Radial drift of solid particles in gaseous discs

Let's consider a gaseous discs and a solid body imbedded into it. Both gas and the body feel the gravitational attraction of a central mass M. However the fact that the body does not have the same internal pressure forces as the gas results in a different dynamics. Specifically, whether the body is small or large (and we will define it), it experience radial inward drifting ultimately due to gas drag. *We want to calculate here which is the maximum radial drag that a body can experience.*

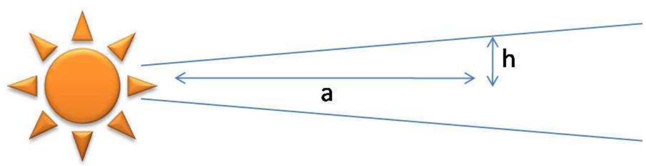

Figure 1: Sketch of the (thin) disc geometry, where H is the presuure scale hight and a is the radial distance from the central object

To calculate it let's first evaluate the difference in azimuthal velocity between the body and the gas particles. Gas in discs are partially supported against gravity by a radial (outward) gradient of pressure. To deduce its circular velocity $V_{\rm gas}$ at a distance a from M, we write approximated radial

$$\frac{GM}{a^2} - \frac{P_{\rm c}}{a\rho_{\rm c}} \simeq \frac{V_{\rm gas}^2}{a}, \tag{1}$$

and vertical

$$\frac{1}{\rho_{\rm c}}\frac{P_{\rm c}}{H} \simeq \frac{GM}{a^2}\frac{H}{a}, \tag{2}$$

equations of motion, where H is the disc thickness at the location (one pressure scale height) and the subscript c refers to quantities evaluated at the mid-plane. The second term of eq. 1 is an approximation for the radial pressure gradient, which counteracts together with the centrifugal force (right hand side term) the radial gravitational force. In eq. 2 the left hand side term is the *vertical* pressure gradient which is assumed to balance the vertical component of the gravitational force, arising from the central star. Combining eq. 1 and eq. 2 we get the classic result,

$$V_{\rm gas} = V_{\rm K}\left(1-\left(\frac{H}{a}\right)^2\right)^{1/2} \simeq V_{\rm K}\left(1-\frac{1}{2}\left(\frac{H}{a}\right)^2\right), \tag{3}$$

where the second equality holds for thin discs $H/a \ll 1$. Eq. tells us that the gas moves slower than a purely Keplerian velocity $V_{\rm K} = \sqrt{GM/a}$, in which only the centrifugal force balances the gravitational force. In the following we consider two extreme regimes, and we will see that at the transition point we have the maximum of the inward drag.

1 Small Friction–Large Bodies

In this regime the azimuthal velocity of the body is Keplerian, since the drag is just a small perturbation, $V_\phi \sim V_{\rm K}$. However, the difference in velocities between the gas and the body results in a (azimuthal) frictional deceleration $a_{\rm f}$ and corresponding loss of angular momentum at a rate

$$\frac{d(aV_{\rm K})}{dt} \simeq -\frac{a}{t_{\rm f}}\left(V_{\rm K} - V_{\rm gas}\right), \tag{4}$$

where the friction time is defined as

$$t_{\rm f} = \frac{\Delta V}{|a_{\rm f}|}, \tag{5}$$

where ΔV is the difference in velocity between the gas and the body and we remind the reader that $a_{\rm f}$ is the acceleration given by the friction force (see below §3.1). In this case the acceleration is mainly azimuthal and $t_{\rm f} = \left(V_{\rm K} - V_{\rm gas}\right)/a_{\rm f}$. Observing that $da = V_{\rm r}dt$ and combined with eq. 3, we get

$$\frac{V_{\rm r}}{V_{\rm K}} \simeq \frac{1}{\Omega t_{\rm f}}\left(\frac{H}{a}\right)^2. \tag{6}$$

where Ω^{-1} is proportional to the orbital period. We note that the stronger the drag is, the shorter $t_{\rm f}$ and the larger is the radial drift. This formula is valid up to $t_{\rm f} \sim \Omega^{-1}$, where $V_{\rm r} = V_{\rm gas} \sim V_{\rm K}(H/a)^2$, and the drag acceleration becomes radial.

2 Large Friction-Small Bodies

In this regime, instead, the small body is swept with the flow and moves with $V_\phi = V_{\rm gas}$. Now the body is forced to rotate with a sub-keplerian velocity that is not enough to counteract the gravitational force. The difference between this two forces results in a inward acceleration, which in turns arises *radial* viscous forces

$$\frac{V_{\rm r} - V_{\rm r,gas}}{t_{\rm f}} \simeq \frac{V_{\rm r}}{t_{\rm f}} = \frac{V_{\rm gas}^2}{a} - \frac{V_{\rm K}^2}{a}, \tag{7}$$

(radial equation of motion for the body) where we have assume that the radial gas velocity $V_{\rm r,gas}$ given by viscous processes is negligible. Equation 7 together with eq. 3 thus give

$$\frac{V_{\rm r}}{V_{\rm K}} \simeq \Omega t_{\rm f} \left(\frac{H}{a}\right)^2. \tag{8}$$

In this regime, *the smaller* is the drag, the faster the body drifts inward.

3 The maximum radial drift

Comparing eq. 6 and eq. 8, we conclude that the larger radial drifting is obtained from bodies whose friction time (the time it takes by friction to change their velocity by a factor of 2) its equal the local dynamical time. For $t_{\rm f} \simeq \Omega^{-1}$,

$$V_{\rm r,max} = V_{\rm K} \times \left(\frac{H}{a}\right)^2, \tag{9}$$

and the body suffers a deceleration

$$a_{\rm f,max} \sim V_{\rm r,max}\Omega = \frac{c_{\rm s}^2}{a}, \tag{10}$$

where $c_{\rm s}$ is the gas sound speed and we use the fact that $H/a = c_{\rm s}/V_{\rm K}$.

3.1 Friction timescale

In all above equations we treat the friction time as a parameter and we alluded to the two regimes as involving "large" and "small" bodies, without explanation. We are now going to explicitly derive the dependence of the friction time-scale on the body (linear) size s, assuming that the body density is actually constant.

The friction force can be in general written as

$$\mathbf{F}_{\rm f} \propto -\pi \mathbf{s^2} \rho |\mathbf{\Delta V}|^\mathbf{2}, \tag{11}$$

where ρ is the density of the surrounding gas. The acceleration is thus $a_{\rm f} = F_{\rm f}/m$, where $m \propto s^3$. From equation 5 we thus get

$$t_{\rm f} \propto s. \tag{12}$$

Therefore, the *larger* the body the *longer* the time-scale to arise the difference in velocity between the body and the surrounding gas. It turns out that for for proto-planetary condition, $t_{\rm f} \simeq \Omega^{-1}$ for body that are a meter size. They will drift in with the maximum radial velocity given by eq. 9.

4 The maximum binary radius

Let's consider two body interacting gravitationally. We saw that bodies with different sizes experience different drag forces. There is thus a maximum binary radius $R_{\rm s}$ below which they can form a binary. For larger separations the drag will tear the binary apart. Let's take a meter size body orbiting around a body with a very different radial drag. The drag force per unit mass suffer by the binary is thus given by eq. 10. We now impose that this acceleration is counter-balanced by the mutual gravity

$$\frac{c_{\rm s}^2}{a} = \frac{Gm}{R_{\rm s}^2}, \tag{13}$$

where m is the mass of the one meter size body. Let's introduce the Hill radius $R_{\rm H} = a(m/M)^{1/3}$, which is the larger separation at which a binary can counter act the *central object*'s tides,

$$\frac{Gm}{R_{\rm H}}^2 = \frac{GM}{a}\frac{R_{\rm H}}{a}. \tag{14}$$

From eq. 13 we get

$$\frac{R_{\rm s}}{R_{\rm H}} = \left(\frac{m}{M}\right)^{1/6}(a/H) \simeq 10^{-3}. \tag{15}$$

The binaries must indeed be very close to survive disruption from gas drag !

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