Is the early universe in termal equilibrium?
We can assume that after the Planck time $T(t) \approx T_P \dfrac{a(t_P)}{a(t)}$

$\Rightarrow$ early on all particles all become relativistic

What is the equilibrium distribution of a particle species i? This depends on whether it is a fermion or a boson, and how many spin or helicity states it possesses, g_i — statistical weight

The number density is given by

$$n_i = g_i \left(\frac{k_B T}{\hbar c}\right)^3 \int_0^\infty \frac{x^2\,dx}{e^x \pm 1} = \begin{pmatrix} 3/4 \\ 1 \end{pmatrix} \frac{g_i}{\pi^2} \zeta(3) \left(\frac{k_B T}{\hbar c}\right)^3$$

fermion ($3/4$), boson (1)

(+ for fermions and −1 for bosons) — Riemann ζ function, $\zeta(3) \sim 1.202$

The energy density is $$\rho_i(T)c^2 = \frac{g_i k_B^4 T^4}{2\pi^2 \hbar^3 c^3} \int_0^\infty \frac{x^3\,dx}{e^x \pm 1} = \begin{pmatrix} 7/8 \\ 1 \end{pmatrix} \frac{g_i}{2} \sigma_r T^4$$

$\Rightarrow$ the total energy density is $$\rho(T)c^2 = \left(\sum_B g_{iB} + \frac{7}{8}\sum_F g_{iF}\right)\frac{\sigma_r T^4}{2} \equiv g^*(T)\,\frac{\sigma_r T^4}{2}$$

effective degrees of freedom
$g^*(T) < 200$ or so

To get total energy density one should add particles that have decoupled and are no longer in thermal equilibrium, or no longer relativistic; but this is negligible in early universe

$$n_B = \frac{4}{3} n_F \approx \frac{\rho_B c^2}{3kT} \approx \frac{\rho_F c^2}{3kT} \approx \frac{\zeta(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3$$

The average separation between particles is $d \sim [g^*(T) n_B]^{-1/3} \approx n_B^{-1/3} \sim \dfrac{\hbar c}{k_B T}$ — thermal length

The cross section of all the particles $(T \to T_P)$: $\sigma_a \simeq \alpha^2 \left(\dfrac{\hbar c}{kT}\right)^2$ where $\alpha \sim \frac{1}{50}$

$\Rightarrow$ collision time is $\tau_{coll} = \dfrac{1}{n \sigma_a c} \simeq \dfrac{\hbar}{g^*(T)\alpha^2 k_B T}$

This can be compared to the expansion time scale $\tau_H = \dfrac{a}{\dot a}$

$$\tau_H = 2t \simeq \left(\frac{3}{32\pi G\rho}\right)^{1/2} \approx \frac{0.3\,\hbar T_P}{g^*(T)^{1/2} k_B T^2} \simeq \frac{2.42\times10^{-6}}{\sqrt{g^*(T)}}\left(\frac{T}{\mathrm{GeV}}\right)^{-2}\ \mathrm{s} \qquad (1\,\mathrm{GeV} \mathrel{\hat=} 1.16\times10^{13}\,\mathrm{K})$$

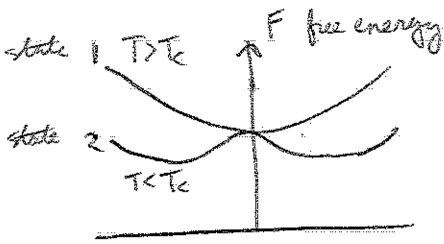
$\Rightarrow \dfrac{\tau_{coll}}{\tau_H} \simeq \dfrac{1}{\sqrt{g^*}\,\alpha^2}\dfrac{T}{T_P} \ll 1 \Rightarrow$ thermal equilibrium

The early phase is often called the era of phase transitions

↓

rearrangement of the microphysics in which a particular symmetry is created or destroyed:

- freezing, melting, evaporation → location of particles
- ferromagnetism → orientation of particles



$$F = U - TS$$

U: internal energy; T: temperature; S: entropy

At high T an increase in entropy leads to decrease in free energy
After the phase transition there is spontaneous symmetry breaking
and the system has to choose a new state

After each phase transition the effective physics changes
Phase transitions can leave defects if different regions pick a different state

Kibble mechanism: different horizon sized volumes choose their ground states independently
(no causal connection between them)
This will create seeds of defects, one per few horizon volumes
As the universe expands and cools the fields decay to their ground states
over most of space, but trapped energy domains remain as defects

↓

This is a general prediction

Possible defects

- monopole (point defect)
- cosmic string (line defect)
- domain wall (surface defect)
- textures (higher dimensional defects)

Phase transitions of the Universe

Between $T \sim 10^{19}$ and 10^{15} GeV quantum gravity effects decrease in importance and interactions are described by a GUT. Baryon number is not conserved in GUTs, so no assymmetry between matter and antimatter (any excess of one kind can be removed through interaction)

Near $T \sim 10^{15}$ K ($t = 10^{-37}$ s) the GUT symmetry breaks into $SU(3) \times SU(2) \times U(1)$ ~~of~~ the standard model of particles; the GUT phase transition typically results in the formation of magnetic monopoles. For typical GUTs $m_M \simeq 10^{16}$ GeV and predict a number density $n_M > 10^{-10}\, n_\gamma \Rightarrow n_{0,M} > 10^{-10}\, n_{\gamma,0} \simeq n_{b,0}$

But $\Omega_M > \frac{m_M}{m_p} \Omega_b \simeq 10^{16}$ which does not match observations; this is the monopole problem

A GUT that unifies the electroweak interactions with the strong interactions puts leptons and hadrons on the same footing and thus allows processes that do not conserve baryon number

$T_{GUT} > T > \underset{}{T_{EW}}^{\approx 10^{-11} s}$ when temperature falls below 10^{15} GeV the unification of the strong and elektroweak interactions no longer holds. Towards the end of this period the universe is filled with an ideal gas of leptons and antileptons, the four vector bosons, quarks, antiquarks and gluons ($g^* \simeq 100$) γ, W^+, W^-, Z

The horizon is 1 cm and contains $\sim 10^{19}$ particles

$T_{EW} \gtrsim T > T_{QH}^{\approx 10^{-5} s} \simeq 200$ MeV : at $T \sim 100$ GeV elektro-weak symmetry is broken and we get separate elektromagnetic and weak forces

quark-hadron phase

All the leptons acquire mass

at $t_{QH} = 10^{-5}$ s the universe is ~ 1 km in size

at $T_{QH} \sim 200$-300 MeV we have the final phase transition and the strong interaction leads to the confinement of quarks into hadrons.

Problems of the standard hot Big Bang model

The standard cosmology model has achieved four important successes:

- the predicted abundances of light-element abundances produced during nucleosynthesis agree with observations (we will discuss in a later lecture)
- the cosmic microwave background is naturally explained as a relic of the hot initial phase
- it accounts naturally for the expansion of the Universe
- it provides a framework to understand the formation of cosmic structure

These are also several problems (some of which can be addressed by incorporating "new physics"

- the origin of the Universe (or the evolution before the Planck time)
- the horizon problem (why is the Universe so isotropic and homogeneous)
- the flatness problem
- the origin of baryon asymmetry
- the unknown evolution at $T > 100\,\mathrm{GeV}$ (monopole problem)
- the origin of the primordial density fluctuations
- the nature of dark matter
- the nature of dark energy

We already saw that models with $w \geq 0$ give rise to particle horizon
If the scale factor tends to 0 at early times as t^β then the particle horizon at time t

$$R_H(t) = a(t) \int_0^t \frac{c\,dt'}{a(t')} \quad \text{exists if } \beta < 1 .$$

as $\ddot{a} = -\frac{4}{3}\pi G\left(\rho + 3\frac{p}{c^2}\right)a = \beta(\beta-1)\,a/t^2 \Rightarrow \beta(\beta-1) = -\frac{4}{3}\pi G\left(\rho + 3\,p/c^2\right)t^2 \propto \ddot{a}$

We have a Big Bang singularity if $0 < \beta < 1$ and therefore there is a particle horizon
This is hard to rhyme with the cosmological principle

An early period of inflation (= accelerated expansion) solves the flatness, horizon, and monopole problem in an elegant fashion; it also can explain the origin of the density fluctuations

If $w = -1$ then $a \propto e^{t/\tau}$ where $\tau = \left(\frac{a}{\dot{a}}\right)_{t=t_i}$

The condition for inflation can be expressed as $\ddot{a} = a(H^2 + \dot{H}) > 0$
standard inflation: $\dot{H} = 0$

Inflation

For inflation to be able to causally disconnect regions that were before in causal contact the expansion must be so rapid that there exists an event horizon at a finite distance from any point

$\Rightarrow$ The Hubble radius in comoving coordinates must shrink with time

$$\frac{d}{dt}\left(\frac{c}{aH}\right) < 0 \qquad \text{as } H = \frac{\dot a}{a} \Rightarrow \frac{d}{dt}\left(\frac{c}{\dot a}\right) < 0 \qquad \Rightarrow \ddot a > 0$$

Given the acceleration equation we need a substance with sufficient negative pressure.

The inflation field; in physics we encounter scalar fields to describe the potential energy associated with a particular force; the force is the gradient of the potential energy scalar field. Other examples are the temperature or pressure field
In quantum field theory a scalar field is associated with spin-0 particles
The Higgs field is an example

The only fundamental scalar field we know that has been observed

Imagine the early Universe was filled with a scalar field $\varphi(\vec{x},t)$ called the inflaton field and that $\varphi(\vec{x},0) = \varphi_0 > 0$, i.e. not in the ground-state

at Big Bang

In this case it may lead to accelerated expansion; after a while the field decays into particles (causing reheating)

The Lagrangian of a scalar field is $\mathcal{L} = -\frac{1}{2}c^2 \partial_\mu \partial^\mu \varphi - V(\varphi)$

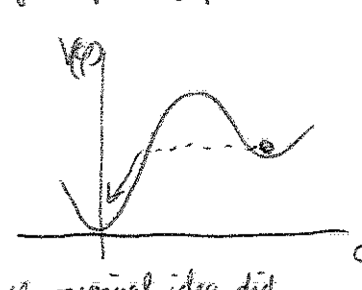
If we assume homogeneity and isotropy we can define effective density and pressure (this is not a real fluid)

$$\rho c^2 = \frac{1}{2}\dot\varphi + V(\varphi)$$

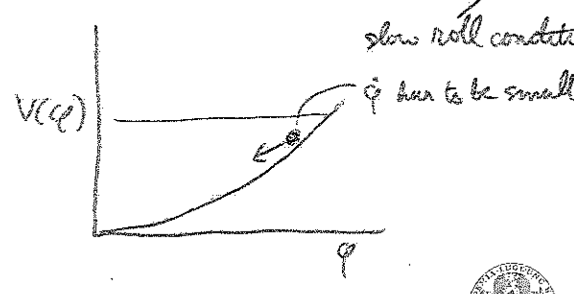
$$P = \frac{1}{2}\dot\varphi^2 - V(\varphi)$$

To get $P < -\frac{1}{3}\rho c^2 \Rightarrow \frac{1}{2}\dot\varphi^2 - V(\varphi) < -\frac{1}{3}\left(\frac{1}{2}\dot\varphi^2 + V(\varphi)\right) \Rightarrow \dot\varphi^2 < V(\varphi)$

slow roll condition



the original idea did not work because the false vacuum areas grow too fast



$$[V(\varphi)] = erg/cm^3 \quad \text{and} \quad [\dot{\varphi}^2] = erg/cm^3 \quad \Rightarrow [\varphi] = \left(\frac{erg\, s^2}{cm^3}\right)^{1/2} = \left(\frac{g}{cm}\right)^{1/2}$$

$$\Rightarrow \varphi_{Planck} = \sqrt{\frac{m_{Planck}}{l_{Planck}}} = \frac{c}{\sqrt{G}} \quad (\text{note lack of } \hbar)$$

The form of the potential depends on the adopted theory. Because we do not have a definitive model, people consider various choices of $V(\varphi)$

$V(\varphi) = \lambda(\varphi^2 - M^2)^2$ Higgs potential
$V(\varphi) = \frac{1}{2} m^2 \varphi^2$ Massive scalar field
$V(\varphi) = \lambda \varphi^4$ Self interacting scalar field

Take the Friedmann & fluid equation and insert the expression for pressure and density of the scalar field:

$$H^2 = \frac{8\pi G}{3c^2}\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right) - \frac{kc^2}{a^2}$$

$$\ddot{\varphi} + 3H\dot{\varphi} = -\frac{dV(\varphi)}{d\varphi}$$

If $\ddot{\varphi} = 0$ or if φ (and $V(\varphi)$) do not change much over the period where a increases exponentially then we can assume $k \sim 0$

$$\Rightarrow H^2 = \frac{8\pi G}{3c^2}\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right)$$

and if $\dot{\varphi}^2 \ll V(\varphi) \Rightarrow H^2 = \frac{8\pi G}{3c^2} V(\varphi)$ and $\ddot{\varphi} \ll \frac{dV(\varphi)}{d\varphi}$

$$\Rightarrow 3H\dot{\varphi} = -\frac{dV(\varphi)}{d\varphi}$$

One can show that the slow roll conditions are

$$\varepsilon \equiv \frac{c^2}{24\pi G}\left(\frac{V'(\varphi)}{V(\varphi)}\right)^2 \ll 1 \quad \text{and} \quad \eta \equiv \frac{c^2}{8\pi G} \frac{V''(\varphi)}{V(\varphi)} \ll 1$$

$\Rightarrow$ the potential must be very flat

$\varepsilon < 1$ needed for inflation
$\eta < 1$ needed for prolonged inflation

If we consider $V(\varphi) = \frac{1}{2} m^2 \varphi^2$

$$\Rightarrow \frac{V'}{V} = \frac{2}{\varphi} \quad \text{and} \quad \frac{V''}{V} = \frac{2}{\varphi^2}$$ and the slow roll conditions are

$\varphi \gg \frac{1}{\sqrt{6\pi}} \varphi_{Planck}$ and $\varphi \gg \frac{1}{\sqrt{4\pi}} \varphi_{Planck}$ i.e. equivalent ($\varphi_{Planck} = \frac{c}{\sqrt{G}}$)

Inflation continues until φ drops below $\varphi_{Planck}/\sqrt{4\pi}$; during inflation H slowly decreases because $V(\varphi)$ slowly decreases.

$$\Rightarrow \underset{\text{variation timescale for } H}{\tau_{vary}} \simeq \left(\frac{d \ln H}{dt} \right)^{-1} \gg \frac{1}{H} \equiv t_{exp}$$ or $|\dot{H}| \ll H^2$ as we saw before

$$\Rightarrow \text{indeed} \quad a(t) \propto e^{Ht}$$

If we want $\dot{\varphi}^2 \ll V(\varphi)$ then this gives a lower limit on the duration τ_{infl} : $\tau_{infl} \gg \frac{1}{m}$

$$H^2 = \frac{8\pi G}{3c^2} V(\varphi) \simeq \frac{8\pi G}{6c^2} m^2 \varphi^2 = \frac{G}{c^2} m^2 \varphi^2_{Planck} = \frac{m^2}{3}$$

end of inflation $\varphi = \varphi_{Planck}/\sqrt{4\pi}$

The Hubble constant defines the expansion e-folding timescale $\tau_{exp} = \frac{1}{H}$: for successful inflation

$\tau_{infl} \gg \tau_{exp}$ if we take $\tau_{exp} = \frac{1}{H} = \sqrt{\frac{3}{m}}$ $\Rightarrow m^2 \gg \frac{3}{\tau_{infl}^2}$ or $\tau_{infl} \gg \frac{1}{m}$

Right after the end of inflation the inflaton field should, by definition, have an energy density corresponding to the critical energy density

$$\rho c^2 = \rho_{crit} c^2 = \frac{3H^2c^2}{8\pi G} = \frac{3H^2}{8\pi} \varphi^2_{Planck}$$

If we take $\rho c^2 = V(\varphi) = \frac{1}{2} m^2 \varphi^2$ and use $\varphi = \frac{\varphi_{Planck}}{\sqrt{4\pi}}$ we get for the end of inflation:

$$\rho c^2 \simeq \frac{m^2}{8\pi} \varphi^2_{Planck} \Rightarrow H^2 \simeq \frac{m^2}{3}$$ consistent with previous estimate

If we assume that inflation ends at $\tau_{infl} = 10^{-32}$ this gives a mass scale (including $\hbar$ correctly) $mc^2 \sim 6.5 \times 10^4$ TeV well beyond the 7 TeV of the LHC.

Can inflation solve the flatness problem?

$$H^2 = \frac{8\pi G}{3c^2} V(\varphi) - \frac{Kc^2}{a^2}$$ as $V(\varphi) \sim$ constant the ratio of the two terms $\propto \frac{1}{a^2}$

After inflation $V(\varphi)$ the field decays into ρc^2 of matter, which behaves relativistically $\rightarrow \propto \frac{1}{a^4}$

We now observe $|\Omega_k| << 0.01$ so at $t_{infl} \sim 10^{-32}$ s (end of inflation) or $a \simeq 10^{-26}$
it must have been as small as $|\Omega_k(t_{infl})| \ll 10^{-54}$
because $\Omega_k \propto a^2$ ever since inflation

Assuming that at the start of inflation Ω_k was ~ 1 the inflation must have increased a by at least $10^{27} = e^{62}$

$$\Rightarrow \tau_{infl} \geq 62\,\tau_{exp} = 62 \frac{1}{H_{infl}}$$

What about homogeneity?

To solve the homogeneity problem, we need to make sure there is enough time before inflation
$\rightarrow$ light must have been able to travel the distance of what is now the observable universe
$\rightarrow$ i.e. the shrinking of the comoving Hubble radius $\frac{1}{aH}$ has to be at least as large as the subsequent increase

If the universe was $\sim 4\times 10^{-25}$ cm (at least the part we see today) then at the start of inflation $\tau_{before} \geq \frac{4\times 10^{-25}}{c} \sim 10^{-35}$ s

If inflation ~~stocks too~~ soon homogenisation cannot occurs

If we want ~ 62 e-foldings in $\sim 10^{-32}$ s we have $\tau_{exp} \sim 10^{-34}$ s
and the time to homogenise is still 10x shorter $\Rightarrow$ we expect homogeneous universe

Inflation also predicts a nearly scale invariant powerspectrum of initial fluctuations
(i.e. $k^3 P(k) \sim$ constant)

or ~~~~ $\frac{d\ln P}{d\ln k} = n - 3$ for tensors $n_T = -2\varepsilon$ $\leftarrow$ gravitational waves
scalar $n = 1 - 4\varepsilon - 2\eta$ $\leftarrow$ CMB