ORIGINS AND EVOLUTION OF THE UNIVERSE - PROBLEM SET #2
DUE MONDAY NOVEMBER 3, 2014

This is the second problem set for this course. The solutions to the problems in this set will be discussed on Monday November 3 from 13:45 in HL414. Your solutions will be graded if they are received by that time. There are four exercises in this set.

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1 Hubble's law occurs naturally in a Big Bang model for the Universe, where the density decreases and the distances between galaxies increase as the Universe expands. This model is not in agreement with the perfect cosmological principle, in which there are no privileged moments in time. The Big Bang itself is clearly a special moment. For this reason Bondi, Gold and Hoyle proposed the Steady State model in the 1940s. Interestingly, this model also gives rise to the Hubble law. In the Steady State model, the global properties of the Universe, such as H_0 and $\rho_m = \rho_0$, remain constant.

(a) (*5 points*) Show that the distance $r(t)$ increases with time t as

$$r(t) \propto e^{H_0 t}.$$

(b) (*5 points*) How does this model avoid the issues with the Big Bang?

(c) (*5 points*) For the Universe to remain in a steady state, the density of a volume V must remain constant, which means that matter must be created continuously at a rate $\dot{M}_{ss}$. Express $\dot{M}_{ss}$ in terms of H_0, ρ_0 and V. Calculate the rate with which matter would have to be created in our Universe, where $H_0 = 70$ km/s/Mpc and $\rho_0 = 3 \times 10^{-27}$ kg/m^3. Express your result in hydrogen atoms per cubic kilometer per year.

2 Since electrons are the lightest charged particles, the main mechanism for matter-radiation interaction in a plasma is Thomson scattering of electrons. Consider a completely ionised, flat, matter-dominated Universe, where the present day value of the Hubble constant is 70 km/s/Mpc and the current baryon number density is $n_{b,0} = 2.2 \times 10^{-7}$ cm^{-3}.

(a) (*10 points*) Calculate the redshift at which the photon-electron collision time is the same as the age of the Universe.

Now consider relativistic particles in the early Universe with a number density $n \propto gT^3$, where g is the number of spin states of the particle. Assume that these particles interact via the weak interaction, with an interaction cross section $\sigma \propto G_F^2 T^2$ (where G_F is the Fermi coupling constant).

(b) (*5 points*) Derive expressions for how the mean free path λ and the mean inter-particle spacing L of the particles scale with g, G_F and T.

(c) (*5 points*) The condition for an ideal gas requires that $\lambda \gg L$. Use your expressions for λ and L to derive a condition on the temperature below which the ideal gas approximation is valid. How does this temperature depend on g and G_F?

3 Typical Grand Unified Theories predict that magnetic monopoles with masses $M_M \sim 10^{15}$ GeV/c^2 will form in the early Universe when the temperature $k_B T_M \sim M_M c^2$. As discussed in class, on average one monopole will form per particle horizon volume at that time. In this question we will quantify the monopole problem. We start by examining how the temperature evolves in a flat radiation dominated Universe.

(a) (*5 points*) Show that in a flat radiation dominated Universe the temperature T evolves as $T(t) = At^{-1/2}$. Express A in terms of physical quantities.
(b) (*5 points*) What was the temperature one second after the Big Bang?
(c) (*5 points*) Derive the expression for the particle horizon at time t, i.e. the proper distance at time t to the edge of the volume containing all particles that have been in causal contact with the observer, for a flat radiation dominated Universe.
(d) (*10 points*) Assuming that one monopole forms in each horizon volume at the time when the temperature is T_M, use your expression for $T(t)$ you derived in question (a) to derive the number density of monopoles that are formed at this time. Express your result in terms of physical quantities.
(e) (*5 points*) Recall that the photon number density at temperature T is

$$n_\gamma = \frac{2.4}{\pi^2}\left(\frac{k_B T}{\hbar c}\right)^3.$$

Calculate the ratio of the number density of monopoles to the photon number density at the time when the monopoles form.
(f) (*5 points*) If this ratio of number densities does not evolve with time once the monopoles have formed, calculate the present day mass density of monopoles. Compare this with the present day critical density. Why is this a problem?

4 Consider a scalar field ϕ with density $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ and pressure $p/c^2 = \frac{1}{2}\dot{\phi}^2 - V(\phi)$.

(a) (*5 points*) Use the fluid equation to show that the evolution of this scalar field obeys:

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{\partial V}{\partial \phi} = 0.$$

In the slow roll approximation we assume that $\ddot{\phi}$ is negligible compared to the other two terms in the equation that describes the evolution of ϕ. Furthermore, the kinetic term is assumed to be small compared to the potential energy, i.e. $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$.

(b) (*8 points*) Consider a scalar field theory with a quadratic potential $V(\phi) = m^2\phi^2$. Using the equation you derived in part (a) and the Friedmann equation, derive the time evolution of this scalar field $\phi(t)$ and the scale factor $a(t)$ in the slow roll approximation.

The requirements for slow roll can be quantified using the slow roll parameters ϵ and η, which are functions of the first and second derivatives of the potential. The condition $\epsilon \ll 1$ ensures exponential expansion, whereas $\eta \ll 1$ ensures that the inflationary periods lasts sufficiently long.

(c) (*7 points*) Assuming slow roll, derive the expression relating $\dot{\phi}^2$ to $V' = \partial V/\partial\phi$ and V. Use this expression to show that the assumption that $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$ is equivalent to assuming that

$$\epsilon \equiv \frac{c^2}{48\pi G}\left(\frac{V'}{V}\right)^2 \ll 1.$$

(d) (*5 points*) For inflation to last, $\ddot{\phi}$ should be small, so we stay in the slow roll regime. Show that

$$\ddot{\phi} = -\frac{1}{3H}V''\dot{\phi} + \frac{1}{H}\frac{V'^2}{6V}\dot{\phi}.$$

(e) (*5 points*) Show that the assumption $\ddot{\phi} \ll V'$ is equivalent to the assumption that

$$\eta \equiv \frac{c^2}{24\pi G}\frac{V''}{V} \ll 1.$$