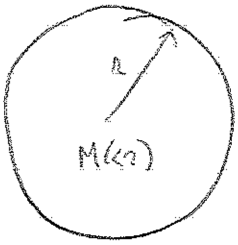
1 a)



$$F = \ddot{r}$$

$$\Rightarrow \ddot{r} = \frac{-GM}{r^2} + \frac{\Lambda}{3} r$$

The mass inside the sphere $M(<r) = \frac{4}{3}\pi \rho r^3$

$$\Rightarrow \ddot{r} = -\frac{4}{3}\pi G \rho r + \frac{\Lambda}{3} r$$

$r = a(t) x_0 \Rightarrow \ddot{r} = \ddot{a} x_0 \qquad \Rightarrow \frac{\ddot{a}}{a} = -\frac{4}{3}\pi G\rho + \frac{\Lambda}{3}$

↑ comoving coordinate

If we multiply both sides by $(a\dot{a})$ $\Rightarrow \dot{a}\ddot{a} = \left(\frac{-4\pi}{3} G\rho + \frac{\Lambda}{3}\right) a\dot{a}$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}(\dot{a}^2) = \left(-\frac{4\pi G}{3}\rho + \frac{\Lambda}{3}\right) a \frac{da}{dt}$$

$$\Leftrightarrow \frac{1}{2} d(\dot{a}^2) = \left(-\frac{4\pi}{3} G\rho + \frac{\Lambda}{3}\right) a\, da$$

For pressureless matter $\rho \propto \frac{1}{a^3}$ or $\rho = \frac{\rho_0}{a^3}$ $\qquad \rho_0 = \rho(a=1)$ present density

$$\frac{1}{2} d(\dot{a}^2) = \left(\frac{-4\pi G}{3}\rho_0 / a^3 + \frac{\Lambda}{3}\right) a\, da \quad \Leftrightarrow \quad d(\dot{a}^2) = \left(-\frac{8\pi G}{3}\rho_0 \frac{1}{a^2} + \frac{2\Lambda a}{3}\right) da$$

Integrate both sides : $\dot{a}^2 = \underbrace{\frac{8\pi G}{3}\rho_0 / a}_{\rho a^2} + \frac{\Lambda}{3} a^2 - k$

↑ integration constant

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{a^2}$$

b) For a static solution $\dot{a} = 0 \Rightarrow \frac{8\pi G\rho}{3} - \frac{k}{a^2} + \frac{\Lambda}{3} = 0$

This yields $\rho = \frac{1}{8\pi G}\left(\frac{3k}{a^2} - \Lambda\right)$

Similarly $\ddot{a} = 0 \Rightarrow \frac{\ddot{a}}{a} = 0 = -\frac{4}{3}\pi G\rho + \frac{\Lambda}{3} \Rightarrow \rho = \frac{\Lambda}{4\pi G}$

This static model is only a solution if

$$\rho = \frac{\Lambda}{4\pi G} = \frac{1}{8\pi G}\left(\frac{3k}{a^2} - \Lambda\right) \Leftrightarrow \frac{3k}{a^2} - \Lambda = 2\Lambda \quad \text{or} \quad 3k = 3\Lambda a^2$$

$\Rightarrow a = \sqrt{\frac{k}{\Lambda}}$, but as a is real (and $\Lambda > 0$) $\Rightarrow k > 0$

1c) $a = a_0(1+\varepsilon(t))$

$$\frac{\ddot{a}}{a} = \frac{-4\pi G}{3}\rho + \frac{\Lambda}{3} \quad \text{but } \Lambda = \frac{4\pi G\rho_0}{a_0^3} \quad \rho = \frac{\rho_0}{a_0^3(1+\varepsilon)^3}$$

$$\Rightarrow \frac{\ddot{a}}{a_0(1+\varepsilon)} = \frac{\ddot{\varepsilon}}{1+\varepsilon} = -\frac{4\pi G}{3}\frac{\rho_0}{a_0^3(1+\varepsilon)^3} + \frac{4\pi G\rho_0}{3a_0^3}$$

$$\Rightarrow \ddot{\varepsilon} = \frac{-4\pi G}{3}\frac{\rho_0}{a_0^3}\frac{1}{(1+\varepsilon)^2} + \frac{4\pi G\rho_0}{3a_0^3}(1+\varepsilon) \approx \frac{-4\pi G}{3}\frac{\rho_0}{a_0^3}(1-2\varepsilon) + \frac{4\pi G}{3}\frac{\rho_0}{a_0^3}(1+\varepsilon)$$

$$= 4\pi G\rho_0/a_0^3\,\varepsilon = 4\pi G\rho\,\varepsilon$$

$$\Rightarrow \varepsilon = \varepsilon_0 e^{\pm\sqrt{4\pi G\rho}\,t}$$ growing mode with $t_{growth}\sqrt{\frac{1}{4\pi G\rho}}$

any perturbation will grow exponentially $\Rightarrow$ unstable

2a) Matter density $\rho_m = \frac{\rho_{m,0}}{a^3}$ and $\rho_{rad} = \frac{\rho_{rad,0}}{a^4}$.

The ratio of matter to radiation density is $\frac{\rho_m}{\rho_{rad}} = \frac{\rho_{m,0}}{\rho_{rad,0}}a = \frac{\Omega_{m,0}}{\Omega_{rad,0}}a$

At matter-radiation equality $\rho_m/\rho_{rad} = 1$

$$\Leftrightarrow \frac{\Omega_{m,0}}{\Omega_{rad,0}}a = 1 \quad \text{as } a = \frac{1}{1+z} \Rightarrow z_{eq} = \frac{\Omega_{m,0}}{\Omega_{rad,0}} - 1$$

($\Omega_{m,0} \leftarrow 0.3175$, $\Omega_{rad,0} \leftarrow 0.0001$)

$\Rightarrow z = 3174$ at matter-radiation equality

b) For vacuum energy $\rho_\Lambda = \text{constant}$ $\frac{\rho_m}{\rho_\Lambda} = \frac{\rho_{m,0}}{\rho_{\Lambda,0}}a^{-3} = \frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\frac{1}{a^3} = 1$

$$\Rightarrow \frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\frac{1}{a^3} = 1 \Leftrightarrow z = \left(\frac{\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)^{1/3} - 1 \quad \Rightarrow z = 0.291$$

($\Omega_{\Lambda,0} \leftarrow 0.6824$)

3a) $$\left(\frac{H}{H_0}\right)^2 = \frac{\Omega_m}{a^3} + \frac{1-\Omega_m}{a^2}$$

Expansion stops if $H = 0$ $\Rightarrow \frac{\Omega_m}{a_{max}^3} + \frac{1-\Omega_m}{a_{max}^2} = 0$

↑ this is always >0 — so this expression can only be zero if $1-\Omega_m < 0$ or $\Omega_m > 0$

$$\Rightarrow a_{max} = \frac{\Omega_m}{\Omega_m - 1}$$

b) $$\left(\frac{da}{dt}\right)^2 = H_0^2\left[\frac{\Omega_m}{a} + (1-\Omega_m)\right] \Leftrightarrow H_0\,dt = \frac{da}{\sqrt{\frac{\Omega_m}{a} - (\Omega_m - 1)}}$$

$$\Leftrightarrow H_0\,dt = \frac{da}{\sqrt{\Omega_m - 1}}\frac{1}{\sqrt{\frac{\Omega_m}{\Omega_m-1}\frac{1}{a} - 1}} \Leftrightarrow \sqrt{\Omega_m - 1}\,H_0\,dt = \frac{da}{\sqrt{\frac{a_{max}}{a} - 1}}$$

$$y = \frac{a}{a_{max}} \Rightarrow da = a_{max}\,dy \Rightarrow \underbrace{\frac{\sqrt{\Omega_m - 1}}{a_{max}} H_0\,dt}_{d\tau} = \frac{dy}{\sqrt{\frac{1}{y} - 1}} = d\tau = \frac{\sqrt{y}}{\sqrt{1-y}}\,dy$$

c) try substitution $y = \sin^2\varphi$ $\quad (\varphi = \arcsin\sqrt{y})$

$$dy = 2\sin\varphi\cos\varphi\,d\varphi \Rightarrow d\tau = 2\sin^2\varphi\,d\varphi = (1-\cos 2\varphi)\,d\varphi$$

$$\Rightarrow \tau = \varphi - \frac{1}{2}\sin 2\varphi$$ if we define $\theta = 2\varphi \Rightarrow \tau = \frac{1}{2}[\theta - \sin\theta]$

$$\Rightarrow t(\theta) = \frac{1}{2H_0}\frac{a_{max}}{\sqrt{\Omega_m - 1}}[\theta - \sin\theta] = \frac{1}{2H_0}\frac{\Omega_m}{(\Omega_m - 1)^{3/2}}[\theta - \sin\theta]$$

d) $$a = a_{max}\,y = a_{max}\sin^2\varphi = \frac{a_{max}}{2}(1-\cos\theta)$$

$2\sin^2\varphi = 1-\cos 2\varphi = 1-\cos\theta$

e) Big Crunch $\theta = 2\pi \rightarrow t_{crunch} = \frac{\pi}{H_0}\frac{\Omega_m}{(\Omega_m - 1)^{3/2}}$

4a) In an empty universe $\left(\frac{\dot{a}}{a}\right)^2 = \frac{-kc^2}{a^2}$

$\Rightarrow \dot{a} = |k|c$ or $a \propto t$: $a(t) = \left(\frac{t}{t_0}\right)$

$\Rightarrow$ proper distance $d_p(t_0) = ct_0 \int_{t_e}^{t_0} \frac{dt}{t} = c t_0 \ln\left(\frac{t_0}{t_e}\right)$

as $\frac{t_0}{t_e} = 1+z$ and $t_0 = \frac{1}{H_0}$ $\Rightarrow d_p(t_0) = \frac{c}{H_0} \ln(1+z)$

b) at the time of emission $d_p(t_e) = \frac{c}{H_0} \frac{\ln(1+z)}{1+z}$

$$\frac{d_p(t_e)}{dz} = \frac{c}{H_0}\left[\frac{1}{(1+z)^2} - \frac{\ln(1+z)}{(1+z)^2}\right] = 0 \iff \ln(1+z) = 1 \iff 1+z = e \iff z = e-1$$

We see these ~~distant~~ high redshift galaxies when the universe was very young and their proper distance to the observer very small

5 a) $1+z = \frac{a(t_0)}{a(t_e)}$ $\quad \dot{z} = \frac{dz}{dt_0} = \frac{\dot{a}(t_0)}{a(t_e)} - \frac{a(t_0)}{a(t_e)^2}\frac{da(t_e)}{dt_0} = \frac{\dot{a}}{a(t_e)} - \frac{a(t_0)}{a^2(t_e)}\frac{da(t_e)}{dt_e}\frac{dt_e}{dt_0}$

~~Note that~~ $\frac{dt_e}{dt_0} = \frac{a(t_e)}{a(t_0)} = \frac{1}{1+z}$

$$\Rightarrow \dot{z} = \frac{a(t_0)}{a(t_e)}\left[\frac{\dot{a}(t_0)}{a(t_0)} - \dot{a}(t_e)\right] = (1+z)\left[H_0 - \dot{a}(t_e)\right]$$

but $\dot{a}(t_e)$ can be obtained from the Friedmann eqn

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\frac{\Omega_m}{a^3} + \frac{(1-\Omega_m)}{a^2}\right] = H_0\left[\Omega_m(1+z)^3 + (1-\Omega_m)(1+z)^2\right]$$

$$= H_0^2(1+z)^2\left[1+\Omega_m z\right]$$

$$\Rightarrow \dot{a}(t) = H_0\sqrt{1+\Omega_m z}$$

$$\Rightarrow \dot{z} = H_0(1+z)\left[1-\sqrt{1+\Omega_m z}\right.$$

b) 10 years $= 3.16 \times 10^8$ s $\qquad$ 1 Mpc $= 3.09 \times 10^{19}$ km $\Rightarrow H_0 = h \times 3.2 \times 10^{-18}\ s^{-1}$

Take $H_0 = 100$ km/s/Mpc and $\Omega_m = 1$, $z = 1$

$$\Rightarrow \dot{z} = 3.2 \times 10^{-18}\left[-0.83\right] = -2.65 \times 10^{-18}\ s^{-1}$$

in 10 years $\Delta z = -8.38 \times 10^{-10}$ $\qquad \Delta z = \Delta\left(\frac{\lambda_{obs}}{\lambda}\right)$

If detection is 3σ $\Rightarrow \left(\frac{\Delta\lambda}{\lambda}\right)_{needed} \lesssim 2.8 \times 10^{-10}$ $\qquad$ Random accelerations from nearby structures are much larger!