Gravitational Lensing

March 8, 2010

1 Angular Deflection

The weak lensing regime can be defined as observing only one image of the source (thus strong lensing is when there are 2 or more images). If we wish to calculate the deflection of light due to gravity, it can be estimated in the following way (we are assuming a thin lens, meaning that the width of the lens is much smaller then the distance to the source or the observer). Since classically the equation of motion of a particle in a gravitational field are independent of its mass, we can write down the equation of motion for a photon just like it was a massive particle traveling with the speed of light. If the photon has an impact parameter b, the time of the interaction will be approximately:

$$t = \frac{b}{c} \tag{1}$$

During this time, the photon will 'feel' an acceleration of:

$$a = \frac{GM}{b^2} \tag{2}$$

The photon will then receive an impulse which will give it a radial velocity:

$$\Delta V_r = a \cdot t = \frac{b}{c}\frac{GM}{b^2} = \frac{GM}{bc} \tag{3}$$

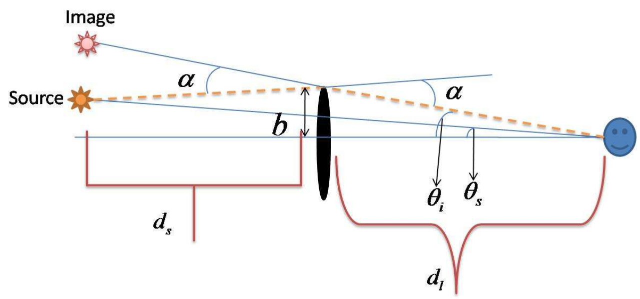
If we would have done the complete integral over the path we would have gotten a factor of 2 and general relativity adds another factor of 2 so the real result is:

$$\Delta V_r = 4\frac{GM}{bc} \tag{4}$$

The deflection angle of the photon (for small angles) is simply:

$$\alpha = \frac{\Delta V_r}{c} = 4\frac{GM}{bc^2} = 2\frac{R_s}{b} \tag{5}$$

The answer is simply the twice the Schwarzschild radius divided by the impact parameter.



2 Critical Surface Density

Now we can try and relate this answer to the critical mass surface density needed for strong lensing. If we assume that we have a thin lens with a constant mass density Σ, then the mass of the lens will be:

$$M = \pi \Sigma b^2 \tag{6}$$

If we also assume that the distance between the observer to the plane of the lens is equal to the distance between the plane of the source and the lens, and that the source lies in the same line of sight as the lens relative to the observer (meaning that $\theta_s = 0$) then the deflection can be rewritten to be:

$$\alpha = \frac{4\pi G \Sigma b}{c^2} = \frac{4\pi G \Sigma d_l \theta_i}{c^2} = \frac{4\pi G \Sigma d_l \alpha}{2c^2} \tag{7}$$

where the last equality arrive from the fact that we have similar triangles. The critical density is when α is approximately unity:

$$\Sigma_{crit} = \frac{c^2}{2\pi G d_l} \tag{8}$$

If the distances were not equal, then the distance between the source and the line going from the observer through middle of the lens is:

$$\theta_s(d_s + d_l) \tag{9}$$

this distance can also be written as:

$$\theta_i(d_s + d_l) - \alpha d_s \tag{10}$$

Combining the two give us:

$$\theta_i - \theta_s \approx \alpha \frac{d_s}{d_s + d_l} \tag{11}$$

Eq.8 can now be rewritten as:

$$\Sigma_{crit} = \frac{c^2 \alpha}{4\pi G b} = \frac{c^2 \alpha}{4\pi G(\theta_i - \theta_s) d_l} = \frac{c^2 (d_s + d_l)}{4\pi G d_s d_l} \tag{12}$$

The general case for the deflection can now be found from eq.5 and eq.11:

$$\alpha = 4\frac{GM}{\theta_i d_l c^2} \Rightarrow \theta_i = \theta_s + 4\frac{GM}{\theta_i d_l c^2}\left(\frac{d_s}{d_s+d_l}\right) \tag{13}$$

This equation is known as the "Lens Equation". This equation is non-linear in θ_i so we can get multipole images.

3 Einstein Ring

If $\theta_s = 0$ then the light from the source can be deformed to make a ring around the lens. The angular opening of the ring will be:

$$\theta_i = \theta_E = 4\frac{GM}{\theta_E d_l c^2}\left(\frac{d_s}{d_s+d_l}\right) \Rightarrow \theta_E = \pm\sqrt{4\frac{GM}{d_l c^2}\left(\frac{d_s}{d_s+d_l}\right)} \tag{14}$$

The Einstein radius is:

$$R_E = d_l\theta_E = \sqrt{4\frac{GMd_l}{c^2}\left(\frac{d_s}{d_s+d_l}\right)} \tag{15}$$

If $d_s = d_l$ we get the simple result:

$$R_E = d_l\theta_E = \sqrt{2\frac{GMd_l}{c^2}} \tag{16}$$

meaning that the Einstein radius is simply the geometrical mean between the Schwarzschild radius and the distance to the lens.

4 Magnification

Gravitational lensing preserves surface brightness (brightness per unit area) so the amount of the magnification is given by the ratio of the area of the image to the source.

The change in the area of the image compared to the source is the multiplication of the change on the y axis and the change on the x axis as can be seen from the figure. In the y axis the projected distance of the source in the sky-plane(see Fig.1) is:

$$R_s = (d_s + d_i)\theta_s \tag{17}$$

while the projected distance of the image is:

$$R_i = (d_s + d_i)\theta_i \tag{18}$$

By looking at Fig.3 we can see that the angle Ω is the same for both the source and the image, so:

$$dy_i = (d_s + d_i)\theta_i\Omega \tag{19}$$

$$dy_s = (d_s + d_i)\theta_s\Omega \tag{20}$$

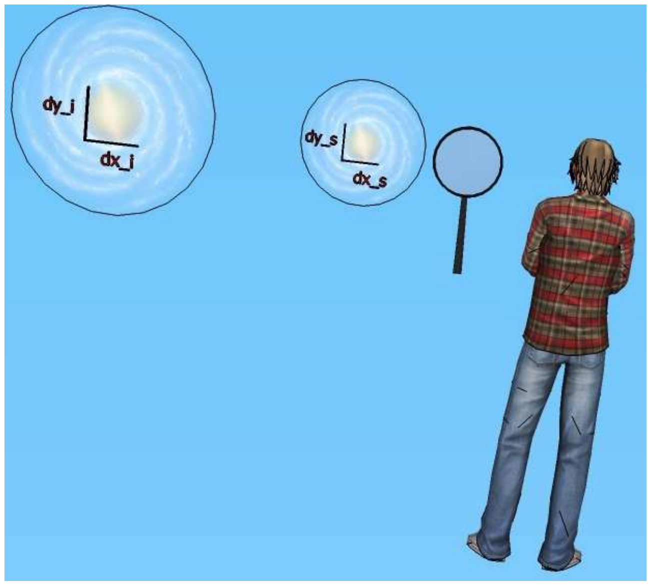


Figure 1: Plane of sky view of the system

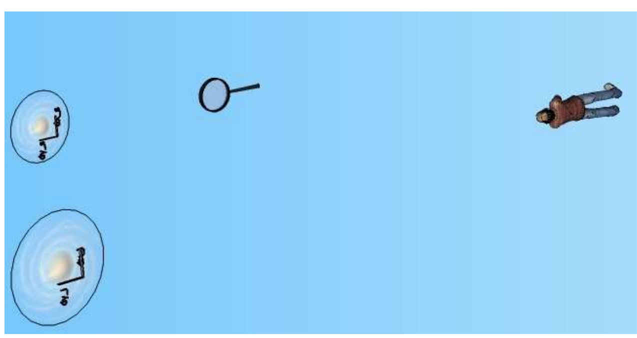


Figure 2: View of the system from above

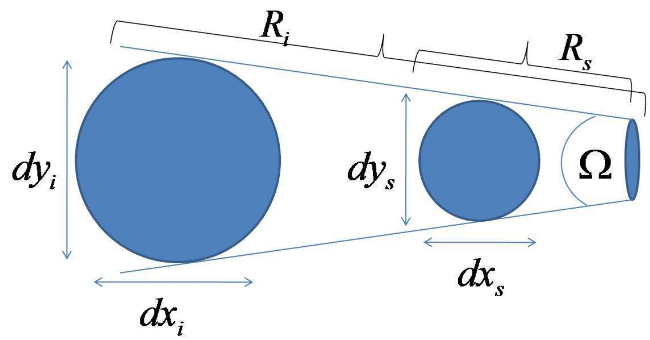


Figure 3: Angles in the plane of the sky

This gives us the relation:

$$dy_i = dy_s \frac{\theta_i}{\theta_s} \tag{21}$$

Two points which are separated by $d\theta_s$ in the plane of the sky, will be separated by $d\theta_i$ in the image so:

$$dx_i = dx_s \frac{d\theta_i}{d\theta_s} \tag{22}$$

The magnification is:

$$\mu \equiv \frac{A_i}{A_s} = \frac{\theta_i}{\theta_s}\frac{d\theta_i}{d\theta_s} \tag{23}$$

In general there are two solutions for the lens equation which will give us two images located at:

$$\theta_{i\pm} = \frac{1}{2}\left(\theta_s \pm \sqrt{\theta_s^2 + 4\theta_E^2\theta_s}\right) \tag{24}$$

The total magnification is the sum of the two images:

$$\mu = |\mu_-| + |\mu_+| \tag{25}$$

From the lens equation we have:

$$\mu_\pm = \left(1 - (\frac{\theta_E}{\theta_{i\pm}})^4\right)^{-1} = \frac{u^2+2}{2u\sqrt{u^2+4}} \pm \frac{1}{2} \tag{26}$$

where we defined $u \equiv \frac{\theta_s}{\theta_i}$. The total magnification is:

$$\mu = \frac{u^2+2}{2u\sqrt{u^2+4}} \tag{27}$$