1 a) Hubble law: $v = \frac{dr}{dt} = H_0 r$

as H_0 is constant $\Rightarrow r(t) = A e^{H_0 t}$

b) The case where $r \to 0$ occurs in the limit $t \to -\infty$; this means that a steady state Universe is infinitely old.

c) If we consider a spherical region of space, its volume increases as

$$V = \frac{4\pi}{3} r^3 \propto e^{3H_0 t}$$

To maintain a constant density matter must be created at a rate

$$\dot{M}_{ss} = \rho_0 \dot{V} = 3 \rho_0 H_0 V$$

For our Universe $H_0 \simeq 70$ km/s/Mpc and $\rho_0 \sim 3 \times 10^{-27}$ kg/m^3

$\Rightarrow \frac{\dot{M}_{ss}}{V} = 3 H_0 \rho_0 \sim 6.4 \times 10^{-28}$ kg/m^3/Gyr $\qquad (H_0 = 2.26 \times 10^{-18}\ s^{-1})$

mass of a hydrogen atom is 1.67×10^{-27} kg $\Rightarrow \frac{\dot{M}_{ss}}{V} = 0.4$ hydrogen atom/km^3/yr
This would be hard to observe!

2a The number of photon-electron collisions per unit time is $n_e \sigma_T c$
$\Rightarrow$ collision time $\tau_T = \frac{1}{c \sigma_T n_e}$

In a flat matter dominated universe $a(t) = \left(\frac{t}{t_0}\right)^{2/3} = \left(\frac{3H_0 t}{2}\right)^{2/3}$

As $a = \frac{1}{1+z}$, the age of the Universe at redshift z is $\left(\frac{3H_0 t}{2}\right)^{2/3} = \frac{1}{1+z}$

$\Rightarrow t = \frac{2}{3H_0}(1+z)^{-3/2}$

This is equal to the collision time when $\frac{2}{3H_0}(1+z)^{-3/2} = \frac{1}{c\sigma_T n_e}$

In a fully ionized plasma $n_e \simeq n_b = \frac{n_{b,0}}{a^3}$ where $n_{b,0}$ is the current baryon number density

$\Rightarrow n_e = n_b (1+z)^3$

$\Rightarrow \frac{2}{3H_0}(1+z)^{-3/2} = \frac{1}{c\sigma_T n_{b,0}(1+z)^3} \iff (1+z)^{3/2} = \frac{3H_0}{2c\sigma_T n_{b,0}}$

$\iff (1+z) = \left(\frac{3H_0}{2c\sigma_T n_{b,0}}\right)^{2/3}$

The Hubble constant $H_0 = 2.27\times10^{-18}\ s^{-1}$, $c = 3\times10^{10}$ cm/s
$\sigma_T = 6.65\times10^{-25}\ cm^2$ and $n_{b,0} = 2.2\times10^{-7}\ cm^{-3}$

$\Rightarrow z = 83$

b) Each particle has a cross section σ. After travelling a distance λ, a particle will sweep out a volume $\sigma\lambda$. The mean free path is defined such that this volume contains one particle

$$\Rightarrow n\sigma\lambda = 1 \quad \Leftrightarrow \quad \lambda = \frac{1}{n\sigma}$$

For these particles $n \propto gT^3$ and $\sigma \propto G_F^2 T^2$

$$\Rightarrow \lambda \propto \frac{1}{g G_F^2 T^5}$$

The mean interparticle spacing is $L = n^{-1/3}$ $\Rightarrow L \propto \frac{1}{g^{1/3} T}$

c) $\lambda \gg L$ is the condition for an ideal gas; comparison of the expression for λ & L suggests that this is the case when $T \ll T_{ideal}$ where T_{ideal} is the ~~temperature~~ where $\lambda = L$

$$\frac{1}{g G_F^2 T_{ideal}^5} \propto \frac{1}{g^{1/3} T} \quad \Leftrightarrow \quad T_{ideal}^4 \propto g^{-2/3} G_F^{-2} \quad \Rightarrow T_{ideal} \propto g^{1/6} G_F^{-1/2}$$

3 a For a flat radiation dominated Universe $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_{rad}$
where $\rho_{rad} = \sigma_2 T^4/c^2$ is the effective mass density of radiation

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2} \quad \Rightarrow \dot{a} = \frac{1}{2}\left(\frac{1}{t_0}\right)^{1/2} t^{-1/2} \quad \Rightarrow \left(\frac{\dot{a}}{a}\right) = \frac{1}{2}\left(\frac{t}{t_0}\right)^{-1/2}\left(\frac{t_0}{t}\right)^{1/2} = \frac{1}{2t}$$

$$\Rightarrow \frac{1}{4t^2} = \frac{8\pi G}{3c^2}\sigma_2 T^4 \quad \Leftrightarrow \quad T^4 = \left(\frac{3c^2}{32\pi G\sigma_T}\right)\frac{1}{t^2}$$

$$\Leftrightarrow T = \left(\frac{3c^2}{32\pi G\sigma_2}\right)^{1/4} t^{-1/2} \quad \& \quad A = \left(\frac{3c^2}{32\pi G\sigma_2}\right)^{1/4}$$

b) at $t = 1s$ $\quad T = A = \left(\frac{3c^2}{32\pi G\sigma_2}\right)^{1/4}$

$c = 3\times10^8$ m/s ; $G = 6.67\times10^{-11}\ m^3/kg/s^2$
$\sigma_2 = 7.56\times10^{-16}\ J\ m^{-3} K^{-4}$

$\Rightarrow T = 1.52\times10^{10}$ K

c) Consider the coordinate distance travelled by a photon from the Big Bang to time t. For photons $ds^2 = 0$; if we consider travelling in the radial distance only then for a flat universe, using the FRW metric

$$c^2 dt^2 - a^2(t)\, dr^2 = 0 \qquad \Leftrightarrow \frac{dr}{dt} = \frac{c}{a}$$

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2} \Rightarrow \frac{dr}{dt} = c\, t_0^{1/2} t^{-1/2}$$

$$\Rightarrow \int_0^r dr' = c t_0^{1/2} \int_0^t t'^{-1/2} dt' \quad \Leftrightarrow \quad r = 2c\sqrt{t_0}\, t^{1/2}$$

This is the coordinate distance travelled by the photon at time t. The proper distance is obtained by multiplying by the scale factor; this yields the particle horizon

$$\Rightarrow d_p = a(t)\, r = 2c\, t_0^{1/2} t^{1/2} \left(\frac{t}{t_0}\right)^{1/2} = 2ct$$

d) $$T = \left(\frac{3c^2}{32\pi G \sigma_r}\right)^{1/4} t^{-1/2} \Rightarrow t = \left(\frac{3c^2}{32\pi G \sigma_r}\right)^{1/2} \frac{1}{T^2}$$

The monopoles form when $T_M = M_M c^2 / k_B \Rightarrow t_M = \left(\frac{3c^2}{32\pi G\sigma_r}\right)^{1/2} \frac{k_B^2}{M_M^2 c^4}$

The particle horizon is then $d_p = 2ct_M = 2\left(\frac{3}{32\pi G\sigma_r}\right)^{1/2} \frac{k_B^2}{M_M^2 c^2}$

The corresponding volume is $V_M = \frac{4}{3}\pi d_p^3 = \frac{4}{3}\pi \cdot 8 \left(\frac{3}{32\pi G\sigma_r}\right)^{3/2} \frac{k_B^6}{M_M^6 c^6}$

$$= \sqrt{\frac{3}{32\pi}}\left(\frac{1}{G\sigma_r}\right)^{3/2} \frac{k_B^6}{M_M^6 c^6}$$

If we assume that one monopole forms per horizon volume then the number density of monopoles at time t_M is:

$$n_M = \frac{1}{V_M} = \left(\frac{32\pi}{3}\right)^{1/2} (G\sigma_r)^{3/2} \left(\frac{M_M c}{k_B}\right)^6$$

e) The photon number density is $n_\gamma = \frac{2.4}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3$

but $kT = M_M c^2$ when the monopoles formed $\Rightarrow n_\gamma = \frac{2.4}{\pi^2}\left(\frac{M_M c}{\hbar}\right)^3$

$$\Rightarrow \frac{n_M}{n_\gamma} = \left(\frac{32\pi}{3}\right)^{1/2} \frac{\pi^2 \hbar^3}{2.4\, k_B^6} (G\sigma_r)^{3/2} (M_M c)^3$$

In cgs $\hbar = 1.05 \times 10^{-27}$ erg s
$k_B = 1.38 \times 10^{-16}$ erg K^{-1}
$G = 6.67 \times 10^{-8}$ $cm^3 g^{-1} s^{-2}$
$\sigma_r = 7.57 \times 10^{-15}$ erg $cm^{-3} K^{-4}$
$M_m = 10^{15}$ $GeV/c^2 = 1.78 \times 10^{-9}$ g
$c = 3 \times 10^{10}$ cm/s

$$\Rightarrow \frac{n_M}{n_\gamma} = 6,96 \times 10^{-12}$$

f) If this ~~ratio~~ ratio does not evolve with time, the present day number density of monopoles is

$$n_{M,0} = 7 \times 10^{-12}\, n_{\gamma,0}$$

$$\Rightarrow \rho_{M,0} = M n_{M,0} = 7 \times 10^{-12} M n_{\gamma,0} = 7 \times 10^{-12} M \frac{2.4}{\pi^2} \left(\frac{k_B T_{\gamma,0}}{\hbar c} \right)^3$$

$$T_{\gamma,0} = 2.73\,K \quad \Rightarrow \quad \rho_{M,0} = 5 \times 10^{-18}\, g/cm^3$$

The critical density is $9,2 \times 10^{-30}$ g/cm^3 $\Rightarrow \Omega_{Monopole} = 5 \times 10^{11}$

The mass density of monopoles is many orders of magnitude too large; some mechanism needs to reduce dramatically the number density of monopoles.

4a) We start with the fluid equation: $\dot\rho + 3\frac{\dot a}{a}\left(\rho + \frac{P}{c^2}\right) = 0$
Insert the expression for the scalar field

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot\varphi^2 + V(\varphi) \right) + 3\left(\frac{\dot a}{a}\right)\dot\varphi^2 = 0$$

$$\Leftrightarrow \dot\varphi\ddot\varphi + \dot V + 3\left(\frac{\dot a}{a}\right)\dot\varphi^2 = 0 \qquad \text{use that } \dot V = \frac{\partial V}{\partial t} = \frac{\partial V}{\partial\varphi}\frac{\partial\varphi}{\partial t} = \frac{\partial V}{\partial \varphi}\dot\varphi$$

$$\Rightarrow \dot\varphi\ddot\varphi + \dot\varphi\frac{\partial V}{\partial\varphi} + 3\left(\frac{\dot a}{a}\right)\dot\varphi^2 = 0 \Rightarrow (\dot\varphi \neq 0) \quad \ddot\varphi + 3\left(\frac{\dot a}{a}\right)\dot\varphi + \frac{\partial V}{\partial\varphi} = 0$$

b) Assuming slow roll: $3\left(\frac{\dot a}{a}\right)\dot\varphi + \frac{\partial V}{\partial\varphi} = 0$, furthermore $\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho$

as $\frac{1}{2}\dot\varphi^2 \ll V(\varphi) \Rightarrow \rho \sim V(\varphi) \Rightarrow \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3} V(\varphi)$

We consider a scalar field ~~theory~~ with a potential $V = m^2\varphi^2$

$$\Rightarrow \frac{\dot a}{a} = \sqrt{\frac{8\pi G}{3}}\, m\varphi \qquad \text{and} \qquad \frac{\partial V}{\partial\varphi} = 2m^2\varphi$$

$$\Rightarrow 3\left(\frac{\ddot{a}}{a}\right)\dot{\varphi} + \frac{\partial V}{\partial \varphi} = 0 \quad = 3\left(\sqrt{\frac{8\pi G}{3}}\, m\varphi\right)\dot{\varphi} + 2m^2\varphi$$

$$\Leftrightarrow \sqrt{24\pi G}\, m \frac{\partial \varphi}{\partial t} = -2m^2 \quad \Leftrightarrow \frac{\partial \varphi}{\partial t} = \frac{-m}{\sqrt{6\pi G}}$$

$$\Rightarrow \varphi(t) = \varphi_{init} - \frac{m}{\sqrt{6\pi G}}\, t$$

↑ initial value of the field at $t=0$

Plug this solution in the Friedmann eqn: $\left(\frac{\dot{a}}{a}\right) = \sqrt{\frac{8\pi G}{3}}\, m \left(\varphi_{init} - \frac{m}{\sqrt{6\pi G}} t\right)$

$$\Leftrightarrow \int \frac{da}{a} = \sqrt{\frac{8\pi G}{3}}\, m \int dt \left(\varphi_{init} - \frac{m}{\sqrt{6\pi G}} t\right) dt$$

$$\Leftrightarrow \ln\left(\frac{a}{a_i}\right) = \sqrt{\frac{8\pi G}{3}}\, m \left(\varphi_{init} t - \frac{m}{\sqrt{24\pi G}} t^2\right)$$ where a_i is the scale factor at $t=0$

$$\Rightarrow a(t) = a_i \exp\left[\sqrt{\frac{8\pi G}{3}}\, m \left(\varphi_{init} t - \frac{m}{\sqrt{24\pi G}} t^2\right)\right]$$

c) Note that $\frac{1}{2}\dot{\varphi}^2 \ll V(\varphi) \Rightarrow H^2 = \frac{8\pi G}{3} V(\varphi) \Rightarrow a(t) = a_{init}\, e^{Ht}$ where $H = \sqrt{\frac{8\pi G}{3} V(\varphi)}$

Slow roll $\Rightarrow 3 \frac{\dot{a}}{a}\dot{\varphi} = 3H\dot{\varphi} = -\frac{\partial V}{\partial \varphi} = -V' \quad \Rightarrow 9H^2\dot{\varphi}^2 = (V')^2$

$$\Rightarrow \frac{24\pi G}{c^2}\dot{\varphi}^2 V = V'^2 \quad \Rightarrow \dot{\varphi}^2 = \frac{c^2}{24\pi G}\frac{V'(\varphi)^2}{V(\varphi)}$$

The slow roll condition can thus be expressed as

$$\frac{1}{2}\dot{\varphi}^2 \ll V(\varphi) \quad \Leftrightarrow \frac{c^2}{48\pi G}\frac{V'^2}{V} \ll V \quad \Leftrightarrow \frac{c^2}{48\pi G}\left(\frac{V'}{V}\right)^2 \equiv \varepsilon \ll 1$$

Note that other definitions are used commonly: e.g. $\frac{\ddot{a}}{a} \gg 0 \Rightarrow \dot{H} + H^2 \gg 0 \Leftrightarrow -\frac{\dot{H}}{H^2} \ll 1$

$\Rightarrow \frac{1}{16\pi G}\left(\frac{V'}{V}\right)^2 = \varepsilon \ll 1$ but they only differ in numerical prefactor.

d) We have $3H\dot{\varphi} = -V'$

$$\Rightarrow \frac{\partial \dot{\varphi}}{\partial t} = \ddot{\varphi} = \frac{d}{dt}\left(\frac{-V'}{3H}\right) \quad \Leftrightarrow \ddot{\varphi} = -\frac{V''}{3H}\dot{\varphi} + \frac{V'}{3}\frac{\dot{H}}{H^2}$$

but we have $H^2 = \frac{8\pi G}{3c^2}V(\varphi) \Rightarrow \frac{dH^2}{dt} = 2\dot{H}H = \frac{8\pi G}{3c^2}V'\dot{\varphi} = H^2\frac{V'}{V}\dot{\varphi}$

$$\Rightarrow \frac{\dot{H}}{H^2} = \frac{1}{2H}\frac{V'}{V}\dot{\varphi}$$

Inserting in our expression for $\ddot{\varphi}$ yields $\ddot{\varphi} = -\frac{1}{3H}V''\dot{\varphi} + \frac{1}{H}\frac{V'^2}{6V}\dot{\varphi}$

e) If we use that $\ddot{\varphi} \ll V' \Rightarrow -\frac{1}{3H}V''\dot{\varphi} + \frac{1}{H}\frac{V'^2}{6V}\dot{\varphi} \ll V' = -3H\dot{\varphi}$

$$\Rightarrow \frac{1}{3H^2}V'' - \frac{1}{H^2}\frac{V'}{6V} \ll 3$$

Using again that $H^2 = \frac{8\pi G}{3c^2}V(\varphi) \Rightarrow \frac{c^2}{8\pi G}\frac{V''}{V} - 3\varepsilon \ll 3$

$\Leftrightarrow \frac{c^2}{24\pi G}\frac{V''}{V} - \varepsilon \ll 1$, but as $\varepsilon \ll 1 \Leftrightarrow \eta \equiv \frac{c^2}{24\pi G}\frac{V''}{V} \ll 1$