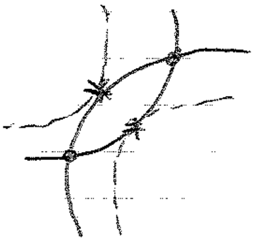
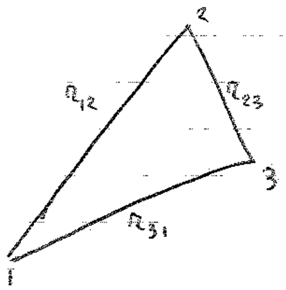
Cosmological Principle + Isotropy → Homogeneity



"*" locations must be the same as "o" → homogeneity

Homologous expansion



let the triangle expand such that $r_{12}(t) = a(t)\, r_{12}(t_0)$
$r_{23}(t) = a(t)\, r_{23}(t_0)$
$r_{31}(t) = a(t)\, r_{31}(t_0)$

an observer at location 1 will see galaxy 2 recede with a velocity $V_{12}(t) = \dot{a}\, r_{12}(t_0) = \left(\frac{\dot{a}}{a}\right) r_{12}(t)$

Similarly $V_{31}(t) = \left(\frac{\dot{a}}{a}\right) r_{31}(t)$

$$\Rightarrow V = \left(\frac{\dot{a}}{a}\right) r \equiv H r \quad \text{with} \quad H \equiv \left(\frac{\dot{a}}{a}\right)$$

Accounting for peculiar motions:

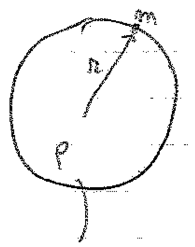
$$V_{obs} = V_{pec} + H_0 r \qquad \tilde{H}_0 = \frac{V_{obs} - V_{pec}}{r} \qquad |\Delta H_0| = \frac{|V_{pec}|}{r}$$

$$\Rightarrow \frac{|\Delta H_0|}{V} = \frac{|V_{pec}|}{V r} \iff \left|\frac{\Delta H_0}{H_0}\right| = \left|\frac{V_{pec}}{V}\right|$$

to get a 1% error in H_0, the peculiar velocity needs to be <1% of the Hubble velocity

$$H_0 = 70 \text{ km/s/Mpc} \Rightarrow 0.01 = \frac{600 \text{ km/s}}{70 \text{ km/s/Mpc}\ r_{min}} \Rightarrow r_{min} \gtrsim 857 \text{ Mpc} \ (\text{or } z > 0.2)$$

Derivation of the Friedmann equation



total mass $M = \frac{4\pi}{3}\rho r^3 \rightarrow F = \frac{GMm}{r^2} = \frac{4\pi G}{3}\rho m r$

The particle has potential energy $V = \frac{-GMm}{r} = -\frac{4\pi G\rho}{3} m r^2$

The velocity of the particle is $\dot{r}$ $\Rightarrow$ kinetic energy is $T = \frac{1}{2} m \dot{r}^2$

Energy conservation $U = T + V = \text{constant} = \underbrace{\frac{1}{2} m \dot{r}^2 - \frac{4\pi}{3} G\rho m r^2}$ (∗)

this describes the evolution of the separation r between two particles

Homogeneity implies this holds for all particles → we can change to comoving coordinates

$$\vec{r} = a(t)\,\vec{x}$$

physical distance   scale factor   comoving distance

only function of time because of homogeneity

Plug into (∗) noting that $\dot{x} = 0$ by definition as objects are fixed in comoving coordinates

$$U = \frac{1}{2} m \dot{a}^2 x^2 - \frac{4\pi}{3} G\rho a^2 x m$$

multiply by $\frac{2}{m a^2 x^2}$

$$\Rightarrow \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

where $k = \frac{-2U}{mc^2 x^2}$ ← conserved, is time independent

H^2

has units $\frac{1}{\text{length}^2}$ fixed coordinates

$k = 0$ $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\frac{\rho_0}{a^3} \Leftrightarrow \dot{a} = \sqrt{\frac{8\pi G\rho_0}{3a}}$ $\dot{a} \to 0$ as $a \to \infty$

$k < 0$ $\left(\frac{\dot{a}}{a}\right)^2 \to -\frac{kc^2}{a^2} \Rightarrow \dot{a}^2 = |k|c^2$ $\dot{a} > 0$ as $a \to \infty$

$k > 0$ $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$ $\dot{a}^2$ is initially positive but once $a_{max} = \sqrt{\frac{3kc^2}{8\pi G\rho}}$ then $\dot{a} = 0$ and expansion stops

↓

what will happen?

as $\dot{a} = \sqrt{\frac{8\pi G\rho_0}{3a} - kc^2} \Rightarrow \ddot{a} = \frac{1}{2\dot{a}} \cdot -\frac{8\pi G}{3}\frac{\rho_0}{a^2}\dot{a}$

$$\Leftrightarrow \ddot{a} = \frac{-4\pi G}{3}\rho_0 \frac{1}{a^2} < 0$$

$\dot{a}$ has to become negative and the sphere will contract

Note that $\ddot{a} < 0$ independent of $k \rightarrow$ a was smaller in the past and $\rightarrow 0$ as $t \rightarrow 0$

$k=0$: $\dot{a}^2 = \frac{8\pi G\rho_0}{3}\frac{1}{a(t)}$

We can solve this differential equation: (can be solved formally as this is a separable equation; but we will try a powerlaw which is a common solution in cosmology)

$a \propto t^q \Rightarrow \dot{a}^2 \propto t^{2q-2}$ but r.h.s $\frac{1}{a} \propto t^{-q}$

$$\Rightarrow 2q - 2 = -q \Leftrightarrow q = 2/3$$

$\Rightarrow a(t) = \left(\frac{t}{t_0}\right)^{2/3}$ $\quad \rho(t) = \frac{\rho_0}{a^3} = \frac{\rho_0 t_0^2}{t^2}$

current age

The Universe expands forever and $H = \frac{\dot{a}}{a} = \frac{2}{3t} \rightarrow 0$ at $t \rightarrow \infty$
Note that $t_0 = \frac{2}{3}H_0^{-1}$

At present day $a=1 \Rightarrow H_0^2 = \frac{8\pi G\rho_0}{3}$ or $\rho_0 = \frac{3}{8\pi G}H_0^2 \equiv \rho_{crit}$

The critical density separates eternal expansion from recollapse; for $H_0 = 70$ km/s this corresponds to ~5 protons per m^3 or $2.78 \times 10^{11}\ h^2\ M_\odot\ Mpc^{-3}$

H_0 in units of 100 km/s/Mpc

With the definition of ρ_{crit} we can write

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left[\frac{\rho(t)}{\rho_{crit}} - \underbrace{\frac{k}{H_0^2 a^2}}_{\Omega_k/a^2}\right] = H^2(a)$$

← curvature contribution

$\Omega(a)$ if we consider a particular ingredient

Now we need more than Friedmann eqn because we need to describe how the density evolves. We will also look into the GR interpretation of the Friedmann equation