Consider now the growth of decoupled matter perturbations in a Universe where the expansion is driven by a relativistic component.

Assume $k=0 \Rightarrow \ddot{\delta} + 2\left(\frac{\dot{a}}{a}\right)\dot{\delta} - 4\pi G \rho_m \delta = 0$ for matter component

We already examined the evolution for $t \gg t_{eq}$ (matter-radiation equality)

But at earlier times a and ρ evolve differently!

If we define $y = \frac{\rho_m}{\rho_r} = \frac{a}{a_{eq}}$ increases with time; $y=1$ at $z = z_{eq} \sim 3500$

$\delta = \frac{\delta\rho_m}{\rho_m}$; rewrite the perturbation equation from function of t in one of y

$\Rightarrow \dot{\delta} = \delta' \frac{\dot{a}}{a_{eq}}$ and $\ddot{\delta} = [\delta'' \dot{a}^2 + \delta' \ddot{a}] / a_{eq}^2$ $\quad (\delta' = \frac{d}{dy}\delta)$

$\rho_m = \frac{y}{1+y}\rho$ and $\rho_r = \frac{1}{1+y}\rho$ and $p = \frac{1}{3}\rho_r c^2$

Friedmann eqn $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_r)$

Acceleration eqn $\ddot{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)a = -\frac{8\pi G}{3}\left(\rho + \frac{1}{1+y}\rho\right)a = -\frac{1}{2}\,\frac{2+y}{1+y}\left(\frac{\dot{a}}{a}\right)^2$

Then $\ddot{\delta} + 2\left(\frac{\dot{a}}{a}\right)\dot{\delta} - 4\pi G \rho_m \delta = 0$ can be written as

$$\delta'' + \frac{2+3y}{2y(1+y)}\delta' - \frac{3}{2y(1+y)}\delta = 0$$

This has two solutions, one growing, one decaying. The growing mode

$$\delta_+ \propto 1 + \frac{3}{2}y \sim 1 + \frac{5000}{1+z}$$

Before z_{eq} we have that $y<1$ and the growing mode is frozen

This Meszaros effect applies to cold dark matter density fluctuations (not coupled to the radiation via pressure) on large scales

The total growth from 0 to t_{eq} is $\frac{\delta_+(y=1)}{\delta_+(y=0)} = \frac{5}{2}$ and afterwards by another factor $1+z_{eq}$

The physical reason for this slow growth is that before t_{eq} the Jeans time is longer than the expansion time. The energy in radiation causes the Universe to expand so fast that the matter has no time to respond.

Before decoupling the baryon dynamics is coupled to that of the radiation
$\Rightarrow \delta_{bar}$ oscillates like the radiation, but after t_{eq} $\delta_{DM} \propto a$

As a result after decoupling $\delta_{DM} \gg \delta_{bar}$

After decoupling $\ddot{\delta}_{bar} + \frac{4}{3t}\dot{\delta}_{bar} = 4\pi G(\bar{\rho}_{bar}\delta_{bar} + \bar{\rho}_{DM}\delta_{DM})$

$$\ddot{\delta}_{DM} + \frac{4}{3t}\dot{\delta}_{DM} = 4\pi G(\bar{\rho}_{bar}\delta_{bar} + \bar{\rho}_{DM}\delta_{DM})$$

If we use that $\delta_m = \frac{\bar{\rho}_{bar}\delta_{bar} + \bar{\rho}_{DM}\delta_{DM}}{\bar{\rho}_{bar} + \bar{\rho}_{DM}} \simeq \delta_{DM}$ and $\Delta \equiv \delta_{DM} - \delta_{bar}$

$$\Rightarrow \ddot{\Delta} + \frac{4}{3t}\dot{\Delta} = 0 \qquad \Rightarrow \Delta = \text{const or } \Delta \propto t^{-1/3}$$

$$\delta_m \propto t^{2/3} \propto a \qquad \Rightarrow \frac{\delta_{DM}}{\delta_{bar}} = \frac{\bar{\rho}_m\delta_m + \bar{\rho}_{bar}\Delta}{\bar{\rho}_m\delta_m - \bar{\rho}_{DM}\Delta} \rightarrow 1$$

The initial non-zero value of δ_{bar} at decoupling leaves a small effect on the δ_m at late times $\Rightarrow$ these are the baryon acoustic oscillations

The power spectrum is defined as

$$\langle \delta(\vec{k}) \delta^*(\vec{k}') \rangle = (2\pi)^3 \, \delta_D(\vec{k}-\vec{k}') \, P(k)$$

isotropy implies that $P(k)$ can only depend on $|k|$

- If δ is a Gaussian field (as predicted by many theories) then $P(k)$ completely specifies the statistical properties.

$P(k)$ quantifies the amount of clustering for each k-mode

The two-point correlation function gives the excess probability of finding pairs of objects at a separation $\vec{r}$. It is defined as $\xi(r) = \langle \delta(x_1) \delta(x_2) \rangle$ and thus is related to the power spectrum through its Fourier transform

$$\xi(r) = \langle \delta(x_1) \delta(x_2) \rangle = \int \frac{d^3k}{(2\pi)^3} e^{ikr} P(k)$$

The power spectrum has units of length^3 and its is convenient to define a dimensionless version

$$\Delta^2(k) = \frac{4\pi k^3 P(k)}{(2\pi)^3}$$

The primordial power spectrum is $P(k) = Ak^n$; if $n=1$ the model is the Harrison-Zeldovich spectrum, where fluctuations are scale-invariant in the gravitational potential Φ

What does a scale invariant spectrum for fluctuations in the potential imply for the matter power spectrum?

$$\Delta^2_\Phi \propto k^3 P_\Phi(k) \propto \text{constant}$$

but $\nabla^2 \Phi = 4\pi G \bar{\rho} \, \delta a^2 \xrightarrow[\text{transform}]{\text{Fourier}} k^2 \Phi(k) \propto \delta(k)$

$$\Rightarrow k^4 P_\Phi(k) \propto P_\delta(k) \quad \text{using the definition of } P(k)$$

$$\Leftrightarrow P_\delta(k) \propto k \, \Delta^2_\Phi(k) \propto k \quad \Rightarrow n=1$$

The observed power spectrum is quite different: $P(k) = Ak^n \, T^2(k)$

Transfer function

$T(k)$ captures the growth of fluctuations, in and outside the horizon

For a matter dominated universe at $(1+z) \gg \frac{1}{\Omega_{m,0}}$ the horizon size is

$$d_H(z) = \frac{2c}{\sqrt{\Omega_{m,0} H_0^2}} (1+z)^{-3/2}$$

A feature of some length l grows as $l \propto a$, but as the horizon grows as $d_H \propto a^{3/2}$ larger features come into causal contact with each other at later times

$\Rightarrow$ we therefore ~~see~~ expect the transfer function to depend on $\Omega_m h^2$ and k

We already found that for $y \equiv \frac{\rho_m}{\rho_r} = \frac{a}{a_{eq}}$ that $\delta_m \propto y + \frac{2}{3}$ (we actually found $\delta \propto 1 + \frac{3}{2} y$ which is equivalent)

$\Rightarrow \delta_m \propto$ constant for $a \ll a_{eq}$ and $\delta_m \propto a$ for $a \gg a_{eq}$

In the radiation-dominated era perturbation modes with $l < \delta_H(z_{eq})$ enter the horizon but δ is constant

In the matter dominated era scales $l > \delta_H(z_{eq})$ enter and $\delta \propto a$ and thus δ grows

$\Rightarrow$ the power spectrum must have a break at the length scale of the horizon at matter-radiation equality: $d_H(z_{eq}) \approx \frac{16}{\Omega_{m,0} h^2}$ Mpc or $k_{eq} \sim 0.06\, \Omega_m h^2$ Mpc^{-1}

For $k < k_{eq}$ (large scales) fluctuations entered the horizon during the matter-dominated era and grow $\propto a$ preserving the initial power law $P(k) \propto k$

For $k > k_{eq}$ (small scales) fluctuations entered during the radiation dominated era and could not grow $\Rightarrow P(k) \propto k^{-3}$

After matter-radiation equality the power spectrum grows as $P(k) \propto \delta^2 \propto a^2$

The ~~[illegible]~~ dependence of k_{eq} can be used to constrain Ω_m

$\Delta^2(k) \propto k^4$ for small k and $\Delta^2(k) \propto k^0$ $\Rightarrow$ hierarchical structure formation where smaller overdensities go non-linear and collapse earlier.

The baryon acoustic oscillations are superimposed on these dark matter fluctuations

Before recombination the baryons and radiation where tightly coupled
The entropy per unit mass of a fluid composed of matter and radiation in a volume V has a very high value because of the large value of σ_r (the entropy per baryon)

$\Rightarrow$ entropy is carried almost entirely by radiation

$$S = \frac{4}{3}\sigma T^3 V \propto \sigma_{rad} \propto \frac{T^3}{\rho_m} \propto \frac{\rho_r^{3/4}}{\rho_m} \qquad \sigma_{rad} = \frac{4 m_p \sigma_r T^3}{3 k_B \rho_m}$$

An adiabatic perturbation leaves S invariant and consists of fluctuations in both ρ_m and ρ_r such that

$$\frac{\delta S}{S} = 0 = \frac{\delta\sigma_{rad}}{\sigma_{rad}} = \frac{3}{4}\frac{\delta\rho_r}{\rho_r} - \frac{\delta\rho_m}{\rho_m} = 3\frac{\delta T}{T} - \frac{\delta\rho_m}{\rho_m}$$

$$\Rightarrow \delta_m \equiv \frac{\delta\rho_m}{\rho_m} = 3\frac{\delta T}{T} = \frac{3}{4}\frac{\delta\rho_r}{\rho_r} \equiv \frac{3}{4}\delta_r$$

We discussed earlier that the value of σ_{rad} might be related to the ~~microscopic~~ microscopic physics of a GUT or electroweak phase transition; if that is correct then we expect fluctuation to have the same value for σ_{rad} $\Rightarrow$ we expect adiabatic perturbations

The power spectrum of temperature fluctuations can be related to the density fluctuations

The usual way is to expand the distribution of T on the sky as a sum of spherical harmonics

$$\frac{\delta T(\theta,\varphi)}{T} = \sum_{l=0}^{\infty}\sum_{m=-l}^{m=l} a_{lm} Y_{lm}(\theta,\varphi)$$

Note that this expansion is analogous to the plane-wave Fourier expansion of the density perturbations δ

The a_{lm} are complex numbers and satisfy $\langle a^*_{l'm'} a_{lm}\rangle = C_l \delta^K_{ll'} \delta^K_{mm'}$

$C_l \equiv \langle |a_{lm}|^2\rangle$ is the angular power spectrum

What we observe is the autocovariance function of the temperature fluctuations

$$C(\vartheta) = \left\langle \frac{\delta T}{T}(\hat{n}_1) \frac{\delta T}{T}(\hat{n}_2) \right\rangle \qquad \text{where } \cos\vartheta = \hat{n}_1 \cdot \hat{n}_2$$

unit vectors towards directions on the sky.

$C(\vartheta)$ is related to C_l through

$$C(\vartheta) = \frac{1}{4\pi}\sum_{l=2}^{\infty}(2l+1) C_l P_l(\cos\vartheta)$$

$P_l(x)$ Legendre polynomial

Even before recombination matter and radiation are not perfectly coupled: radiation leaks out of the perturbation, which leads to dissipation of the perturbations

This process occurs because photons bounce around (following a random walk) during recombination; for small scale fluctuations the hot and cold photons can mix $\Rightarrow$ on the scales corresponding to the distances photons can travel the fluctuations are damped.

The dissipation scale $\lambda_D \simeq 2c\sqrt{t_{\gamma e}\, t}$ at time t

mean time between Thomson scattering : $t_{\gamma e} \propto \frac{1}{n_e} \propto \frac{1}{(1+z)^3}$

Before t_{eq} $\lambda_D \propto (1+z)^{-5/2}$ and $\lambda_D \propto (1+z)^{-9/2}$ after t_{eq}

The corresponding mass scale is $\rho(z)\lambda_D^3$, which gives $M_D \sim 10^{12}(\Omega_m h^2)^{-5/4} M_\odot$ at recombination : ~ cluster of galaxy scale

Without accounting for this Silk damping the amplitude of an acoustic wave on a mass scale $< M_D$ would remain constant during radiation domination and decay $\propto t^{-1/6}$ after t_{eq}; such structures are obliterated by photon diffusion