^2 k^2 - 4\pi G\rho_{bg})\delta = 0$$

The problem focusses on matter fluctuations, which are pressureless $\Rightarrow c_s^2 = 0$

$$\Rightarrow \ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - 4\pi G\rho_{bg}\delta = 0$$

$$u \equiv \frac{\delta(t)}{H(t)} \Rightarrow \delta = uH$$

Linear growth equation $$\frac{d^2}{dt^2}(uH) + 2\underbrace{\frac{\dot{a}}{a}}_{=H}\frac{d}{dt}(uH) \underbrace{-4\pi G\rho_{bg}}_{=\dot{H}} uH = 0$$

$$\Rightarrow \frac{d^2}{dt^2}(uH) + 2H\frac{d}{dt}(uH) + uH\frac{dH}{dt} = 0$$

c) Chain rule $\frac{d}{dt} = \frac{da}{dt}\frac{dz}{da}\frac{d}{dz}$ as $a = \frac{1}{1+z} \Rightarrow \frac{da}{dz} = \frac{-1}{(1+z)^2}$

$$\Rightarrow \frac{d}{dt} = -\dot{a}(1+z)^2\frac{d}{dz} = -\frac{\dot{a}}{a}(1+z)\frac{d}{dz} = -H(1+z)\frac{d}{dz}$$

$$\Rightarrow H(1+z)\frac{d}{dz}\left(H(1+z)\frac{d}{dz}(uH)\right) - 2H^2(1+z)\frac{d}{dz}(uH) - uH^2(1+z)\frac{dH}{dz} = 0$$

First divide by $H(1+z)$ $$\Rightarrow \frac{d}{dz}\left(H(1+z)\frac{d}{dz}(uH)\right) - 2H\frac{d}{dz}(uH) - uH\frac{dH}{dz} = 0$$

$$\Leftrightarrow \frac{d}{dz}\left(H^2(1+z)\frac{du}{dz} + uH(1+z)\frac{dH}{dz}\right) - 2H^2\frac{du}{dz} - 2uH\frac{dH}{dz} - uH\frac{dH}{dz} = 0$$

$$\Leftrightarrow 2H(1+z)\frac{du}{dz}\frac{dH}{dz} + H^2\frac{du}{dz} + H^2(1+z)\frac{d^2u}{dz^2} + H(1+z)\frac{du}{dz}\frac{dH}{dz} + u(1+z)\left(\frac{dH}{dz}\right)^2 + uH\frac{dH}{dz} + uH(1+z)\frac{d^2H}{dz^2}$$
$$- 2H^2\frac{du}{dz} - 3uH\frac{dH}{dz} = 0$$

Divide by $H^2(1+z)$ gives:

$$\frac{d^2u}{dz^2} + \frac{du}{dz}\left[\frac{dH}{dz}\frac{3}{H} - \frac{1}{1+z}\right] - \frac{2u}{H(1+z)}\frac{dH}{dz} + \frac{u}{H^2}\left(\frac{dH}{dz}\right)^2 + \frac{u}{H}\frac{d^2H}{dz^2} = 0$$

$$\Leftrightarrow \frac{d^2u}{dz^2} + \left(\frac{3}{H}\frac{dH}{dz} - \frac{1}{1+z}\right)\frac{du}{dz} + \frac{u}{H}\left[\frac{d^2H}{dz^2} + \frac{1}{H}\left(\frac{dH}{dz}\right)^2 - \frac{2}{1+z}\frac{dH}{dz}\right] = 0$$

4d) $3H^2 + 2\dot{H} = \Lambda$

$$\frac{d\Lambda}{dt} = 0 = 6H\dot{H} + 2\ddot{H} \iff 3H\dot{H} + \ddot{H} = 0$$

but $\frac{d}{dt} = -H(1+z)\frac{d}{dz} \Rightarrow -3H^2(1+z)\frac{dH}{dz} + (1+z)H\frac{d}{dz}\left[(1+z)H\frac{dH}{dz}\right] = 0$

$(/(1+z)H)$

$$\Rightarrow -3H\frac{dH}{dz} + \frac{d}{dz}\left[(1+z)H\frac{dH}{dz}\right] = 0$$

$$\iff -3H\frac{dH}{dz} + H\frac{dH}{dz} + (1+z)\left(\frac{dH}{dz}\right)^2 + (1+z)H\frac{d^2H}{dz^2} = 0$$

$$\iff \frac{d^2H}{dz^2} + \frac{1}{H}\left(\frac{dH}{dz}\right)^2 - \frac{2}{1+z}\frac{dH}{dz} = 0$$

$$\Rightarrow \frac{d^2u}{dz^2} + \left(\frac{3}{H}\frac{dH}{dz} - \frac{1}{1+z}\right)\frac{du}{dz} = 0$$

e) One solution is $u = \text{constant} \Rightarrow \frac{\delta(z)}{H(z)} = \text{const} \iff \delta(z) \propto H(z)$

As $H(z)$ ~~decreases~~ increases this is a decaying mode with time

For the growing mode rewrite as

$$\frac{\frac{d}{dz}\left(\frac{du}{dz}\right)}{\frac{du}{dz}} = \frac{1}{1+z} - \frac{3}{H}\frac{dH}{dz}$$

$$\Rightarrow \ln\left(\frac{du}{dz}\right) = \ln(1+z) - 3\ln(H) + \text{const.}$$

integration constant

$$\iff \ln\left(\frac{du}{dz}\right) = \ln\left(\frac{1+z}{H^3}\right) + \text{const} \quad \text{or} \quad \frac{du}{dz} = A\left(\frac{1+z}{H^3}\right)$$

$$\Rightarrow u(z) = \int_z^\infty \frac{1+z'}{H(z')^3}\,dz' \qquad \text{as } \delta(z) = u(z)H(z)$$

$$\Rightarrow \delta(z) \propto H(z)\int_z^\infty \frac{1+z'}{H^3(z')}\,dz'$$

4f) For $\Lambda = 0$ we have a flat matter dominated Universe with scale factor $a \propto t^{2/3}$

$$\Rightarrow H = \frac{\dot{a}}{a} = \frac{2}{3t} \propto a^{-3/2}$$

The decaying mode evolves as $\delta \propto H$ or $\delta \propto a^{-3/2}$

The growing mode $\delta \propto H \int_z^\infty \frac{1+z'}{H^3(z')} dz' \propto a^{-3/2} \int_a^0 \frac{dz'}{a} \cdot \frac{1}{a^{-9/2}}$

as $a = \frac{1}{1+z} \Rightarrow da = \frac{-dz}{(1+z)^2}$ or $dz = -(1+z)^2 da = -\frac{da}{a^2}$

$$\Rightarrow \delta \propto a^{-3/2} \int_a^0 a^{9/2} \cdot \frac{da}{a^3} = a^{-3/2} \int_a^0 a^{3/2} da \propto a^{-3/2} \cdot a^{5/2} \propto a$$