

ORIGINS AND EVOLUTION OF THE UNIVERSE - SAMPLE EXAM

Read all questions carefully. Write your name and student number on the first page and number all pages.

The Friedmann Robertson-Walker metric is given by

$$ds^2 = -c^2 dt^2 + a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + (\sin \theta)^2 d\phi^2) \right].$$

The particle number density for a non-relativistic gas in thermal equilibrium at a temperature T is given by:

$$n(T) = g \left(\frac{mk_B T}{2\pi\hbar^2} \right)^{3/2} e^{-mc^2/k_B T},$$

where k_B is Boltzmann's constant and g the statistical weight of the particle. For a relativistic gas in thermal equilibrium the number density is given by:

$$n(T) = 0.122 \times g \times F \left(\frac{k_B T}{\hbar c} \right)^3,$$

where $F = 1$ for bosons and $F = 3/4$ for fermions. The corresponding energy density of such a relativistic gas is given by:

$$\rho(T)c^2 = F' \times g \times \frac{\sigma_r T^4}{2},$$

where $\sigma_r = 7.565 \times 10^{-15} \text{erg/cm}^3/\text{K}^4$ is the radiation density constant and $F' = 1$ for bosons and $F' = 7/8$ for fermions. Other useful numbers are $1 \text{ eV} = 1.16 \times 10^4 \text{K}$ and the hydrogen ionisation potential is 13.6 eV .

1 [40 points] Answer the following questions in a few sentences only, or with a diagram - no derivations

- (a) (5 points) State the cosmological principle and explain the terms used.
- (b) (5 points) What is Silk damping?
- (c) (5 points) How does the scale factor evolve with time in a radiation-dominated universe?
- (d) (5 points) Why does the power spectrum of the temperature fluctuations in the CMB radiation show peaks?
- (e) (5 points) What is the Kibble mechanism?
- (f) (5 points) Since when is the temperature of the neutrino background different from the CMB radiation?
- (g) (5 points) Reconcile the following statements:
 - the Friedmann equations tell us that an empty universe has negative curvature.
 - from General Relativity we know that curvature is caused by mass-energy.
- (h) (5 points) Relate the present-day baryon-to-photon ratio to fundamental particle physics.

2 [15 points] Consider an isotropic and homogenous universe.

- (a) (5 points) Show that the luminosity distance to a source at coordinate distance r whose light was emitted at time t_e and observed at time t_0 is $ra(t_0)^2/a(t_e)$.
- (b) (5 points) Derive an integral that relates the luminosity distance only to redshift, for a curved isotropic and homogeneous universe with scale factor $a(t)$.
- (c) (5 points) For a flat isotropic and homogeneous universe, calculate the luminosity distance to redshift $z = 1$ for matter domination and for dark energy domination. How can this be used as a cosmological test?

3 [15 points] Observations of the temperature fluctuations from the surface of last scattering provide important constraints on cosmological parameters.

- (a) (5 points) What is the important physical change that the matter in the universe underwent at the time when the cosmic microwave background radiation last scattered?
- (b) (5 points) Describe the dependence of this process on temperature by deriving the Saha equation.
- (c) (5 points) If the baryon-to-photon ratio were 1, roughly at what temperature would last scattering occur? Why is the baryon-to-photon ratio important?

4 [20 points] Consider a spherical volume within a homogeneous expanding universe, containing only pressureless matter of density ρ . At the present time the density is ρ_0 and the scale factor $a = 1$.

- (a) (5 points) Derive the acceleration equation for the expansion factor $a(t)$ from Newtonian physics.
- (b) (5 points) Integrate the acceleration equation to yield the Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{C}{a^2}.$$

- (c) (5 points) Define the critical density and rewrite the Friedmann equation in terms of the density parameter Ω_m .
- (d) (5 points) Consider a universe that consists of only matter with a present density that is 5 times the critical density. How much larger can such a universe grow?

5 [20 points] The energy E in an expanding comoving volume $V(t)$ obeys $dE = -pdV$ for adiabatic changes, where p is the pressure.

- (a) (5 points) Show that the energy density ρc^2 and the pressure obey the fluid equation:

$$\dot{\rho} + 3\left(\frac{\dot{a}}{a}\right)^2(\rho + p/c^2) = 0.$$

Consider a homogeneous scalar field ϕ with a potential $V(\phi)$. Then its energy density is $\dot{\phi}^2/2 + V(\phi)$ and its pressure is $\dot{\phi}^2/2 - V(\phi)$.

- (b) (5 points) When ϕ is small, what equation of state does the scalar field have? What is the effect of such an equation of state on the expansion of the universe?

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- (c) (5 points) Use the fluid equation to derive the equation for the evolution of the scalar field:

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + V'(\phi) = 0.$$

- (d) (5 points) Explain how a scalar field with a potential of the form $V(\phi) = (A^2 - \phi^2)^2$ can cause a finite period of inflation.