3. Galactic Dynamics

Aim: understand equilibrium of galaxies

1. What are the dominant forces?
2. Can we define some kind of equilibrium?
3. What are the relevant timescales?
4. Do galaxies evolve along a sequence of equilibria? 
   [like stars]?

Equilibrium of stars as comparison:
- gas pressure versus gravity
  isotropic pressure at a given point
- hydrostatic equilibrium, thermal equilibrium
- timescales corresponding to the above, plus nuclear burning
- structure completely determined by mass plus age (and metallicity)
  very little influence from “initial conditions"

galaxies:

   Much harder: what are galaxies are made of?

Solution: assume ensemble of point masses, moving under influence of their own gravity

Point masses can be: stars, mini BHs, planets, very light elementary particles ...

Questions addressed in this and the next handout:
- What are equations of motion?
- Do stars collide?
- Use virial theorem to estimate masses, binding energy, etc.
- How many equilibrium solutions for each mass?
- What are the relevant time scales?
  dynamical timescale, particle interaction timescale

Basic reference:

chapters 1 to 1.1; 2 selected parts; 3 selected parts, 4 selected parts
3.1 Equations of motion

assume a collection of masses $m_i$ at location $x_i$, and assume gravitational interaction. Hence the force $\vec{F}_i$ on particle $i$ is given by

$$\vec{F}_i = m_i \frac{d^2}{dt^2} \vec{x}_i = \sum_{j \neq i} \frac{\vec{x}_j - \vec{x}_i}{|\vec{x}_j - \vec{x}_i|^3} Gm_i m_j$$

- Only analytic solution for 2 point masses
- "Easy" to solve numerically (brute force) - but slow for $10^{11}$ particles [see http://astrogrape.org/

- Analytic approximations necessary for a better understanding of solution

In the following we investigate properties of gravitational systems without explicitly solving the equations of motion.

3.2 Do stars collide ?

(BT 1 to 1.1)

Is it safe to ignore non-gravitational interactions ?

Calculate number of collisions between $t_1$ and $t_1 + dt$

of star coming in from the left, in galaxy with homogeneous number density $n$.

- incoming star has velocity $v$
- suppose all stars have radius $r_*$.
- all stars in volume $V_1$ will cause collision $\rightarrow$
  $$V_1 = \pi (2r_*)^2 \times v \times dt$$
- number of stars in $V_1$ : $N_1 = nV_1$
- number of collisions per unit time = $4\pi r_*^2 n v$

Typical values:

$r_\odot = 7 \times 10^{10}$ cm

$n = 10^{10} / [3 kpc]^3 = 1.3 \times 10^{-56} cm^{-3}$

$v = 200$ km/sec $= 2 \times 10^7$ cm/sec

which gives a collision rate of $1.6 \times 10^{-26}$ sec$^{-1} = 5 \times 10^{-19}$ yr$^{-1} \rightarrow$ very rare indeed

Hence we can ignore these collisions without too much trouble.
3.3 Virial Theorem: (not in BT)

relation for global properties: Kinetic Energy and Potential energy.

Again consider our system of point masses $m_i$ with positions $\vec{x}_i$.

Construct $\sum_i \vec{p}_i \vec{x}_i$ and differentiate w.r.t. time:

$$\frac{d}{dt} \sum_i \vec{p}_i \vec{x}_i = \frac{d}{dt} \sum_i m_i \frac{d\vec{x}_i}{dt} \vec{x}_i = \frac{d}{dt} \sum_i \frac{1}{2} \frac{d}{dt} (m_i x_i^2)$$

$$= \frac{1}{2} \frac{d^2 I}{dt^2}$$

where $I = \sum_i m_i x_i^2$, which is the moment of inertia.

However, we can also write:

$$\frac{d}{dt} \sum_i \vec{p}_i \vec{x}_i = \sum_i \frac{d\vec{p}_i}{dt} \vec{x}_i + \sum_i \vec{p}_i \frac{d\vec{x}_i}{dt}$$

Then

$$\sum_i \vec{p}_i \frac{d\vec{x}_i}{dt} = \sum_i m_i \vec{v}_i^2 = 2K$$

with $K$ the kinetic energy.

Since $\frac{d\vec{p}_i}{dt} = \vec{F}_i$ we have

$$\frac{1}{2} \frac{d^2 I}{dt^2} = \sum_i \vec{F}_i \vec{x}_i + 2K.$$
Hence

\[ \sum_i \sum_{j \neq i} \vec{F}_{ij} \vec{x}_i = \sum_i \sum_{j > i} (\vec{F}_{ij} \vec{x}_i + \vec{F}_{ji} \vec{x}_j) \]

This is simply a change in how the summation is done, it does not use any special property of the force field. Because \( \vec{F}_{ij} = -\vec{F}_{ji} \) (forces are equal and opposite for pairwise forces) the last term can be rewritten

\[ \sum_i \vec{F}_i \vec{x}_i = \sum_i \sum_{j > i} \vec{F}_{ij} (\vec{x}_i - \vec{x}_j) \]

For gravitational force \( \vec{F}_{ij} = -\frac{Gm_im_j}{|\vec{x}_i - \vec{x}_j|^2} \)

\[ \sum_i \vec{F}_i \vec{x}_i = \sum_i \sum_{j > i} \frac{Gm_im_j}{|\vec{x}_i - \vec{x}_j|^2} (\vec{x}_i - \vec{x}_j)(\vec{x}_i - \vec{x}_j) \]

which equals

\[ = -\sum_i \sum_{j > i} \frac{Gm_im_j}{|\vec{x}_i - \vec{x}_j|} = -\frac{1}{2} \sum_i \sum_{j \neq i} \frac{Gm_im_j}{|\vec{x}_i - \vec{x}_j|} = W \]

with \( W \) the total potential energy. Therefore for a galaxy in quasi-static equilibrium:

\[ K = -\frac{1}{2} W, \]

which is the virial theorem for quasi-static systems. The more general expression for non-static systems is:

\[ \frac{1}{2} \frac{d^2 I}{dt^2} = W + 2K. \]

**Homework assignments:**

1) derive the relation for the potential energy

\[ W = -\frac{1}{2} \sum_i \sum_{j \neq i} \frac{Gm_im_j}{|\vec{x}_i - \vec{x}_j|} \]

Proceed by assuming that each particle has mass \( fm_i \), \( 0 \leq f \leq 1 \). Slowly add an extra mass \( m_i df \) to each particle from infinity and calculate how much energy is released. Integrate \( f \) from 0 to 1. Explain each step in this calculation.

2) Derive how empty galaxies are. First calculate the distance between neighbor stars, by assuming they are located on a 3-dimensional grid. Next get the radius of the sun from literature. Take the ratio of the two. Compare this to the same ratio of galaxies: calculate the average distance between neighboring galaxies in the same way, and get the radius of galaxies.
3.4 Applications (BT pages 213, 214)

Consider a system with total mass $M$

Kinetic energy $K = \frac{1}{2} M \langle v^2 \rangle$ with

$\langle v^2 \rangle = \text{mean square speed of stars (assumption: speed of star not correlated with mass of star)}$

Define gravitational radius $r_g$

$$ W = - \frac{G M^2}{r_g} $$

Spitzer found for many systems that $r_g = 2.5 r_h$, where $r_h$ is the radius which contains half the mass

Virial theorem implies:

$$ M \langle v^2 \rangle = \frac{G M^2}{r_g} $$

$$ M = \langle v^2 \rangle r_g G^{-1} $$

Hence, we can estimate the mass of galaxies if we know the typical velocities in the galaxy, and its size!

Homework Assignments:

3) calculate the mass of our galaxy for two assumed radii:
   a.) 10 kpc (roughly the distance to the sun)
   b.) 350 kpc (halfway out to Andromeda)

Take as a typical velocity the solar value of 200 km/s.

4) The kinetic energy is given by

$$ 1/2 M \langle v^2 \rangle $$

if the velocities of the stars are NOT correlated with their mass. Give the correct expression for the total kinetic energy if the velocities of stars ARE correlated with their mass. This is a simple, single expression without additions of various terms.
3.5 Binding Energy and Formation of Galaxies

The total energy $E$ of a galaxy is

$$E = K + W = -K = 1/2W$$

- Bound galaxies have negative energy - cannot fall apart and dissolve into a very large homogeneous distribution
- A galaxy cannot just form from an unbound, extended smooth distribution $\rightarrow E_{total} = E_{start} \approx 0$, $E_{gal} = -K$, so energy must be lost or the structure keeps oscillating:

Possible energy losses through
- Ejection of stars
- Radiation (before stars would form)

3.6 Scaling Relations (not in BT)

Consider a steady state galaxy with particles $m_i$ at location $x_i(t)$.

Can the galaxy be rescaled to other steady state galaxies?
- scaled particle mass $\hat{m}_i = a_m m_i$
- scaled particle location $\hat{x}_i = a_x x_i(a_t t)$
- $a_m, a_x, a_t$ are scaling parameters

In the 'rescaled' galaxy we have

$$\hat{F}_i = \hat{m}_i \left( \frac{d^2 \hat{x}_i}{dt^2} \right) = a_m m_i a_x^2 \vec{x}(a_t t) = a_m a_x a_t^2 \vec{F}_{i,orig}$$

The gravitational force is equal to

$$\hat{F}_G = \sum_{j \neq i} a_x x_j - a_x x_i \left| a_x x_j - a_x x_i \right| G a_m m_i a_m m_j = \frac{a_{m}^2}{a_x^2} \vec{F}_{G,orig}$$

Equilibrium is satisfied if the two terms above are equal

$$\hat{F}_i = \hat{F}_G$$

Since we have equilibrium when all scaling parameters are equal to 1, the scaled system is in equilibrium if the scaling parameters satisfy:

$$a_m a_x a_t^2 = \frac{a_m^2}{a_x^2},$$

or

$$a_m = a_x^3 a_t^2.$$
Now how do velocities scale?
\[
\hat{v} = \frac{d\hat{x}}{dt} = \frac{d}{dt}a_x x(a_t t) = a_x a_t v_{\text{orig}}
\]

Hence, the scaling of the velocities satisfies: 
\[a_v = a_x a_t\]. We can write the new scaling relation as 
\[a_m = a_x a_v^2\]

Hence ANY galaxy can be scaled up like this!

Notice that it is trivial to derive this from the virial theorem - the expression for the mass has exactly the same form.

As a consequence, if we have a model for a galaxy with a certain mass and size, we can make many more models, with arbitrary mass, and arbitrary size.

**Homework Assignments:**
5) Show explicitly that the virial theorem produces the same result for the scaling relations.

6) How does this compare to stars? What happens to a star when we scale it up in size by a factor of 2, and reduce the temperature by a factor of 2? Which equilibrium is maintained, which equilibrium is lost?