**ORIGINS AND EVOLUTION OF THE UNIVERSE - PROBLEM SET #3**
**DUE MONDAY DECEMBER 1, 2014**

This is the third problem set for this course. The solutions to the problems in this set will be discussed on Monday December 1 from 13:45 in HL414. Your solutions will be graded if they are received by that time. There are four exercises in this set.

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1 In the standard model of particle physics neutrinos have no mass and are therefore, like photons, still relativistic.

(a) (*5 points*) By considering entropy conservation, derive the present day temperature of the neutrino background.
(b) (*5 points*) Derive the present day density parameter $\Omega_{\nu,0}$ of these neutrinos, under the assumption that there are three families of neutrinos. How does this compare to the present density in photons?
(c) (*5 points*) Express the number density of a neutrino species in terms of the number density of photons, just before the annihilation of electrons and positrons. How does the number density of photons change by the annihilation of electrons and positrons?

Observations of neutrino oscillations imply that neutrinos are not massless. The actual masses are still unknown, but better observations of the growth of large-scale structure may be able to determine the masses in the coming years.

(d) (*10 points*) Suppose that one of the neutrino families has a non-zero mass $m_\nu = 0.04$ eV, so that it was relativistic when it decoupled, but is now non-relativistic. What is the present day density parameter of these neutrinos? How does this compare to the density of relativistic neutrinos calculated in part (b)?

2 Before the weak force decouples the number densities of protons and neutrons are in thermal equilibrium. In this exercise we assume that the weak force decouples at a temperature of $k_B T = 1$ MeV. The mass difference between a neutron and a proton is 1.29 MeV/c$^2$.

(a) (*5 points*) What is the ratio of neutrons to protons when the weak force decouples?
(b) (*10 points*) The protons and neutrons begin forming $^4$He at time $t = 200$s. If the half-life of neutrons is $\tau_{1/2} = 620$s, derive the ratio of neutrons to protons at this time. What is the mass fraction of $^4$He that is produced during BBN?
(c) (*5 points*) Calculate the mass fraction of $^4$He is the half life of neutrons is doubled. Comment on the result.

3 (*15 points*) The ionisation fraction $x = n_p/(n_p + n_H)$ can be related to the baryon-to-photon ratio $\eta$, the temperature and the ionisation energy for hydrogen $Q = 13.6$ eV through

$$\frac{x^2}{1-x} = \frac{4.11}{\eta}\left(\frac{m_e c^2}{2\pi k_B T}\right)^{3/2} e^{-\frac{Q}{k_B T}}.$$

The observed baryon-to-photon ratio is $\eta = 5.5 \times 10^{-10}$, which yields a recombination temperature (i.e. the temperature where $x = 0.5$) of $T_{\rm rec} = 3740$K.

What is the value for $T_{\rm rec}$ if t the baryon-to-photon ratio is increased to $\eta = 2 \times 10^{-8}$? At what redshift does this occur? Note that the Saha equation cannot be solved analytically for $T$, so you will need to solve it numerically.

4 Consider perturbations $\delta(t)$ of the cold dark matter component in a flat Universe containing matter with a mean density $\rho_{\rm bg}$ and a cosmological constant $\Lambda$.

(a) (*5 points*) Use the Friedmann and acceleration equations to show that

$$\dot{H} = -4\pi G \rho_{\rm bg} \tag{1}$$

and

$$\Lambda = 3H^2 + 2\dot{H}, \tag{2}$$

where $H(t) = \dot{a}/a$ is the Hubble parameter at time $t$.

Starting from the equations of motion describing a collisional fluid, we can introduce perturbations about a homogeneously expanding background and linearise these equations. If we consider solutions of the form $\delta\rho = D(t)e^{i\vec{k}\cdot\vec{r}}$, where $D(t) = \rho_{\rm bg}\delta(t)$, we obtain the linear growth equation in a homogeneously expanding background

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} + (c_s^2 k^2 - 4\pi G \rho_{\rm bg})\delta = 0.$$

(b) (*5 points*) We can introduce a new variable $u \equiv \delta(t)/H(t)$. Using the linear growth equation, show that $u$ satisfies:

$$\frac{d^2}{dt^2}(uH) + 2H\frac{d}{dt}(uH) + uH\frac{dH}{dt} = 0. \tag{3}$$

(c) (*10 points*) By expressing derivatives with respect to $t$ as derivatives with respect to redshift $z$, show that equation 3 can be written as:

$$\frac{d^2u}{dz^2} + \left(\frac{3}{H}\frac{dH}{dz} - \frac{1}{1+z}\right)\frac{du}{dz} + \frac{u}{H}\left[\frac{d^2H}{dz^2} + \frac{1}{H}\left(\frac{dH}{dz}\right)^2 - \frac{2}{1+z}\frac{dH}{dz}\right] = 0. \tag{4}$$

(d) (*5 points*) By differentiating equation 2, show that:

$$\frac{d^2H}{dz^2} + \frac{1}{H}\left(\frac{dH}{dz}\right)^2 - \frac{2}{1+z}\frac{dH}{dz} = 0 \tag{5}$$

Hence $u$ satisfies:

$$\frac{d^2u}{dz^2} + \left(\frac{3}{H}\frac{dH}{dz} - \frac{1}{1+z}\right)\frac{du}{dz} = 0. \tag{6}$$

(e) (*10 points*) Using equation 6, show that there are two solutions for the dark matter overdensity: a decaying mode $\delta(z) \propto H(z)$ and a growing mode:

$$\delta(z) \propto H(z) \int_z^{\infty} \frac{1+z'}{H^3(z')} dz'. \tag{7}$$

(f) (*5 points*) If $\Lambda = 0$, show that these modes become: $\delta \propto a^{-3/2}$ and $\delta \propto a$.