1 Multipole Radiation

An easy way to derive the equation for energy emission from a multipole (up to a numerical factor) is the following way. Radiation is dominated by two different regimes, near field and far field, and the transition between the two happens at one wavelength of the radiation. The field that the multipole generates is:

$$F = \frac{P^{(n)}}{r^{n+1}} \tag{1}$$

where $P^{(n)}$ is the multipole of the n_{th} order. The energy density that the field creates is:

$$U = \frac{F^2}{8\pi} \tag{2}$$

The energy that flows through a shell with a radius of one wavelength per unit time is:

$$\frac{dE}{dt} = 4\pi U r^2 c \tag{3}$$

Evaluating the expression above with the radius taken as one wavelength gives:

$$\frac{dE}{dt} = 4\pi\frac{1}{8\pi}(\frac{P^{(n)}}{r^{n+1}})^2 r^2 c = \frac{1}{2}(\frac{P^{(n)}}{r^n})^2 c = \frac{1}{2}(\frac{P^{(n)}}{(\frac{c}{f})^n})^2 c = \frac{1}{2}(f^n P^{(n)})^2 c^{1-2n} \tag{4}$$

For example if we want EM dipole radiation:

$$\frac{dE}{dt} = \frac{1}{2} f^4 d^2 c^{-3} \tag{5}$$

Notice that if we want to calculate gravitational waves we need to add a factor of $\frac{1}{G}$ since the unit of the charge are measured differently for gravity and the field is produced by $GP^{(n)}$ instead of just $P^{(n)}$. Another example is the quadrupole radiation emitted from two objects in a circular orbit around their common CM.

$$\frac{dE}{dt} = \frac{1}{2G}(f^3 G Q)^2 c^{-5} \tag{6}$$

In the CM frame we have the relation:

$$m_1 r_1 = m_2 r_2 \tag{7}$$

$$r = r_1 + r_2 = r_1(1 + \frac{m_1}{m_2}) \tag{8}$$

$$f^2 = \frac{G(m_1 + m_2)}{r^3} \tag{9}$$

$$Q = m_1 r_1^2 + m_2 r_2^2 = m_1 r_1^2 (1 + \frac{m1}{m_2}) = \frac{m_1 m_2 r^2}{m_1 + m_2} \tag{10}$$

Plugging in this relations into eq.6:

$$\frac{dE}{dt} = \frac{1}{2Gc^5}\frac{G^3(m_1+m_2)^3}{r^9}\frac{G^2(m_1 m_2)^2 r^4}{(m_1+m_2)^2} = \frac{G^4}{2c^5}\frac{(m_1+m_2)(m_1 m_2)^2}{r^5} \tag{11}$$