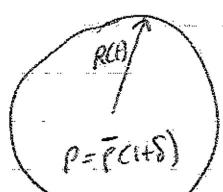
Growth of ~~structure~~

We start with tiny fluctuations in the background radiation temperature, which are related to density fluctuations. These grow into the very clustered Universe we see today. We therefore need to study the evolution of density perturbation.

Gravitational Jeans instability: Jeans showed that starting from a homogeneous and isotropic "mean" fluid, small fluctuations in the density $\delta\rho$ and velocity δv can evolve with time

The simple criterion to decide whether a fluctuation will grow with time is that the typical lengthscale of a fluctuation should be larger than the Jeans length λ_J

Consider a static, homogeneous matter-only Universe in which ~~there~~ is a spherical region that is overdense



$$\Rightarrow \delta(t) = \frac{\rho-\bar\rho}{\bar\rho} \ll 1$$

$$\ddot R = -\frac{G\,\Delta M}{R^2} = -\frac{G}{R^2}\left(\frac{4\pi}{3}R^3\bar\rho\,\delta\right) \quad \Leftrightarrow \quad \frac{\ddot R}{R} = -\frac{4\pi G\bar\rho}{3}\,\delta(t)$$

Hence a mass excess $\delta>0$ will cause the sphere to collapse

Conservation of mass gives $M = \frac{4\pi}{3}\bar\rho\,[1+\delta(t)]\,R(t)^3$ = constant during collapse

$$\Rightarrow R(t) = R_0\,[1+\delta]^{-1/3} \quad \text{where } R_0 = \left(\frac{3M}{4\pi\bar\rho}\right)^{1/3}$$

If $\delta \ll 1$ then $R(t) \simeq R_0\left[1-\frac{1}{3}\delta(t)\right] \Rightarrow \ddot R = -\frac{1}{3}R_0\ddot\delta \approx -\frac{1}{3}R\ddot\delta$ (if $\delta\ll1$, so is the change in R)

$$\Rightarrow \text{mass conservation yields } \frac{\ddot R}{R} \approx -\frac{1}{3}\ddot\delta \quad (\delta\ll1)$$

$$\Rightarrow \ddot\delta = 4\pi G\bar\rho\,\delta \quad \text{which has solutions } \delta(t) = A_1 e^{t/t_{dyn}} + A_2 e^{-t/t_{dyn}}$$

where $t_{dyn} = \frac{1}{(4\pi G\bar\rho)^{1/2}}$ is the dynamical time for collapse

If the overdense sphere starts at rest, $\dot\delta = 0$ at $t=0$ $\Rightarrow A_1 = A_2 = \delta(0)/2$

After a few dynamical times only the growing mode matters $\Rightarrow$ the fluctuations grow exponentially with time.

However as the sphere collapses, pressure will build up. When a sphere of gas is compressed by its own gravity a pressure gradient will build up that tends to counter the effects of gravity (e.g in a star)

If the pressure gradient balances gravity we have hydrostatic equilibrium

The pressure gradient steepening takes time; any change in pressure travels with the speed of sound c_s; therefore the time to build up a pressure gradient in a sphere of radius R is $t_{pressure} \sim R/c_s$

$$c_s = c\left(\frac{dP}{d\rho}\right)^{1/2} = \sqrt{w}\,c$$

For hydrostatic equilibrium to develop the gradient must build up before collapse:

$$\frac{R}{c_s} = t_{pressure} < t_{dyn} = \frac{1}{\sqrt{G\bar{\rho}}} \Rightarrow R < c_s t_{dyn} \approx \frac{c_s}{\sqrt{G\bar{\rho}}} \equiv \lambda_J$$

The Jeans length

A more accurate derivation yields $\lambda_J = c_s\sqrt{\frac{\pi}{G\bar{\rho}}} = 2\pi c_s t_{dyn}$

Consider a spatially flat Universe with mean density $\bar{\rho}$ $\Rightarrow \frac{1}{H} = \left(\frac{3}{8\pi G\bar{\rho}}\right)^{1/2} = \sqrt{\frac{3}{2}}\, t_{dyn} \approx 1.22 t_{dyn}$

$\Rightarrow$ The Jeans length in an expanding Universe will then be $\lambda_J = 2\pi c_s t_{dyn} = 2\pi\left(\frac{2}{3}\right)^{1/2}\frac{c_s}{H}$

If we now focus on one component with equation of state w and $c_s = \sqrt{w}\,c$

$$\Rightarrow \lambda_J = 2\pi\left(\frac{2}{3}\right)^{1/2}\sqrt{w}\,\frac{c}{H}$$

For a photon gas $c_s = \frac{c}{\sqrt{3}} \sim 0.58c$ $\Rightarrow \lambda_J = \frac{2\pi\sqrt{2}}{3}\frac{c}{H} \sim 3\frac{c}{H}$

Density fluctuations in the radiative component will be pressure supported if they are smaller than 3× Hubble distance. Such fluctuations will oscillate (only larger ones can collapse)

$\Rightarrow$ A Universe containing only radiation will have density fluctuations with $\lambda_J < \frac{3c}{H}$ but they produce sound waves

To get collapsed structures we need a non-relativistic component with $\sqrt{w} \ll 1$

Prior to decoupling the baryons were coupled to the photons $\Rightarrow$ no collapse possible!
At z_{dec} $c/H(z_{dec}) \sim 0.2$ Mpc and $\varepsilon_\gamma \sim 1.4\,\varepsilon_{baryon}$

$$\Rightarrow \lambda_J\,(\text{before decoupling}) \simeq \frac{3c}{H(z_{dec})} \approx 0.6 \text{ Mpc}$$

$$\Rightarrow \text{baryon Jeans mass before decoupling } M_J \equiv \rho_{bary}\left(\frac{4\pi}{3}\lambda_J^3\right) \simeq 7\times10^{18} M_\odot$$

30,000x Coma cluster!

After decoupling we have two separate gases; for the baryons the sound speed drops to

$$c_s(\text{baryon}) = \left(\frac{kT}{mc^2}\right)^{1/2} c \qquad kT_{dec} \approx 0.26\text{eV}$$

$mc^2 = 1.22\, m_p c^2 \simeq 1140$ MeV $(Y_p = 0.24)$

$$\Rightarrow c_s(\text{baryon}) \simeq 1.5\times10^{-5}\, c \quad \text{or } 5 \text{ km/s}$$

Thus after decoupling the Jeans length decreased by a factor $\dfrac{c_s(\text{baryon})}{c_s(\text{photon})} \approx 2.6\times10^{-5}$

$$\Rightarrow M_J(\text{after}) \simeq 10^5 M_\odot$$

much smaller than the mass of our galaxy
~ the baryonic mass of the smallest dwarf galaxies.

After decoupling baryon density perturbations could start growing!

We can study the Jeans theory in a bit more detail, focussing first on collisional fluids.
The equations of motion are in the Newtonian approximation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \qquad \text{continuity equation}$$

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} + \frac{1}{\rho}\nabla p + \nabla \varphi = 0 \qquad \text{Euler equation}$$

$$\nabla^2 \varphi - 4\pi G \rho = 0 \qquad \text{Poisson equation}$$

We will also neglect any dissipative terms arising from viscosity or thermal conductivity.
Therefore we have conservation of entropy per unit mass

$$\frac{\partial s}{\partial t} + \vec{v} \cdot \nabla s = 0$$

A trivial solution is $\rho = \rho_0$, $\vec{v} = 0$, $s = s_0$, $p = p_0$, $\nabla \varphi = 0$

Note that if $\rho = \rho_0 \neq 0$ then φ must vary spatially because of the Poisson equation $\Rightarrow$ homogeneous distribution of ρ cannot be stationary, similar to what we saw when we derived the Friedmann eqn.

Although the derivation is formally incorrect, the results are qualitatively unchanged, and the results can be "reinterpreted" to give correct results.

Perturb around this: $\rho = \rho_0 + \delta\rho$, $\vec{v} = \delta\vec{v}$, $p = p_0 + \delta p$, $s = s_0 + \delta s$, $\varphi = \varphi_0 + \delta\varphi$

$$\Rightarrow \quad \frac{\partial \delta\rho}{\partial t} + \rho_0 \nabla \cdot \delta\vec{v} = 0$$

$$\frac{\partial \delta\vec{v}}{\partial t} + \frac{1}{\rho_0}\left(\frac{\partial p}{\partial \rho}\right)_s \nabla \delta\rho + \frac{1}{\rho_0}\left(\frac{\partial p}{\partial s}\right)_\rho \nabla \delta s + \nabla \delta\varphi = 0$$

$$\nabla^2 \delta\varphi - 4\pi G \delta\rho = 0$$

$$\frac{\partial \delta s}{\partial t} = 0$$

We will look for solutions in the form of plane waves $\delta u_i = \delta_i \, e^{i\vec{k}\cdot\vec{r}}$

where $\delta u_i = \delta\rho, \delta\vec{v}, \delta\varphi, \delta s$

Given that the unperturbed solutions do not depend on position we can search for solutions

$$\delta_i(t) = \delta_{0,i}\, e^{i\omega t} \qquad \text{amplitudes } D, \vec{V}, \Phi, \Sigma$$

~~Use~~ that $c_s^2 = \left(\frac{\partial p}{\partial \rho}\right)_s$ and $\delta_0 = D/\rho_0$

$$\Rightarrow \omega \delta_0 + \vec{k}\cdot\vec{V} = 0$$

$$\omega \vec{V} + \vec{k}\, c_s^2 \delta_0 + \frac{\vec{k}}{\rho_0}\left(\frac{\partial p}{\partial s}\right)_\rho \Sigma + \vec{k}\,\Phi = 0$$

$$k^2 \Phi + 4\pi G \rho_0 \delta_0 = 0$$

$$\omega \Sigma = 0$$

Let us consider solutions with $\omega \neq 0 \Rightarrow \Sigma = 0$: the perturbations are adiabatic

Also $\vec{k}\cdot\vec{V} \neq 0$ we can decompose into components parallel and vertical to V

$\vec{k}$ vertical to $\vec{V} \Rightarrow \delta_0 = 0,\ \Phi = 0$ these vertical modes do not imply density perturbation

$\Rightarrow \vec{k} \parallel \vec{V}$:

$$\Rightarrow \begin{aligned} &\omega \delta_0 + kV = 0 \\ &\omega V + k c_s^2 \delta_0 + k\Phi = 0 \\ &k^2 \Phi + 4\pi G \rho_0 \delta_0 = 0 \end{aligned} \quad \Leftrightarrow \begin{pmatrix} \omega & k & 0 \\ k c_s^2 & \omega & k \\ 4\pi G\rho_0 & 0 & k^2 \end{pmatrix} \begin{pmatrix} \delta_0 \\ V \\ \Phi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This admits a non-zero solution for δ_0, V and Φ if and only if its determinant vanishes $\Rightarrow$ ω and k must satisfy the dispersion relation

$$\omega^2 - c_s^2 k^2 + 4\pi G \rho_0 = 0$$

The solutions are of two types, depending on whether $\lambda = \frac{2\pi}{k}$ larger or smaller than

$$\lambda_J = c_s \left(\frac{\pi}{G\rho_0}\right)^{1/2}$$

In the case that $\lambda < \lambda_J$, the value of ω is real and $\omega = \pm c_s k \left[1 - \left(\frac{\lambda}{\lambda_J}\right)^2\right]^{1/2}$

These represent two sound waves in directions $\pm\vec{k}$ with a dispersion ω

If $\lambda \gg \lambda_J$ the frequency is imaginary:

$$\omega = \pm i \, (4\pi G \rho_0)^{1/2} \left[1 - \left(\frac{\lambda_J}{\lambda}\right)^2 \right]^{1/2}$$

and the solution for the density is $\dfrac{\delta\rho}{\rho_0} = \delta_0 \, e^{i\vec{k}\cdot\vec{r}} \, e^{\pm|\omega| t}$

The characteristic timescale for the evolution of the amplitude is

$$\tau \equiv |\omega|^{-1} = \frac{1}{\sqrt{4\pi G \rho_0}} \left[1 - \left(\frac{\lambda_J}{\lambda}\right)^2 \right]^{-1/2}$$

for $\lambda \gg \lambda_J$ this corresponds to the dynamical ~~of~~ or free-fall time

Let us now look at the homogeneously expanding solution with expansion factor $a(t)$

$$\rho_{bg} = \rho_0 \left(\frac{a}{a_0}\right)^{-3} \qquad \vec{V}_{bg} = \left(\frac{\dot{a}}{a}\right)\vec{r} \qquad \varphi_{bg} = \frac{2}{3}\pi G \rho_{bg} r^2 \qquad P_{bg} = P(\rho_{bg})$$

We perturb again around the background, using $\dot{\rho}_{bg} = -3\frac{\dot{a}}{a}\rho_{bg}$; $\nabla\cdot\vec{V}_{bg} = 3\left(\frac{\dot{a}}{a}\right)$; $(\delta\vec{v}\cdot\nabla)\vec{v}_{bg} = \frac{\dot{a}}{a}\delta\vec{v}$

$$\Rightarrow \quad \delta\dot{\rho} + 3\frac{\dot{a}}{a}\delta\rho + \frac{\dot{a}}{a}(\vec{r}\cdot\vec{\nabla})\delta\rho + \rho_{bg}(\vec{\nabla}\cdot\delta\vec{V}) = 0$$

$$\delta\dot{\vec{V}} + \frac{\dot{a}}{a}\delta\vec{v} + \frac{\dot{a}}{a}(\vec{r}\cdot\vec{\nabla})\delta\vec{v} = -\frac{1}{\rho_{bg}}\vec{\nabla}\delta P - \vec{\nabla}\delta\varphi$$

$$\nabla^2\delta\varphi - 4\pi G\delta\rho = 0$$

We can drop the $\vec{r}\cdot\vec{\nabla}$ terms because these are a coordinate dependent artefact of the Newtonian formulation

As before $\delta u_i = u_i(t)\, e^{i\vec{k}\cdot\vec{r}}$ $\qquad \left(\text{note that } k = \dfrac{2\pi}{\lambda} = \dfrac{2\pi}{\lambda_0}\dfrac{a_0}{a} = k_0\dfrac{a_0}{a}\right)$

$$\Rightarrow \dot{D} + 3\frac{\dot{a}}{a}D + i\rho_{bg}\vec{k}\cdot\vec{V} = 0$$

$$\dot{\vec{V}} + \frac{\dot{a}}{a}\vec{V} + i c_s^2\vec{k}\frac{D}{\rho} + i\vec{k}\cdot\bar{\Phi} = 0$$

$$k^2\bar{\Phi} + 4\pi G \underset{\rho_{bg}\delta}{D} = 0$$

as before only $\vec{k} \parallel \vec{V}$ is of interest

$$\dot{D} + 3\frac{\dot{a}}{a}D + i\rho_{bg}kV = 0$$

$$\dot{V} + \frac{\dot{a}}{a}V + ik\left(c_s^2 - \frac{4\pi G\rho_{bg}}{k^2}\right)\frac{D}{\rho} = 0$$

as $D = \rho_{bg}\delta$ $\quad (\dot{D} = \dot{\rho}_{bg}\delta + \rho_{bg}\dot{\delta})$ $\quad \Rightarrow \dot{\delta} + ikV = 0$

which upon differentiation gives $\ddot{\delta} + ik\left(\dot{V} - \frac{\dot{a}}{a}V\right) = 0$

$$\Rightarrow \ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} + (c_s^2k^2 - 4\pi G\rho)\delta = 0$$

This is the generalization of the static case, and gives the evolution of density perturbations of waves with wavenumber k as long as $\delta \ll 1$

To solve this equation we need a prescription of a, ρ and c_s.

* Flat matter dominated Einstein - de Sitter model

$$\rho = \frac{1}{6\pi G t^2} \quad \text{and} \quad a = a_0\left(\frac{t}{t_0}\right)^{2/3} \quad \text{and} \quad \frac{\dot{a}}{a} = H = \frac{2}{3t}$$

If we assume that the matter comprises monoatomic particles of mass m, then the sound speed is

$$c_s = \left(\frac{5k_BT_m}{3m}\right)^{1/2} = \left(\frac{5k_BT_{0,m}}{3m}\right)^{1/2}\frac{a_0}{a}$$

$$\Rightarrow \ddot{\delta} + \frac{4}{3}\frac{\dot{\delta}}{t} - \frac{2}{3t^2}\left(1 - \frac{c_s^2k^2}{4\pi G\rho}\right)\delta = 0$$

In the case c_sk very small (long wavelengths, low sound speed) try a solution $\delta \propto t^n$

$(k \simeq 0)$ $\quad \Rightarrow \left[n(n-1) + \frac{4}{3}n - \frac{2}{3}\right]t^{n-2} = 0 \quad \Leftrightarrow \quad n(n-1) + \frac{4}{3}n - \frac{2}{3} = 0$

This has two solutions $n = -1$ or $n = 2/3$

Growing mode $\delta_+ \propto t^{2/3} \propto a$; decaying mode $\delta_- \propto t^{-1}$

For large k (short wavelengths) and under the assumption that c_s varies slowly

$$\ddot{\delta} + \frac{4}{3}\frac{\dot{\delta}}{t} - \frac{2}{3t^2}\left(\frac{c_sk^2}{4\pi G\rho}\right)\delta = 0$$ looks like damped harmonic oscillator with frequency $\sim 1/t$ *

If we try again $\delta \propto t^n$ we find solutions for $n^2 + \frac{n}{3} - \frac{2}{3}\left[1 - \frac{c_s^2 k^2}{4\pi G\rho}\right] = 0$

$$n = -\frac{1}{6} \pm \frac{1}{6}\sqrt{25 - 6\frac{c_s^2 k^2}{\pi G \rho}}$$

hence instability when $k \lesssim \frac{\sqrt{G\rho}}{c_s}$ and oscillations large k
Once more Jeans criterion!

The densities grow $\delta \propto t^{2/3} \propto a(t) \propto \frac{1}{1+z}$ as long as $|\delta| \ll 1$
When $\delta \sim 1$ the evolution can no longer be treated as a linear perturbation and numerical simulations are commonly used to study the later evolution
When a region reaches an overdensity ~ 1 it breaks away from the Hubble flow and collapses

If the Universe consisted only of baryonic matter then density fluctuations could only have started to grow at $z_{dec} \sim 1100$ $\Rightarrow$ grown by factor 1100
But DM perturbations started to grow effectively at z_{eq} (matter-radiation transition) ~ 3500
$\Rightarrow$ DM causes a headstart in structure formation

If we consider a low Ω_m universe where curvature dominates then $a \propto t$
and
$$\ddot{\delta} + \frac{2\dot{\delta}}{t} \simeq 0$$
which has solutions $\delta_- \propto t^{-1}$ and $\delta_+ \propto t^0$
$\Rightarrow$ no growth in low density universe

If Λ dominates $\ddot{\delta} + 2H\dot{\delta} \sim 0$ with solutions $\delta_- \propto e^{-2Ht}$ and $\delta_+ \propto t^0$
$\Rightarrow$ no growth in Λ dominated universe.

In the case of a radiation dominated Universe the derivation needs to include the pressure in the energy density: $\rho \to \rho + P/c^2$ and one can show that

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} + \left[c_s^2 k^2 - \frac{32}{3}\pi G \rho_{bg}\right]\delta = 0 \qquad \left(c_s = \frac{c}{\sqrt{3}}\right)$$

For a flat radiation-dominated Universe $a \propto t^{1/2}$ and $\rho = \frac{3}{32\pi G t^2}$, $\frac{\dot{a}}{a} = \frac{1}{2t}$

$$\Rightarrow \ddot{\delta} + \frac{\dot{\delta}}{t} - \frac{1}{t^2}\left(1 - \frac{3c_s^2 k^2}{32\pi G\rho}\right)\delta = 0$$

For $k \to 0$ the solution $\delta \propto t^n$ with $\delta_+ \propto t$ and $\delta_- \propto 1/t$
As before damped oscillations for large k, with transition near Jeans length

Consider now the growth of decoupled matter perturbations in a Universe where the expansion is driven by a relativistic component.

Assume $k=0$ $\Rightarrow$ $\ddot{\delta} + 2\left(\frac{\dot{a}}{a}\right)\dot{\delta} - 4\pi G \rho_m \delta = 0$ for matter component

We already examined the evolution for $t \gg t_{eq}$ (matter-radiation equality)

But at earlier times a and ρ evolve differently!

If we define $y = \frac{\rho_m}{\rho_r} = \frac{a}{a_{eq}}$ increases with time; $y=1$ at $z = z_{eq} \sim 3500$

$\delta = \frac{\delta \rho_m}{\rho_m}$; rewrite the perturbation equation from function of t in one of y.

$\Rightarrow \dot{\delta} = \delta' \frac{\dot{a}}{a_{eq}}$ and $\ddot{\delta} = \left[\delta'' \dot{a}^2 + \delta' \ddot{a}\right] / a_{eq}^2$ $\quad \left(\delta' = \frac{d}{dy}\delta\right)$

$\rho_m = \frac{y}{1+y}\rho$ and $\rho_r = \frac{1}{1+y}\rho$ and $p = \frac{1}{3}\rho_r c^2$

Friedman eqn $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_r)$

Acceleration eqn $\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)a = -\frac{4\pi G}{3}\left(\rho + \frac{1}{1+y}\rho\right)a = -\frac{1}{2}\frac{2+y}{1+y}\left(\frac{\dot{a}}{a}\right)^2$

Then $\ddot{\delta} + 2\left(\frac{\dot{a}}{a}\right)\dot{\delta} - 4\pi G \rho_m \delta = 0$ can be written as

$$\delta'' + \frac{2+3y}{2y(1+y)}\delta' - \frac{3}{2y(1+y)}\delta = 0$$

This has two solutions, one growing, one decaying. The growing mode

$$\delta_+ \propto 1 + \frac{3}{2}y \sim 1 + \frac{5000}{1+z}$$

Before z_{eq} we have that $y<1$ and the growing mode is frozen

This Meszaros effect applies to cold dark matter density fluctuations (not coupled to the radiation via pressure) on large scales

The total growth from 0 to t_{eq} is $\frac{\delta_+(y=1)}{\delta_+(y=0)} = \frac{5}{2}$ and afterwards by another factor $1+z_{eq}$

The physical reason for this slow growth is that before t_{eq} the Jeans time is longer than the expansion time. The energy in radiation causes the Universe to expand so fast that the matter has no time to respond.

Before decoupling the baryon dynamics is coupled to that of the radiation
$\Rightarrow \delta_{bar}$ oscillates like the radiation, but after t_{eq} $\delta_{DM} \propto a$

As a result after decoupling $\delta_{DM} \gg \delta_{bar}$

After decoupling $\ddot{\delta}_{bar} + \frac{4}{3t}\dot{\delta}_{bar} = 4\pi G(\bar{\rho}_{bar}\delta_{bar} + \bar{\rho}_{DM}\delta_{DM})$

$$\ddot{\delta}_{DM} + \frac{4}{3t}\dot{\delta}_{DM} = 4\pi G(\bar{\rho}_{bar}\delta_{bar} + \bar{\rho}_{DM}\delta_{DM})$$

If we use that $\delta_m = \frac{\bar{\rho}_{bar}\delta_{bar} + \bar{\rho}_{DM}\delta_{DM}}{\bar{\rho}_{bar} + \bar{\rho}_{DM}} \simeq \delta_{DM}$ and $\Delta \equiv \delta_{DM} - \delta_{bar}$

$$\Rightarrow \ddot{\Delta} + \frac{4}{3t}\dot{\Delta} = 0 \qquad \Rightarrow \Delta = \text{const or } \Delta \propto t^{-1/3}$$

$$\delta_m \propto t^{2/3} \propto a \qquad \Rightarrow \frac{\delta_{DM}}{\delta_{bar}} = \frac{\bar{\rho}_m\delta_m + \bar{\rho}_{bar}\Delta}{\bar{\rho}_m\delta_m - \bar{\rho}_{DM}\Delta} \Rightarrow 1$$

The initial non-zero value of δ_{bar} at decoupling leaves a small effect on the δ_m at late times $\Rightarrow$ these are the baryon acoustic oscillations