ORIGINS AND EVOLUTION OF THE UNIVERSE - PROBLEM SET #1
DUE MONDAY OCTOBER 6, 2014

This is the first problem set for this course. The solutions to the problems in this set will be discussed on Monday October 6 from 13:45 in HL414. Your solutions will be graded if they are received by that time. There are five exercises in this set.

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1 In class we derived the Friedmann equation by considering the Newtonian force exerted on a test mass by a sphere of mass M at a distance r. This derivation can be extended by adding a cosmological constant. This is done by adding an additional force. To ensure an isotropic and homogeneous universe, this force $\vec{F}_\Lambda$ has to be proportional to $\vec{r}$. The convention is to express the force as

$$\vec{F}_\Lambda = \frac{\Lambda}{3}\vec{r}.$$

(a) (*5 points*) Derive expressions for $(\dot{a}/a)^2$ and $\ddot{a}/a$, i.e. the Friedmann equation and the acceleration equation for this Newtonian cosmology with pressureless matter.

(b) (*10 points*) Show that these equations permit a static universe, which is why Einstein introduced the cosmological constant. How is Λ related to the density? What condition must the curvature satisfy?

(c) (*10 points*) This universe is only static if $\dot{a} = 0$ to begin with, but is nonetheless unstable. This can be seen if we consider a small perturbation $\epsilon(t)$ to the scale factor: $a(t) = a_0[1 + \epsilon(t)]$, which results in a differential equation for $\epsilon(t)$. Derive this expression and find the solution for $\epsilon(t)$. Comment on the result.

2 Consider a flat universe in which the dark energy, cold pressureless matter and radiation contribute 68.24, 31.75 and 0.01 percent of the total energy density respectively.

(a) (*5 points*) At what redshifts is the energy density of matter equal to the energy density of radiation?

(b) (*5 points*) At what redshift is the contribution of matter to the energy density equal to that of dark energy in this universe?

3 Consider a universe with only matter and curvature.

(a) (*5 points*) Use the Friedmann equation to show that expansion can only stop if $\Omega_{m,0} > 1$; what is the value of $a_{\max}$, the scale factor at maximum expansion?

(b) (*10 points*) Using the substitution $y = a/a_{\max}$ and $d\tau = a_{\max}^{-1} H_0 \sqrt{\Omega_{m,0} - 1}\, dt$ show that

$$\frac{\sqrt{y}}{\sqrt{1-y}}dy = d\tau.$$

(c) (*10 points*) Show that this results in an parametric expression for the time $t(\theta)$ where $\theta = 2\arcsin(\sqrt{y})$ given by

$$t(\theta) = \frac{1}{2H_0}\frac{\Omega_{m,0}}{(\Omega_{m,0}-1)^{3/2}}[\theta - \sin(\theta)].$$

(d) (*10 points*) Show that the scale factor is described by

$$a(\theta) = \frac{a_{\max}}{2}[1 - \cos(\theta)].$$

(e) (*5 points*) How old is this universe when the Big Crunch occurs?

4 Consider a galaxy with redshift z that is observed in an empty expanding universe whose geometry is described by a FRW metric (this is clearly a thought experiment).

(a) (*5 points*) Show that the proper distance to this galaxy is given by

$$d_p(t_0) = \frac{c}{H_0}\ln(1+z).$$

(b) (*5 points*) Show that the proper distance at the time of emission has a maximum for objects with a redshift $z = e - 1$.

5 So far we have considered the redshift of an object to be fixed in time, but since it is a ratio of the scale factors now at at emission, the redshift will change. In this problem we consider a matter dominated universe.

(a) (*10 points*) Differentiate the definition of redshift and use the Friedmann equation to show that

$$\dot{z} = \frac{dz}{dt_o} = H_0(1+z)[1 - \sqrt{1+\Omega_{m,0}z}].$$

(b) (*5 points*) What fractional precision in the observed wavelength would be required to detect cosmological deceleration in a decade?