Calculation of Ω_{rad} from CMB

Current temperature $T_0 = 2.755\,K$

energy density $\varepsilon_{rad} = \rho_{rad} c^2 = \alpha T^4$ where $\alpha \equiv \frac{\pi^2 k_B^4}{15 \hbar^3 c^3} = 7.565 \times 10^{-16}\,J\,m^{-3} K^{-4}$

$\Rightarrow \varepsilon_{rad}(T_0) = 4.17 \times 10^{-14}\,J\,m^{-3}$ $\Rightarrow \Omega_{rad} = 2.47 \times 10^{-5}\,h^{-2}$

Distances

if A and B are close then $d_A - d_B$ is the distance that A would measure to B, provided the measurement is done now, so A must make the measurement at the same proper time; this is not practical as one cannot measure simultaneously all distances

Imagine a distant galaxy (such that we can adopt the FRW metric)
$\downarrow$
How far is this galaxy

In an expanding universe distances increase with time $\Rightarrow$ we need to specify the time t when the distance is the correct one.

galaxy at (r, θ, φ) : the proper ~~time~~ distance $d_p(t)$ is the length of the spatial geodesic when the scale factor is fixed at $a(t)$

$$ds^2 = a^2(t) \left[dr^2 + S_k^2(r) \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \right]$$
$$= a^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right) \right]$$

The angle (θ, φ) is constant $\Rightarrow ds = a(t)\,dr$ for flat case

The proper distance is then $d_p(t) = a(t) \int_0^r dr = a(t)\, r$ $\leftarrow$ comoving ~~distance~~ coordinate

$$\dot{d}_p = \dot{a} r = \frac{\dot{a}}{a} d_p \Rightarrow V_p(t_0) = H_0 \, d_p(t_0) \quad \text{and} \quad H_0 = \left(\frac{\dot{a}}{a} \right)_{t = t_0}$$

More general
$$d_p = \int_0^r \frac{a\, dr'}{\sqrt{1 - kr'^2}} = a(t) \underbrace{\begin{cases} \arcsin(\sqrt{k}\, r)/\sqrt{k} & k > 0 \\ r & k = 0 \\ \operatorname{arcsinh}(\sqrt{-k}\, r)/\sqrt{-k} & k < 0 \end{cases}}_{f_k(r)}$$

The proper distance is the distance measured by the time ~~it takes~~ takes for the light from the object to reach us.

Universiteit Leiden

B

Instead of the proper distance, we <u>can</u> measure the redshift. What does that tell us?

Light travels along null-geodesic $ds=0$ $\cos(d\varphi, d\theta)=0$ $\Rightarrow c^2dt^2 = a^2(t)dr^2$

$$\Rightarrow \frac{c\,dt}{a(t)} = \frac{dr}{\sqrt{1-kr^2}}$$

The time it takes the ray to get from $r=0$ to $r=r_0$ we integrate

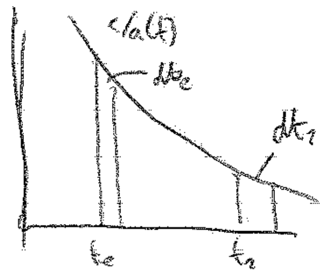
$$\text{received} \nearrow \int_{t_e}^{t_r} \frac{c\,dt}{a(t)} = \int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}} \equiv f_k(r_0)$$

emission

Imagine a light ray a short interval later:

$$\int_{t_e+dt_e}^{t_r+dt_r} \frac{c\,dt}{a(t)} = \int_0^{r_0} \frac{dr}{\sqrt{1-kr^2}}$$ ← the galaxy has not moved!

$$\Rightarrow \int_{t_e}^{t_r} \frac{c\,dt}{a(t)} = \int_{t_e+dt_e}^{t_r+dt_r} \frac{c\,dt}{a(t)}$$



We can subtract $\int_{t_e+dt_e}^{t_r} \frac{c\,dt}{a(t)}$ from each side

$$\Rightarrow \int_{t_e}^{t_e+dt_e} \frac{c\,dt}{a(t)} = \int_{t_r}^{t_r+dt_r} \frac{c\,dt}{a(t)}$$ (both slices have the same size)

area = width × height $\Rightarrow \frac{dt_r}{a(t_r)} = \frac{dt_e}{a(t_e)}$ as $a(t_r) > a(t_e) \Rightarrow dt_r > dt_e$

Imagine that there were successive crests of a single wave $\Rightarrow \lambda \propto dt \propto a(t)$

$$\Rightarrow \frac{\lambda_r}{\lambda_e} = \frac{a(t_r)}{a(t_e)}$$ If $t_r = t_0$ (we receive it now) at $a(t_0)=1$

$$\Rightarrow a(t_e) = \frac{1}{1+z}$$

The redshift tells us the scale factor at the time of emission.

For a single component Universe $a(t) = \left(\frac{t}{t_0}\right)^{2/(3+3w)}$ $\quad w \neq -1$

$$H_0 = \left(\frac{\dot a}{a}\right)_{t=t_0} = \frac{2}{3(1+w)} \frac{1}{t_0}$$ if $w > -1/3$ the universe is younger than the Hubble time

$$t_0 = \frac{2}{3(1+w)} \frac{1}{H_0}$$

In a flat, single component universe observe an object with redshift z

$$1+z = \frac{a(t_0)}{a(t_e)} = \left(\frac{t_0}{t_e}\right)^{2/(3+3w)}$$

$$\Rightarrow t_e = \frac{2}{3(1+w)H_0} \frac{1}{(1+z)^{3(1+w)/2}}$$

The current proper distance to the galaxy is

$$d_p(t_0) = c\int_{t_e}^{t_0} \frac{dt}{a(t)} = c\, t_0 \frac{3(1+w)}{1+3w}\left[1 - \left(\frac{t_e}{t_0}\right)^{(1+3w)/(3+3w)}\right] \quad w \neq -1/3$$

or in terms of z and H_0

$$d_p(t_0) = \frac{c}{H_0} \frac{2}{1+3w}\left[1 - (1+z)^{-(1+3w)/2}\right]$$

The most distant object you can see in theory is the one for which the light was emitted at $t=0$ and now reaches us. The proper distance to such an objects is called the horizon distance

$$d_{Hor}(t_0) = c\int_0^{t_0} \frac{dt}{a(t)}$$

In a flat universe the horizon has a finite value if $w > -1/3$:

$t \to 0$ or $z \to \infty$ $\quad d_{Hor}(t_0) = c\, t_0 \frac{3(1+w)}{1+3w} = \frac{c}{H_0} \frac{2}{1+3w}$

The portion of the Universe within the horizon is the "visible universe" and consists of all points that are <u>causally connected to the observer</u>

Note that for matter dominated flat Universe $d_{Hor} = 3c\, t_0$ or $2R_H \left(= 2\frac{c}{H_0}\right)$
R_H is the Hubble distance in the present Universe, but at earlier times when the Universe was smaller it was easier for light to make progress.

Particle horizon: boundary of that part of the universe from which light rays can have reached us in the age of the Universe. At the present epoch it is the observable universe

Imagine a sphere centered on us; the proper radius of the sphere is now

$$a(t_0) f_k(r) = a(t_0) \int_{t_e}^{t_0} \frac{c\,dt}{a(t)} \qquad \text{or for flat case } r = \int_{t_e}^{t_0} \frac{c\,dt}{a}$$

At early times $a(t) \propto t^{1/2}$ (radiation dominated) or $t^{2/3}$ (matter dominated)

If the integral $t_e \to 0$ converges then there is a particle horizon

For flat matter dominated case: $r = \int_{t_e}^{t_0} c\left(\frac{t}{t_0}\right)^{-2/3} dt = 3ct_0\left[1 - \left(\frac{t_e}{t_0}\right)^{1/3}\right] = 3ct_0\left[1 - \frac{1}{\sqrt{1+z}}\right]$

For a more general equation of state $a(t_0)\, f_k(r) = a(t_0) \int_{a_e}^{a_0} \frac{c\,da}{a(t)\,\dot{a}(t)}$

$$= \frac{a_0(t_0)\, c}{H_0} \int_{a_e}^{a_0} \frac{da}{a^2\sqrt{\Omega_m a^{-3-3w} + (1-\Omega_0)\, a^{-2}}}$$

$$= \frac{c}{H_0} \int_0^{z_e} \frac{dz}{(1+z)\sqrt{\Omega_0 (1+z)^{1+3w} + (1-\Omega_0)}} \quad \Rightarrow \text{a horizon exists if } w > -1/3$$

For a flat, matter dominated universe the current horizon is $3ct_0 = R_H$

At other times $R_H = 2c/H(t) = 3ct = 3ct_0(1+z)^{-3/2}$

The proper distance of the particle horizon was smaller in the past by a factor $\frac{t}{t_0} = \left(\frac{a}{a_0}\right)^{3/2} = (1+z)^{-3/2}$

$8\times$ smaller at $z=3$

$\Rightarrow$ $32000\times$ smaller at $z=1023$

This becomes interesting when we compare the distance to a source at redshift z with the horizon radius

coordinate distance $r = 3ct_0\left[1 - \frac{1}{\sqrt{1+z}}\right]$

which at the time of emission is a proper distance $a(t_e)\, r = \frac{r}{1+z} = 3ct_0\left(\frac{1}{1+z} - \frac{1}{(1+z)^{3/2}}\right)$

We already know that the proper radius of the horizon is $3ct = \frac{3ct_0}{(1+z)^{3/2}}$

The comoving horizon / coordinate distance $r = \frac{d_p}{a} = 3ct(1+z) = \frac{3ct_0}{(1+z)^{1/2}}$

$\Rightarrow$ The proper horizon scales linearly with linearly with time, whereas the proper distances scale $\propto t^{2/3}$ so more and more galaxies enter the horizon as time goes on.

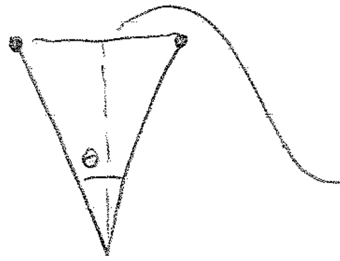
We receive light from distant galaxies ~~that~~ was emitted at a time when they could not have communicated with us

$\downarrow$

So why does the distant universe look so homogeneous?

$\Rightarrow$ we need an explanation

This is called the horizon problem



Consider two measurements of the CMB an angle θ apart

$\to$ comoving distance $2\sin(\theta/2)\,[D_H(0) - D_H(z)]$

where $D_H(z) = \int_z^\infty (1+z)\, c \frac{dt}{dz}\, dz = \frac{2c}{H_0\sqrt{1+z}}$

if the distance is larger than $2 D_H(z)$ then the two points are not causally connected (no event in spacetime is in the past light cone of the two points)

~~For the CMB~~ This occurs when $\sin\theta/2 > \frac{1}{\sqrt{1+z}-1}$

For the CMB this corresponds to $\theta > 3.6°$ $\Rightarrow$ but CMB T uniform to 1 part in 10^5!

Note that horizon size at time t depends on the integral $\frac{c\,dt}{a(t)}$ from early times to t

$\Rightarrow$ we need to change $a(t)$ at early times : objects can be in causal contact early then leave the horizon, and then reenter

$\downarrow$

We need accelerated expansion, which occurs if $\omega < -1/3$ (acceleration eqn)
We already saw that for $\omega < -1/3$ there is no particle horizon (if this persists to $z \to \infty$)

$\Rightarrow$ Inflation is the proposal to have a period of enormous acceleration.
We will return to this later