At the time of decoupling the temperature of the neutrinos coincides with the temperature of the other constituents; after decoupling the neutrinos continue to evolve as a relativistic gas $T \propto \frac{1}{a}$ and $\rho \propto 1/a^4$

The e^+, e^-, γ component follows the same evolution, but at e^+, e^- annihilation the temperature is increased again; this ratio of temperatures persists until today $T_\gamma / T_\nu = \text{const} > 1$

Once e^+/e^- annihilation is over the Universe is dominated by radiation with a thermal Planck spectrum.

Towards the end of the lepton era nuclear physics starts to take place, ultimately resulting in H, He and traces of D, Li. This phase is not important for the thermal history of the Universe, but clearly important for our existence!

The element abundance by mass is the ratio of the mass of a particular element to the total mass in baryons

The abundance of ^4He : $Y = 0.25$ $\rightarrow$ 6 % of the nuclei
^3He : $\sim 10^{-3} Y$
^2H or D : $\sim 2 \times 10^{-3} Y$

Could the helium be produced in stars: no! If our Galaxy maintained a constant luminosity for $\sim 10^{10}$ years the total energy radiated would correspond to the fusion of 1% of the nucleons.

Atmospheres of main-sequence stars consist of ~25% He by mass with only weak trends with age and metallicity $\rightarrow$ This He was not made in stars
Furthermore D is photodissociated in stars; note that essentially all elements heavier than ^7Li are made in stars.

The ratio of neutron and proton number densities is $\frac{n_n}{n_p} = e^{-Q/kT} = e^{-1.5 \times 10^{10} K / T}$ (mass difference $m_n - m_p = 1.29$ MeV)
as long as protons and neutrons are in thermal equilibrium

The equilibrium is maintained by weak nuclear reactions:

$$n + \nu_e \leftrightarrows p + e^- \quad \text{and} \quad n + e^+ \leftrightarrows p + \bar{\nu}_e$$

Because of the mass difference there are more protons than neutrons

The equilibrium is maintained for $T_{ew} > 10^{10}$ K when the neutrinos decouple

When this happens $X_n = \frac{n}{n+p} \simeq \frac{n}{n_{tot}} \approx (1+e^{1.5})^{-1} \simeq 0.17 = X_n(0)$

More detailed calculations show that equilibrium is maintained until $T \sim 1.3 \times 10^9$ K (20s after the Big Bang), after which neutrons decay via beta-decay: $n \to p + e^- + \bar{\nu}_e$ which has a mean lifetime of 900s

$$\Rightarrow X_n(t) = X_n(0)\, e^{-(t-20s)/900s} \approx X_n(0)$$

Note that $m_n - m_p \sim kT_{ew}$ if T_{ew} would be much lower, only protons would be formed, and no nuclear reactions would occur.

Still the scarcity of neutrons relative to protons explains why BBN is so incomplete, leaving 75% of baryons unfused.

$$p + p \to D + e^+ + \nu_e$$ which is a weak force interaction $(\sigma_W \sim 10^{-47} m^2 (\frac{T}{MeV})^2)$

whereas $p + n \leftrightarrows D + \gamma$ is a strong force interaction

As $p+n \to D+\gamma$ is so much more efficient ($n+n \to D+e^- + \bar{\nu}_e$ is also a weak reaction) BBN proceeds until of neutrons are bonded into nuclei

$\Rightarrow$ This allows us to compute the maximum value for Y, if we take $n_n/n_p = 0.2$ and consider a group of 2 neutrons and 10 protons $\to 1\,{}^4He + 8p$

$$\Rightarrow Y_{max} = \frac{4}{12} = \frac{1}{3}$$ generally if $f = n_n/n_p \Rightarrow Y_{max} = \frac{2f}{1+f}$

The observed value is indeed smaller; this is because nucleosynthesis takes a while and some neutrons will decay; some will end up in ^3He or D or heavier nuclei such as Li

To compute Y accurately, as well as other abundances ~~also~~ all reactions need to be considered

At $t \simeq 2s$ the proton-neutron freeze-out occurs, the neutrinos are decoupled but the photons are still strongly coupled. To build the heavier nuclei we need a series of 2 particle interactions

$$p + n \leftrightarrows D + \gamma$$

the binding energy $B_D = (m_n + m_p - m_D)c^2 = 2.22\,\mathrm{MeV}$
γ with $E > B_D$ can destroy D.

Around the time of D formation, the relative number densities are given by a Saha-like equation

$$\frac{n_D}{n_p n_n} = \frac{g_D}{g_p g_n}\left(\frac{m_D}{m_p m_n}\right)^{3/2}\left(\frac{kT}{2\pi\hbar^2}\right)^{-3/2} e^{B_d/kT}$$

$g_D = 3$
$g_p = g_n = 2$

as $m_p \simeq m_n \simeq m_D/2$

$$\Rightarrow \frac{n_D}{n_p n_n} = 6\left(\frac{m_n kT}{\pi\hbar^2}\right)^{-3/2} e^{B_d/kT}$$

$\Rightarrow$ deuterium is favoured in the limit $T \to 0$, whereas p^+, n are favoured when $T \to \infty$

Define T_{nuc} is the temperature where $n_D/n_n = 1$ (half the neutrons have been fused)

$$\Rightarrow \frac{n_D}{n_n} = 6 n_p \left(\frac{m_n kT}{\pi\hbar^2}\right)^{-3/2} e^{B_d/kT}$$

Currently 75% of baryons are in the form of H, and before deuterium synthesis 83% were

$$\Rightarrow n_p \approx 0.8\, n_{baryon} = 0.8\,\eta\, n_\gamma = 0.8\,\eta \left[0.243\left(\frac{kT}{\hbar c}\right)^3\right]$$

$$\Rightarrow \frac{n_D}{n_n} \approx 6.5\,\eta\left(\frac{kT}{m_n c^2}\right)^{3/2} e^{B_d/kT}$$

This ratio is unity for T_{nuc} $8 \times 10^8\,\mathrm{K}$ $\Rightarrow$ 200s after the Big Bang.

The time delay until the start of nucleosynthesis is not negligible compared to the decay time of neutrons (900s)

At the start of nucleosynthesis $\frac{n_n}{n_p} \approx \frac{e^{-200/900}}{5.9\,(1 - e^{-200/900})} \approx \frac{0.8}{5.2} \approx 0.15$

$\Rightarrow$ This lowers Y_{max} to ~ 0.27

The ratio n_D/n_n does not remain at the equilibrium values; once there is enough D we get

$$D + p \leftrightharpoons {}^3He + \gamma \quad \text{or} \quad D + n \leftrightharpoons {}^3H + \gamma$$

${}^3H$ decays into ${}^3He + e^- + \bar{\nu}_e$ but a decay time of 18 years $\Rightarrow$ stable during BBN

${}^3H$ and ${}^3He$ are quickly converted into ${}^4He$ trough

$$\left.\begin{aligned} {}^3H + p &\leftrightharpoons {}^4He + \gamma \\ {}^3He + n &\leftrightharpoons {}^4He + \gamma \\ {}^3H + n &\leftrightharpoons {}^4He + \gamma \\ {}^3He + D &\leftrightharpoons {}^4He + p \end{aligned}\right\} \text{these are all strong force reactions (no neutrinos involved)}$$

The binding energy per nucleon is very high for ${}^4He$, whereas there are no stable nuclei with $A=5$ (${}^5He$ and ${}^5Li$ are not stable) $\Rightarrow$ very difficult to make heavier elements

$$\begin{aligned} {}^4He + D &\leftrightharpoons {}^6Li + \gamma \\ {}^4He + {}^3H &\leftrightharpoons {}^7Li + \gamma \\ {}^4He + {}^3He &\rightleftarrows {}^7Be + \gamma \end{aligned}$$

Synthesis of nuclei with $A > 7$ is hindered by the absence of stable nuclei with $A=8$

Initially at $T \gg 10^9 K$ all the baryons are in the form of protons and neutron
As the deuterium density increases ${}^3H$, ${}^3He$ and ${}^4He$ are formed
At $t \sim 10 min$ ($T \sim 4 \times 10^8 K$) BBN is essentially over.

The yields of D, ${}^3He$, ${}^4He$, ${}^6Li$ & ${}^7Li$ depend on η: a high value for η (baryon to photon ratio) increases the value for T_{nuc}

As BBN is a race against the clock, an earlier start means more ${}^4He$ is formed, but less D and ${}^3He$ as leftovers

The deuterium abundance can be used to estimate η

$$\Rightarrow n_{baryon,0} = \eta\, n_{\gamma,0} = 0.23 \pm 0.02 \; m^{-3} \Rightarrow \Omega_{baryon,0} = 0.04 \pm 0.01$$

Also CMB gives constraints on Ω_{baryon} which are in good agreement.

The radiative era begins at the moment of the annihilation of the electron-positron pairs at a $T \sim 5\times10^9\,K$ or $t \sim 10\,s$

The end of the radiative era occurs when the density of matter coincides with that of the relativistic particles, corresponding to a redshift

$$1+z_{eq} = \frac{\rho_{0,c}\,\Omega_0}{K_0\,\rho_{0,r}} \simeq \frac{4.3}{K_0}\times10^4\,\Omega_0\,h^2 \approx 5800 \quad \text{where } K_0 \simeq 1.68 \text{ if } N_\nu = 3$$

neutrino density is $\frac{7}{8}\left(\frac{4}{11}\right)^{4/3}\times N_\nu = 0.68$ times the photon density

$$T_{eq} = T_{0,r}\,(1+z_{eq}) \simeq \frac{10^5\,\Omega_0\,h^2}{K_0}\,K$$

At these temperatures the hydrogen & helium are fully ionized. As the temperature drops the number of neutral atoms and He^+ atoms grows through the equilibrium reactions

$$H^+ + e^- \leftrightarrows H + \gamma \qquad He^{++} + e^- \leftrightarrows He^+ + \gamma \qquad He^+ + e^- \leftrightarrows He + \gamma$$

The number density of the individual components is determined by the Saha equation

Let us focus on hydrogen $p^+ + e^- \leftrightarrows H + \gamma$, the density of the various particles is given by the Boltzmann distribution (non-relativistic)

$$n = g\left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2} e^{-(\mu - mc^2)/k_B T}$$

$g = 4$ for H
$g = 2$ for e^-
$n_e = n_p$ charge neutrality
$m_p \simeq m_H$

μ is the chemical potential = internal energy of the particle, which is released when the particle is destroyed; the chemical potential is conserved of ingoing and outgoing particles when the reaction is in equilibrium

In this case $\mu_H = \mu_p + \mu_e$ (photons have $\mu = 0$)

We furthermore have $(m_e + m_p - m_H)c^2 \simeq 13.6\,eV$

$$\Rightarrow \frac{n_H}{n_p\,n_e} = \frac{g_H}{g_p\,g_e}\left(\frac{m_H}{m_p\,m_e}\right)^{3/2}\left(\frac{kT}{2\pi\hbar^2}\right)^{-3/2} e^{[m_p + m_e - m_H]c^2/kT}$$

$$= \left(\frac{m_e kT}{2\pi\hbar^2}\right)^{-3/2} e^{Q/kT} \qquad \text{where } Q = 13.6\,eV$$

if $X \equiv \frac{n_p}{n_p + n_H} = \frac{n_p}{n_{baryon}}$ is the fractional ionization

$$\Rightarrow \frac{1-X}{X} = n_p\left(\frac{m_e kT}{2\pi\hbar^2}\right)^{-3/2} e^{Q/kT} \qquad \left(\text{where we used that } n_H = \frac{1-X}{X}\,n_p\right)$$

Recall that $\eta = \frac{n_{baryon}}{n_\gamma}$ if we assume that hydrogen is the only element then we can write

$$\eta = \frac{n_p}{X n_\gamma}$$

But the photons have a blackbody spectrum $n_\gamma = \frac{2.404}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3 = 0.243\left(\frac{kT}{\hbar c}\right)^3$

$$\Rightarrow n_p = 0.243\, X \eta \left(\frac{kT}{\hbar c}\right)^3$$

$$\Rightarrow \frac{1-X}{X^2} \simeq 3.84\eta \left(\frac{kT}{m_e c^2}\right)^{3/2} e^{Q/kT}$$

If we define the moment of recombination as the instant when $X = 1/2$ assuming $\eta = 5.5\times 10^{-10}$ the recombination temperature is

$$kT_{rec} = 0.323\ eV = \frac{Q}{42} \ll 13.6\,eV$$

this is the consequence of the large number of photons; something similar happened for Deuterium formation

This corresponds to temperature $T_{rec} \sim 3740K$

$z_{rec} = 1370$

$t_{rec} = 240{,}000$ yr.

Recombination is not instantaneous, but quite rapid : $X=0.9$ at $z=1475$, $X=0.1$ at $z=1255$ $\Big\}\ \delta t = 70{,}000$ yr.

Since the number density of free electrons drops radidly during the epoch of recombination, the time of photon decoupling comes soon after the time of recombination

The rate of photon scattering is $\Gamma(z) = n_e(z)\sigma_T c = X(z)(1+z)^3 n_{baryon,0}\,\sigma_T c$

$\sigma_T = 6.65\times 10^{-25}\ cm^2$, using $\Omega_{bary,0} = 0.04$

$$\Rightarrow \Gamma(z) = 4.4\times 10^{-21}\ s^{-1}\ X(z)(1+z)^3$$

at $z=0$ $H_0 = 2.5\times 10^{-18}\ s^{-1} \Rightarrow \Gamma(z) \ll H$ so photons are decoupled

at $z=1200$ expansion rate $\sim H_0 \Omega_{m0}^{1/2}(1+z)^{3/2} \ll \Gamma(z)$ so before recombination photons were well-coupled to electrons

When recombination takes place the Universe is matter-dominated, so

$$\frac{H^2}{H_0^2} = \frac{\Omega_{m,0}}{a^3} = \Omega_{m,0}\,(1+z)^3$$

If we take $\Omega_{m,0} = 0.3$ $\Rightarrow H(z) = 1.24 \times 10^{-18}\ s^{-1}\,(1+z)^{3/2}$

The redshift for photon decoupling is when expansion rate = scattering rate : $H = \Gamma$

$$\Rightarrow 1+z_{dec} = \frac{43}{X(z_{dec})^{2/3}} \qquad \text{this yields } z_{dec} = 1130$$

The exact redshift is a little bit smaller because the Saha equation assumes ~~that~~ the reaction $H + \gamma \leftrightarrows p + e^-$ is in equilibrium, but when Γ starts to drop ~~below~~ H this is no longer the case $\Rightarrow$ below $z \sim 1200$ the Saha equation underpredicts the number of free electrons and decoupling is in fact delayed

More detailed calculations yield $z_{dec} \sim 1100$ or $T_{dec} \simeq 3000$K or $t_{dec} \sim 350{,}000$ yr.
This is comparable to the redshift of last scattering

Photon decoupling marks the beginning of structure formation!
Now the gas can evolve without being smoothed out by the photons

We start with tiny fluctuations in the background radiation temperature = photon energy density : does this also correspond to matter energy density?
It is on scales larger than θ_H (we discussed this before)

$$\varepsilon(\vec{r}) = \bar{\varepsilon} + \delta\varepsilon(\vec{r}) \qquad \text{then} \quad \nabla^2(\delta\Phi) = \frac{4\pi G}{c^2}\,\delta\varepsilon \qquad \text{Poisson's equation}$$

photons in a potential well need to climb out of the well and are redshifted
In the COBE map the cool (redshifted) parts correspond to minima in $\delta\Phi$ at the time of last scattering, and hot spots to maxima

Sachs & Wolfe showed in 1967 that $\dfrac{\delta T}{T} = \dfrac{1}{3}\,\dfrac{\delta\Phi}{c^2}$ the Sachs-Wolfe effect

On smaller scales we have the more complicated acoustic oscillations.