as $\dot{a} = \sqrt{\frac{8\pi G\rho_0}{3a} - kc^2} \Rightarrow \ddot{a} = \frac{1}{2\dot{a}}\left(-\frac{8\pi G}{3}\frac{\rho_0}{a^2}\dot{a}\right)$

$$\Leftrightarrow \ddot{a} = \frac{-4\pi G}{3}\rho_0\frac{1}{a^2} < 0$$

$\dot{a}$ has to become negative and the sphere will contract

Note that $\ddot{a} < 0$ independent of $k \rightarrow$ a was smaller in the past and $\rightarrow 0$ as $t \rightarrow 0$

$$k=0: \quad \dot{a}^2 = \frac{8\pi G\rho_0}{3}\frac{1}{a(t)}$$

We can solve this differential equation: (can be solved formally as this is a separable equation; but we will try a powerlaw which is a common solution in cosmology)

$$a \propto t^q \quad \Rightarrow \quad \dot{a}^2 \propto t^{2q-2} \quad \text{but r.h.s } \frac{1}{a} \propto t^{-q}$$

$$\Rightarrow 2q - 2 = -q \Leftrightarrow q = 2/3$$

$$\Rightarrow a(t) = \left(\frac{t}{t_0}\right)^{2/3} \qquad \rho(t) = \frac{\rho_0}{a^3} = \frac{\rho_0 t_0^2}{t^2}$$

current age

The Universe expands forever and $H = \frac{\dot{a}}{a} = \frac{2}{3t} \rightarrow 0$ at $t \rightarrow \infty$
Note that $t_0 = \frac{2}{3}H_0^{-1}$

At present day $a=1 \Rightarrow H_0^2 = \frac{8\pi G\rho_0}{3}$ or $\rho_0 = \frac{3}{8\pi G}H_0^2 \equiv \rho_{crit}$

The critical density separates eternal expansion from recollapse; for $H_0 = 70$ km/s this corresponds to ~5 protons per m^3 or $2.78\times10^{11}\, h^2\, M_\odot\, Mpc^{-3}$

H_0 in units of 100 km/s/Mpc

With the definition of ρ_{crit} we can write

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\frac{\rho(t)}{\rho_{crit}} - \underbrace{\frac{k}{H_0^2 a^2}}_{\Omega_k/a^2}\right] = H^2(a)$$

$\leftarrow$ curvature contribution

$\Omega(a)$ if we consider a particular ingredient

Now we need more than Friedmann eqn because we need to describe how the density evolves. We will also look into the GR interpretation of the Friedmann equation

We need to describe how density evolves; so far we looked at pressureless matter, but this is not sufficient in general.

Use the first law of thermodynamics $dE + p\,dV = T\,dS$

expanding volume of unit comoving radius

The volume has a physical radius of a $\Rightarrow E = mc^2 = \frac{4\pi}{3} a^3 \rho c^2$

$$\Rightarrow \frac{dE}{dt} = 4\pi a^2 \rho c^2 \frac{da}{dt} + \frac{4\pi}{3} a^3 \frac{d\rho}{dt} c^2$$

$$\frac{dV}{dt} = 4\pi a^2 \frac{da}{dt}$$

According to the cosmological principle there can be no flow of heat $\Rightarrow dS = 0$
(i.e. reversible adiabatic expansion)

$$\Rightarrow \frac{dE}{dt} + p\frac{dV}{dt} = 0 \qquad \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \qquad \text{fluid equation}$$

dilution in density due to increase in volume

loss of energy because pressure has done work as volume increased → has gone into potential energy

Note there are no pressure forces, as these require gradients.

We need an equation of state $P \equiv P(\rho)$ ← this function depends on material

Baryonic matter has eqn of state $P = w\varepsilon$ ← energy density with $w \approx 0$:

low density, non-relativistic massive particles $P = \frac{\rho}{\mu} kT$ (μ: mean mass)

$\varepsilon \approx \rho c^2$ (only rest mass important) $\Rightarrow P \approx \frac{kT}{\mu c^2} \varepsilon$

For a non-relativistic gas $3kT = \mu \langle v^2 \rangle$ $\Rightarrow P_{non\text{-}rel} = w\,\varepsilon_{non\text{-}rel}$ where $w \approx \frac{\langle v^2 \rangle}{3c^2} \ll 1$

Photons and highly relativistic particles have $w = \frac{1}{3}$ (as $\langle v^2 \rangle \sim c^2$)

w cannot take on arbitrary values: variations in P will travel at sound speed $c_s^2 = c^2\left(\frac{dP}{d\varepsilon}\right)$
A substance with $w > 0$ $c_s = \sqrt{w}\,c$ $\Rightarrow w \leq 1$ (as $c_s \leq c$)

$w < -1/3$ is interesting because $\ddot{a} > 0$; this would be "dark energy"
This can be seen from the "acceleration equation"

We can use the fluid equation and Friedmann equation to derive a third (but not independent) equation describing the acceleration of the scale factor

$$\frac{d}{dt}\left[\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}\right] \Rightarrow 2\left(\frac{\dot{a}}{a}\right)\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2}\right) = 2\left(\frac{\dot{a}}{a}\right)\left(\frac{a\ddot{a} - \dot{a}^2}{a^2}\right) = \frac{8\pi G}{3}\dot{\rho} + \frac{2kc^2\dot{a}}{a^3}$$

Use the fluid equation $\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{P}{c^2}\right) \Rightarrow$ to substitute $\dot{\rho}$ and cancelling $\frac{2\dot{a}}{a}$ in each term

$$\Rightarrow \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 = -4\pi G\left(\rho + \frac{P}{c^2}\right) + \frac{kc^2}{a^2}$$

$\left(\frac{\dot{a}}{a}\right)^2$ is given by Friedmann equation

$$\Rightarrow \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$

in terms of energy density $\frac{\ddot{a}}{a} = \frac{-4\pi G}{3c^2}(\varepsilon + 3P)$

if a component has any pressure, this increases the "gravitational force" and further decreases the rate of expansion

Note that the acceleration equation does not depend on k

The Universe contains different components with different equations of ~~state~~; for sure non-relativistic matter and radiation, but there could be others...

$$\varepsilon = \sum_w \varepsilon_w \quad \leftarrow \text{different components (only } w \text{ matters)} \qquad \Rightarrow P = \sum_w P_w = \sum_w w\,\varepsilon$$

As long as there is no interaction between components the fluid equation holds separately for each component

$$\dot{\varepsilon}_w + 3\frac{\dot{a}}{a}(\varepsilon_w + P_w) = 0 \quad \Leftrightarrow \quad \dot{\varepsilon}_w + 3\frac{\dot{a}}{a}(1+w)\cdot\varepsilon_w = 0$$

$$\Leftrightarrow \frac{d\varepsilon_w}{\varepsilon_w} = -3(1+w)\frac{da}{a}$$

If we assume w = constant $\varepsilon_w(a) = \varepsilon_{w,0}\, a^{-3(1+w)}$ (as $a = 1$ at present)

- For radiation $w = 1/3 \Rightarrow a(t) = \left(\frac{t}{t_0}\right)^{1/2}$ and $\rho(t) = \frac{\rho_0 t_0^2}{t^2} \propto \underbrace{\frac{1}{a^3}}_{\text{volume}} \cdot \underbrace{\frac{1}{a}}_{\text{redshift}}$

Notice that the expansion is slower when radiation dominates due to deceleration caused by pressure ($a \propto t^{2/3}$ for matter)

- Λ-dominated $\dot{a}^2 = \frac{8\pi G\rho_\Lambda}{3c^2}a^2 \Leftrightarrow \dot{a} = H_0 a$ with $H_0 = \left(\frac{8\pi G\rho_\Lambda}{3c^2}\right)^{1/2}$

$$\Rightarrow a(t) = e^{H_0(t-t_0)}$$

Pressureless matter $\rho(t) = \rho_0/a^3$; for relativistic matter $\rho(t) = \rho_0/a^4$
and for vacuum energy $\rho(t) = \Lambda c^2/3 = \text{const}$.

For each component we can define $\Omega = \dfrac{\rho(t)}{\rho_{crit}}$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right) = H_0^2 \left(\frac{\Omega_{matter}}{a^3} + \frac{\Omega_{rad}}{a^4} + \frac{\Omega_\Lambda}{a^0} + \frac{\Omega_{curvature}}{a^2} \right)$$

By definition Ω's sum up to 1 (as $a=1$ at present, and $\dot{a} = H_0$)

$\Rightarrow$ The evolution of the expansion depends on the current composition of the Universe and how these components get diluted during expansion

$\downarrow$

Measuring Ω's central quest of cosmology!

Now we need to connect our results to GR before we continue to examining the implications for observations.

We derived $\left(\frac{\dot a}{a}\right) = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$ where k was interpreted as some "initial" amount of kinetic energy; the correct interpretation requires GR

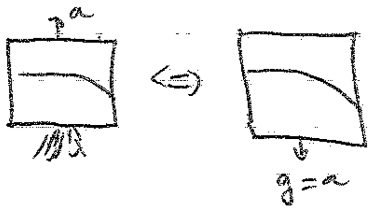
$F = -\frac{G M_g m_g}{r^2}$ ← gravitational mass

Newton's second law $F = m_i\, a$
↑ inertial mass

The assumption is $m_g = m_i$: the equivalence principle ; but why would the strength of gravity acting on an object be the same as the resistance to acceleration by <u>any</u> force?

Current experiments show $m_g = m_i$ to one part in 10^{12}

Einstein started with the equivalence principle



⇒ gravity curves space!

John Wheeler

Newton: mass tells gravity how to exert a force ($F = -GMm/r^2$)
force tells mass to accelerate ($F = ma$)

Einstein: mass-energy tells space how to curve
curved space-time tells mass-energy how to move
↙
gives a natural explanation of the equivalence principle

In GR the curvature of space-time determines how objects move: they move along geodesics (≡ shortest paths) in a curved space-time

Space-time is described by a metric g_{ij} which gives the distance ds between events $\vec{x} = (t, x, y, z)$ and $\vec{x} + d\vec{x} = (t+dt, x+dx, y+dy, z+dz)$

In special relativity (no gravity) we have Minkowski space $ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$

More general $ds^2 = \sum_{ij} g_{ij}\, dx^i dx^j$ or short $ds^2 = g_{ij}\, dx^i dx^j$
where repeated indices imply summation
(Einstein convention)

ds is coordinate independent ⇒ form of g_{ij} depends on coordinates chosen.

2d-plane: $ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$ $\quad (x = r\cos\theta,\ y = r\sin\theta)$

↑ Pythagoras ↑ Cartesian ↑ Polar

In GR the curvature of space time is important: particles move such that

$$\delta \int_{\text{path}} ds = 0 \qquad \text{the integral is stationary}$$

Because of g_{ij} these paths are no longer straight, but the geodesics are described by

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{kl} \frac{dx^k}{ds}\frac{dx^l}{ds} = 0$$

where the Christoffel symbol $\Gamma^i_{kl} = \frac{1}{2} g^{im} \left[\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right]$

In Newtonian & SR: conservation of mass, energy, momentum; with equivalence of mass & energy the conservation laws can be written as

$$\frac{\partial T_{ik}}{\partial x^k} = 0 \quad \text{where } T_{ik} \text{ is the energy-momentum tensor (but not in GR)}$$

T_{ik} describes the matter distribution; for a perfect fluid (no viscosity, heat flow or stresses) with pressure P and energy density ρ it is

$$T_{ik} = (P + \rho c^2)\, U_i U_k - P\, g_{ik}$$

where U_i is the fluid four-velocity $U_i = g_{ik} U^k = g_{ik} \frac{dx^k}{ds}$ → $x^k(s)$ is world line of a fluid element (trajectory)

In GR $\frac{\partial T_{ik}}{\partial x^k} = T_{ik,k} = 0$ cannot be used because the result is not a tensor, but one can define a so-called covariant derivative, which does yield a tensor $T_i{}^k{}_{;k} = 0$

Einstein wanted to find a relation between matter and metric and to equate T_{ik} to a tensor obtained from g_{ik} which contains only the first two derivatives of g_{ik} and has 0 covariant derivative

In the Newtonian limit $T_i{}^k{}_{;k}$ must reduce to Poisson's equation $\nabla^2 \varphi = 4\pi G \rho$

⇒ it should be linear in the second derivative of the metric

When ~~Einstein~~ Einstein was thinking about this, the properties of curved spaces were well-known and for instance it was known ~~that~~ the Riemann-Christoffel ~~tensor~~

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}$$

could be used to determine whether a space ~~that~~ is curved or flat

This tensor is the unique choice (with a very tedious proof)

The Riemann tensor is ~~xxxx~~ fourth order, but can be contracted to form the Ricci tensor $R_{ik} = R^{l}_{\ ilk}$ or even further to the curvature, or Ricci scalar $R = g^{ik} R_{ik}$

With these we can define the Einstein tensor $G_{ik} \equiv R_{ik} - \frac{1}{2} g_{ik} R$

Einstein showed that $G_{i\ ;k}^{\ k} = 0$ and as $T_{i\ ;k}^{\ k} = 0$ as well he proposed

$$G_{ik} \equiv R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T_{ik}$$

to get to Newtonian result in weak gravitational limit

These define Einstein's gravitational field equations

It is possible to write a modified set of field equations that are also consistent with the conservation laws:

$$G_{ik} - \Lambda g_{ik} = \frac{8\pi G}{c^4} T_{ik}$$

cosmological constant: as $g_{i\ ;k}^{\ k} = 0 \Rightarrow T_{i\ ;k}^{\ k}$ still 0

This allows for the construction of a static universe

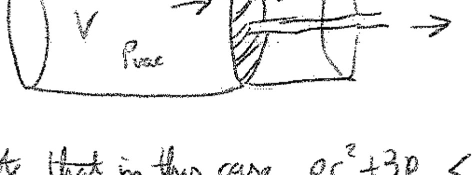
What is the physical meaning of Λ? For this question it is useful to move the Λ term to the r.h.s.

$$\Rightarrow T_{ik}^{vac} = \frac{\Lambda}{8\pi G} g_{ik}$$

Can the vacuum have energy? We do not understand the properties of the vacuum and this may well be critical for understanding the current accelerated expansion or the very beginning (inflation)

T_{ik}^{vac} has to be the same for all observers: there is only one isotropic tensor of rank 2, which is $\eta_{\alpha\beta} = \mathrm{diag}(1,-1,-1,-1)$

$$\Rightarrow P_{vac} = -\rho_{vac} c^2$$



The energy created by withdrawing the piston by a volume dV is $\rho_{vac} c^2 dV$
This must be supplied by work done by vacuum pressure $P_{vac} dV \Rightarrow P_{vac} = -\rho_{vac} c^2$

Note that in this case $\rho c^2 + 3p < 0$

Pressure as a source of gravity

Newtonian gravity is modified in the case of a relativistic fluid (i.e. where we cannot assume $P \ll \rho c^2$). One can show that in that case $R = \frac{8\pi G}{c^4} T$ where $T = c^2\rho - 3p$

This leads to a modified Poisson equation $\nabla^2 \varphi = 4\pi G\left(\rho + \frac{3P}{c^2}\right)$

For a gas of particles moving with the same speed u the effective gravitational mass density is $\rho(1 + u^2/c^2)$ $\Rightarrow$ a radiation dominated fluid generates an ~~attraction~~ that is twice as strong as one would expect from Newtonian arguments.

$\downarrow$

relevant for gravitational lensing

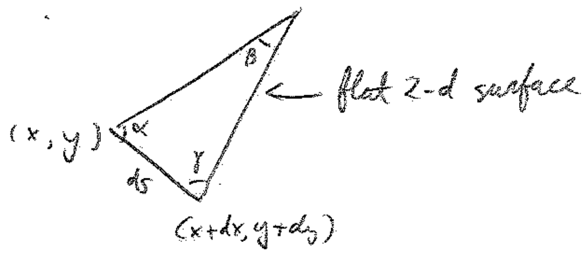
In GR the metric is key: which one describes the Universe and obeys the cosmological principle?

$\downarrow$

Curvature the same everywhere.

Start with 2-d spaces

$\alpha + \beta + \gamma = \pi$ (in radians)

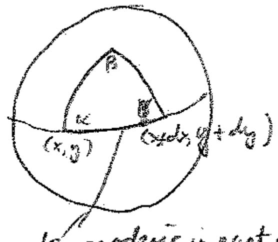


We know that $ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2$

Now a curved space: the surface of a sphere

We now have $\alpha + \beta + \gamma = \pi + \frac{A}{R^2}$ ← area of the triangle; ← radius of the sphere

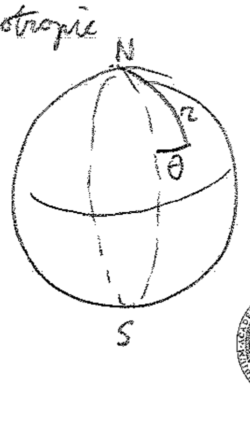
if $\alpha + \beta + \gamma > \pi$: positively curved



ds geodesic is part of a great circle (a circle with a centre that corresponds to the centre of the sphere)

In the case of a sphere the curvature is homogeneous and isotropic

Polar coordinates $ds^2 = dr^2 + R^2 \sin^2\left(\frac{r}{R}\right) d\theta^2$



Note that the surface has a finite area: $4\pi R^2$ and a maximum separation πR

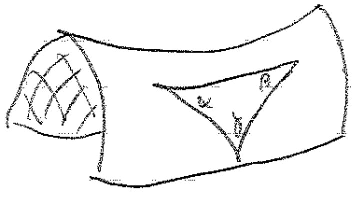
$\downarrow$

quite different from flat space.

Similarly we can define a negatively curved space

$$\alpha + \beta + \gamma = \pi - \frac{A}{R^2}$$

and $ds^2 = dr^2 + R^2 \sinh^2 (r/R)\, d\theta^2$



infinite area, no upper limit on distance

Flat space in 3d: $ds^2 = dx^2 + dy^2 + dz^2$ or $ds^2 = dr^2 + r^2[d\theta^2 + \sin^2\theta\, d\varphi^2]$

positively curved: $ds^2 = dr^2 + R^2 \sin^2 (r/R) [d\theta^2 + \sin^2\theta\, d\varphi^2]$

negatively curved: $ds^2 = dr^2 + R^2 \sinh (r/R) [d\theta^2 + \sin^2\theta\, d\varphi^2]$

All these metrics have constant curvature

$$ds^2 = dr^2 + S_k^2(r)\ [d\theta^2 + \sin^2\theta\, d\varphi^2]$$

where $S_k(r) = \begin{cases} R\sin(r/R) & k=+1 \\ R & k=0 \\ R\sinh(r/R) & k=-1 \end{cases}$ if $r \ll R$ $S_k \approx r$

If we now change the coordinate system such that $r \to x \equiv S_k(r)$

then $ds^2 = \frac{dx^2}{1 - kx^2/R^2} + x^2 [d\theta^2 + \sin^2\theta\, d\varphi^2]$

or as written in the book $ds^2 = \frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$

This is the most general spatial metric with constant curvature; the only change we can make is to allow space to shrink or expand

$\Rightarrow$ Robertson-Walker metric: $ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$

Any change to the time coordinate could be accounted for, so this is the most general form.

Plugging this metric into the Einstein equation yields 2 non-trivial equations

time-time: $\left(\frac{\dot a}{a}\right) + \frac{kc^2}{a^2} = \frac{8\pi G}{3}\rho$ Where k is interpreted as the curvature of the Universe

2nd $2\frac{\ddot a}{a} + \underbrace{\left(\frac{\dot a}{a}\right) + \frac{kc^2}{a^2}}_{= \frac{8\pi G}{3}\rho \text{ according to Friedmann eqn}} = -8\pi G \frac{P}{c^2}$ $\Rightarrow \frac{\ddot a}{a} = \frac{-4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$ acceleration equation