"Classical Cosmology" or a "Search for two numbers"

In the early days of observational cosmology much emphasis was placed on the geometric properties

Sandage: "We need to determine H_0 and q_0"

deceleration parameter

$$a(t) = a(t_0) + \left.\frac{da}{dt}\right|_{t=t_0}(t-t_0) + \frac{1}{2}\left.\frac{d^2a}{dt^2}\right|_{t=t_0}(t-t_0)^2 + \dots$$

divide by $a(t_0)$ $\Rightarrow$ $a(t) \approx 1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2$

where $q_0 \equiv -\left(\frac{\ddot{a}\, a}{\dot{a}^2}\right)_{t=t_0} = -\left(\frac{\ddot{a}}{a H^2}\right)_{t=t_0}$

No physics and a pure kinematical description

Acceleration equation $\frac{\ddot{a}}{a} = \frac{-4\pi G}{3c^2} \sum_w \rho_w (1+3w)$

$$\Rightarrow -\frac{\ddot{a}}{aH^2} = \frac{1}{2}\left[\frac{8\pi G}{3c^2H^2}\right] \sum \rho_w (1+3w) = \frac{1}{2}\sum_w \Omega_w (1+3w) = q_0$$

$$q_0 = \frac{\Omega_m}{2} + \Omega_r - \Omega_\Lambda \text{ for flat universe}$$

In our ΛCDM model $q_0 \approx -0.55$

Recall that proper distance is $d_p = c\int_{t_e}^{t_0} \frac{dt}{a(t)}$

$$\frac{1}{a(t)} \approx 1 - H_0(t-t_0) + \left(\frac{1+q_0}{2}\right) H_0^2 (t-t_0)^2 \Rightarrow d_p(t_0) \approx c(t_0-t_e) + \frac{cH_0}{2}(t_0-t_e)^2$$

We do not know $t_0 - t_e$ but we know the scale factor at emission: $z = \frac{1}{a(t_e)} - 1$

$$\Rightarrow z \approx H_0(t_0-t_e) + \left(\frac{1+q_0}{2}\right) H_0^2 (t_0-t_e)^2 \Rightarrow t_0 - t_e = \frac{1}{H_0}\left[z - \left(\frac{1+q_0}{2}\right) z^2\right]$$

or $d_p(t_0) \approx \frac{c}{H_0} z \left[1 - \frac{1+q_0}{2} z\right]$ linear relation with z only if $z \ll \frac{2}{1+q_0}$

But we cannot measure the proper distance. But we can consider alternative definitions

One way to assign a distance is to use the luminosity.

$$d_L \equiv \left(\frac{L}{4\pi f}\right)^{1/2}$$

The "luminosity" distance is the proper distance in a static and Euclidean universe

If we consider a FRW metric $ds^2 = -c^2 dt^2 + a^2(t)\left[dr^2 + S_k(r)\, d\Omega^2\right]$

$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$

The photons emitted at time t_e spread over a sphere with radius $d_p(\cancel{t_0}) = r$ and proper surface area $A_p(t_0)$

In Euclidean space $A_p(t_0) = 4\pi d_p^2(t_0) = 4\pi r^2$, but more general

$$A_p(t_0) = 4\pi S_k^2(r)$$

$k>0$ then $A_p(t_0) < 4\pi r^2$
$k<0$ then $A_p(t_0) > 4\pi r^2$
spread over larger area!

In addition to these geometric effects the expansion of the Universe causes the flux to be decreased by a factor $\frac{1}{(1+z)^2}$: 1) energy decreases because $\lambda_0 = (1+z)\lambda_e \Rightarrow E_0 = \frac{E_e}{1+z}$

2) time dilation: the ~~time~~ between photon ~~detection~~ detections will be greater by factor $(1+z)$

$$\Rightarrow f = \frac{L}{4\pi S_k^2(r)(1+z)^2} \quad \text{or} \quad d_L \equiv S_k(r)(1+z)$$

If we consider flat geometry $d_L = r(1+z) = d_p(t_0)(1+z)$

~~due to expansion~~

Even in flat space using the inverse square law will ~~overestimate~~ overestimate the proper distance, but all $z \ll 1$ $d_L \simeq d_p$

Angular diameter distance

Distant objects appear smaller; we can use apparent size to define angular diameter distance

~~If~~ If an object of length ℓ subtends angle $\delta\theta \ll 1$ $\Rightarrow \delta\theta = \frac{\ell}{d_A}$

$$\Rightarrow d_A \equiv \frac{\ell}{d\theta}$$

Consider FRW metric; the comoving coordinates of the object where (r,θ_1,φ) and (r,θ_2,φ)
$\delta\theta = |\theta_1 - \theta_2|$

The distance at emission (between ~~the~~ the two ends of the object) $ds = a(t_e)\, S_k(r)\, d\theta = \ell$

$$\Rightarrow \ell = \frac{S_k\, d\theta}{1+z} \qquad \Rightarrow d_A \equiv \frac{\ell}{d\theta} = \frac{S_k(r)}{1+z}$$

Comparison with luminosity ~~distance~~ ~~shows that~~

$$d_A = \frac{d_L}{(1+z)^2}$$

This is the Etherington relation; this is valid in any cosmological background where photons travel on null-geodesics <u>and</u> photon number is conserved
↓
This can be tested with e.g. Euclid + SNe survey

For small z $(z \ll 1)$
$$d_A \simeq \frac{c}{H_0} z \left(1 - \frac{3+q_0}{2} z\right)$$
$$d_L \simeq \frac{c}{H_0} z \left(1 + \frac{1-q_0}{2} z\right)$$

Also interesting is the behaviour for $z \to \infty$

$$d_L(z\to\infty) \simeq z\, d_{Hor}(t_0) \quad \text{increases}$$
$$d_A(z\to\infty) \simeq \frac{d_{Hor}(t_0)}{z} \quad \text{decreases!}$$

Note that $d_p(z_1,z_2) = d_p(0,z_2) - d_p(0,z_1)$

this makes sense because d_p is defined in terms of light travel time; same is true for comoving distance

but $d_A(z_1,z_2) \neq d_A(0,z_2) - d_A(0,z_1)$

$$\frac{r(0,z_2) - r(0,z_1)}{(1+z_2)} \neq \frac{r(0,z_2)}{(1+z_2)} - \frac{r(0,z_1)}{(1+z_1)}$$

The relation between (comoving) distance, Hubble constant and density parameters can also be seen by writing the Friedmann equation as

$$H^2(a) = H_0^2\left[\Omega_\Lambda + \frac{\Omega_m}{a^3} + \frac{\Omega_{rad}}{a^4} - \frac{\Omega - 1}{a^2}\right]$$

The radial equation of motion of a photon is $R\,dr = c\,dt = c\frac{dR}{\dot{R}} = c\frac{dR}{RH}$ where $R = \frac{R_0}{1+z}$

$$\Rightarrow R_0\,dr = \frac{c}{H(z)}\,dz = \frac{c}{H_0}\left[(1-\Omega)(1+z)^2 + \Omega_\Lambda + \Omega_m(1+z)^3 + \Omega_r(1+z)^4\right]^{-1/2} dz$$

To calculate the comoving distance simply integrate this integral numerically

Note that $1 - \Omega(t) = \dfrac{-kc^2}{R_0^2\, a(t)^2\, H^2(t)}$

The current value is $1 - \Omega_0 = \dfrac{-kc^2}{R_0^2 H_0^2}$

If we are in a matter dominated Universe, should we be surprised if $\Omega_m \sim 0.2$?
The answer is "yes"

$$1 - \Omega(t) = \frac{H_0^2(1-\Omega_0)}{H(t)^2 a^2(t)} \qquad \text{at early times} \quad \frac{H^2(t)}{H_0^2} = \frac{\Omega_{r,0}}{a^4} + \frac{\Omega_{m,0}}{a^3}$$

$$\Rightarrow 1 - \Omega(t) = \frac{(1-\Omega_0)a^2}{\Omega_{r,0} + a\Omega_{m,0}}$$

$\Rightarrow$ during radiation domination $|1-\Omega| \propto a^2 \propto t$
during matter domination $|1-\Omega| \propto a \propto t^{2/3}$

$\Rightarrow$ the deviation grows with time and if Ω_m (or Ω in general) $\neq 1$ then it was arbitrarily close early on $(1-\Omega(t = t_{Planck}) = 10^{-60})$

$\Downarrow$

this suggests fine tuning!

We therefore expect a flat Universe ... the initial observation that $\Omega_m = 0.2$ is called the flatness problem.

We already saw that the energy density of the CMB black body radiation is

$$\varepsilon_{rad} \equiv \rho_{rad} c^2 = \alpha T^4 \quad \text{where } \alpha = \frac{\pi^2 k_B^4}{15 \hbar^3 c^3} \quad \text{which yielded } \Omega_{rad} = 2.47 \times 10^{-5} h^{-2}$$

We also know that $\rho_{rad} \propto \frac{1}{a^4}$ this implies $T \propto \frac{1}{a}$ as photons are conserved

The Universe cools as it expands, or it was hotter in the past, and very hot in the early universe?

If the temperature is changing then the ~~thermal distribution~~ must change as well:

black body: $\varepsilon(\nu)d\nu = \frac{8\pi h}{c^3} \frac{\nu^3 d\nu}{e^{h\nu/kT}-1}$

As the Universe expands, photons are redshifted and $\nu \propto \frac{1}{a}$ but $h\nu/kT$ remains the same $\rightarrow$ the black body spectrum is retained!

$\Rightarrow$ As long as at some early stage interactions were frequent to establish a thermal distribution, it will persevere when interactions become infrequent.

The energy density in the CMB $= \varepsilon_{rad}(t_0) = 4.17 \times 10^{-14}\, J\, m^{-3}$ and the mean energy of a CMB photon is $E_{mean} \approx 3 k_B T = 7.05 \times 10^{-4}\, eV$

$$\Rightarrow n_\gamma = 3.7 \times 10^8 / m^3$$

It is interesting to compare this to the number density of baryons: $\Omega_b \sim 0.02 h^{-2} \simeq 0.04$

$$\Rightarrow \varepsilon_{bar} = \rho_{bar} c^2 = \Omega_b \rho_c c^2 \simeq 3.38 \times 10^{-11}\, J/m^3$$

so baryon density is much higher, but baryons have more energy!

The restmass of a proton $\sim 939\, MeV$ $\Rightarrow n_{bar} = 0.22/m^3$

$$\Rightarrow \text{photon / baryon} = \frac{1}{\eta} \approx 1.7 \times 10^9$$

this needs to be explained...

The value of $\frac{1}{\eta}$ coincides with the entropy per baryon, which we will discuss later

At recombination non baryonic dark matter dominated the gravitational potential
If the density of dark matter was not perfectly homogeneous, but varied slightly such that

$$\varepsilon_{DM}(\vec{r}) = \underbrace{\bar{\varepsilon}_{DM}}_{\text{average energy density}} + \delta\varepsilon_{DM}(\vec{r})$$

$\Rightarrow$ variation in gravitational potential $\Rightarrow \nabla^2(\delta\Phi) = \frac{4\pi G}{c^2}\delta\varepsilon$

Imagine a photon in a potential well (a minimum of the potential): as it "climbs" out of the well, it loses energy and is redshifted $\rightarrow$ cool spot
If the photon rolls down a potential hill it is blueshifted $\rightarrow$ hot spot

The cool and hot spots in the COBE temperature map correspond to minima and maxima resp in $d\Phi$ at the time of recombination:

$$\frac{\delta T}{T} = \frac{1}{3}\frac{d\Phi}{c^2} \qquad \text{Sachs-Wolfe effect}$$

This explains the existence of temperature fluctuations on scales $\theta > \theta_H \sim 1°$

For $\theta < \theta_H$ the origin of the temperature fluctuations is complicated by the behaviour of the photons and baryons.

The energy density of the photon-baryon fluid is ~30% of the DM
Its equation of state ~~is~~ w is between 0 and 1/3
If it enters a potential well, the fluid is compressed by gravity but then the pressure rises until it is high enough to cause the fluid to expand outward

If the baryon-photon fluid is at maximum compression in well at decoupling, its density will be higher than average and as $T \propto \varepsilon^{1/4}$ the photons will be hotter than average; the opposite holds for maximum expansion.

If the plasma is in the process of expanding or contracting, the Doppler effect will blue or redshift the photons.

The resulting power spectrum can be computed given $\begin{cases}\text{initial conditions for primordial fluctuations} \\ \text{mix of ingredients}\end{cases}$

The location of the highest (first peak) corresponds to the potential well that just reached maximum compression, which have size $c/H(z_{dec})$ $\Rightarrow$ gives constraints on curvature & Ω
The amplitude depends on the sound speed of the plasma $C_S = \sqrt{w_{plasma}}\, c$ (depends on η)

$\Rightarrow$ observations give $\Omega_{b,0}^{CMB} \simeq 0.04$