Keplerian systems in velocity space

June 12, 2010

We all know that in real space a bound Keplerian system traces an ellipse, but what would it trace in velocity space?

The change in the velocity over a given time Δt is:

$$\Delta v = a\Delta t = \frac{GM}{r^2}\Delta t. \tag{1}$$

Since we conserve angular momentum, the change in the orbital angle for a given time Δt is:

$$\left(\frac{\Delta\theta}{\Delta t}\right) r^2 = h = const. \tag{2}$$

Combining eq.(2) with eq.(1):

$$\Delta v = \frac{GM}{h}\Delta\theta. \tag{3}$$

For a given change in the orbital angle, the change in the velocity is always the same and it is in the radial direction. This means that we add to the velocity a vector which is constant in magnitude but rotates. This addition of vectors traces out a circle whose radius is GM/h.