“Compact Objects and their merger”,

E.M. Rossi, Leiden Observatory

Stellar **mass** ***<u>black holes</u>***

<u>S</u>tudy material:
Chapter 12, Shapiro&Teukolsky
For GWs: https://arxiv.org/pdf/0903.0338.pdf

(stellar) Black Holes

1700s intuition

Michell (1784) & Laplace (1796) considered a particle of light of mass "m". Its mechanical energy is

$$E = m\left(\frac{1}{2}v_0^2 - \frac{GM}{r}\right).$$

light escapes if $E > 0$, equivalently if $\boxed{r > R_g = \frac{2GM}{c^2}.}$ where $v_0 = c$.

this is the radius of a "dark star"

Words from the past

If there should really exist in nature any bodies, whose density is not less than that of the sun, and whose diameters are more than 500 times the diameter of the sun, since their light could not arrive at us; or if there should exist any other bodies of a somewhat smaller size, which are not naturally luminous; of the existence of bodies under either of these circumstances, we could have no information from sight; yet, if any other luminous bodies should happen to revolve about them we might still perhaps from the motions of these revolving bodies infer the existence of the central ones with some degree of probability, as this might afford a clue to some of the apparent irregularities of the revolving bodies, which would not be easily explicable on any other hypothesis; but as the consequences of such a supposition are very obvious, and the consideration of them somewhat beside my present purpose, I shall not prosecute them any further.

— John Michell, 1784[13]

[35]

VII. *On the Means of discovering the Distance, Magnitude, &c. of the Fixed Stars, in consequence of the Diminution of the Velocity of their Light, in case such a Diminution should be found to take place in any of them, and such other Data should be procured from Observations, as would be farther necessary for that Purpose. By the Rev.* John Michell, *B. D. F. R. S. In a Letter to* Henry Cavendish, *Esq. F. R. S. and A. S.*

Read November 27, 1783.

1700s intuition

$$\boxed{r > R_g = \frac{2GM}{c^2}.}$$

Therefore an object with size $R < R_g$ cannot emit light, it is a "*black hole*". From this condition we derive that a BH does not necessarily has R=0 and an infinite density, but $\rho > \rho_g$

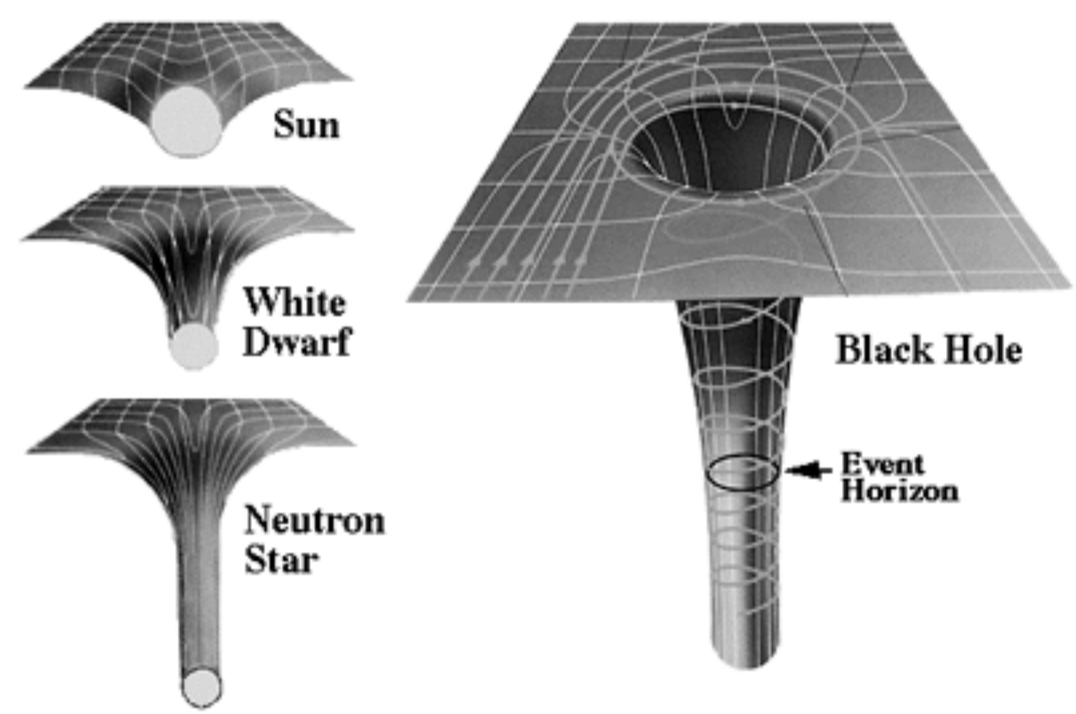
$$\rho_g = \frac{M}{\frac{4\pi}{3} R_g^3} \approx 2 \times 10^{16} \left(\frac{M}{M_{\rm sun}} \right) \ {\rm g\ cm^{-3}}$$

inverse with mass, density very high for solar mass but....

A BH ~10^8 M_{sun} has a density comparable to Sun ~1 g/ cm^{-3}

General relativity

- *1915*: two months after the publication of GR, Karl Schwarzchild calculated the exact solution of Einstein equation in spherical symmetry, in vacuum. It has only one parameter: mass M. It applies to any spherical object. It can be used to describe space-time around (r>0) a black hole placed at r=0.



Stellar evolution

- 1930 Chandrasekhar calculates a maximum mass for WDs, extended to all degenerate objects by Landau.
- Eddington (who did not believe Chandrasekhar's limit) envisages black holes (together with Landau) but he feels that nature must somehow prevents this total collapse to a point

3. « *the star apparently has to go on radiating and radiating and contracting and contracting until, I suppose, it gets down to a few kilometers radius when gravity becomes strong enough to hold radiation and the star can at last find peace.*« (Eddington, 1930).

4. « *I think there should be a law of Nature to prevent the star from behaving in this absurd way.* » (Eddington, 1930).

Stellar evolution

- 1939 Oppenheimer & Snyder showed that as a consequence of the collapse of an homogeneous sphere without pressure in GR, the sphere eventually is not able to exchange information with the rest of the world. It is the first rigorous calculation of a BH formation.

SEPTEMBER 1, 1939 PHYSICAL REVIEW VOLUME 56

On Continued Gravitational Contraction

J. R. OPPENHEIMER AND H. SNYDER
University of California, Berkeley, California
(Received July 10, 1939)

When all thermonuclear sources of energy are exhausted a sufficiently heavy star will collapse. Unless fission due to rotation, the radiation of mass, or the blowing off of mass by radiation, reduce the star's mass to the order of that of the sun, this contraction will continue indefinitely. In the present paper we study the solutions of the gravitational field equations which describe this process. In I, general and qualitative arguments are given on the behavior of the metrical tensor as the contraction progresses: the radius of the star approaches asymptotically its gravitational radius; light from the surface of the star is progressively reddened, and can escape over a progressively narrower range of angles. In II, an analytic solution of the field equations confirming these general arguments is obtained for the case that the pressure within the star can be neglected. The total time of collapse for an observer comoving with the stellar matter is finite, and for this idealized case and typical stellar masses, of the order of a day; an external observer sees the star asymptotically shrinking to its gravitational radius.

Theoretical physics

- end 1950s - beginning 1960s: BHs was an object of study for physicists, with little interest showed by astrophysicists.
  - At the end of fifties Wheeler and co. reconsider the issue of collapse for a massive object (p.s. Wheeler in 1968 came up with the name "black hole").
  - In 1963 Kerr found a family of solutions of the Einstein equations in vacuum, with no charge but relaxing the assumption of asymmetry (spin)
  - In 1963 Newman extends the solution to charged BH (not really expected in nature)

The Kerr- Newman solution is the most complete description of the space-time outside a stationary BH

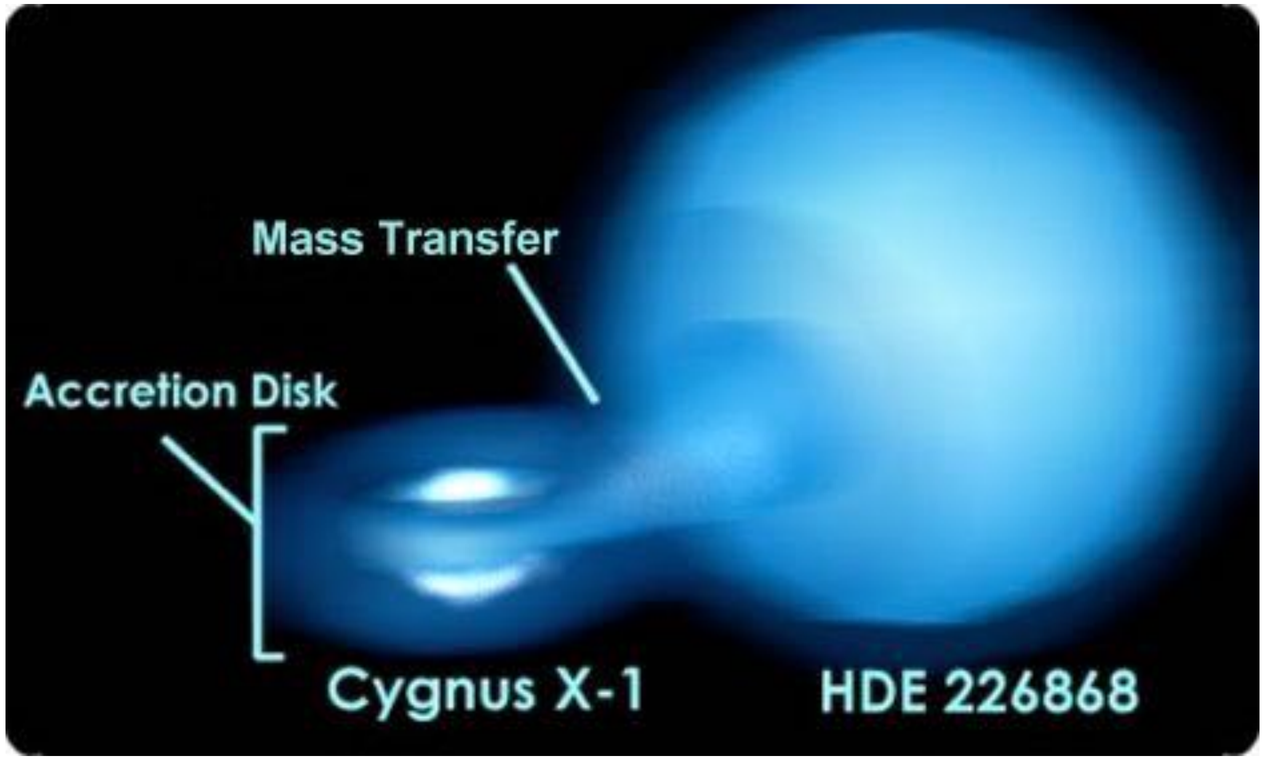
Early 60s: the beginning of X-ray astronomy *and* of the astrophysics of compact objects

In 1962, Riccardo Giacconi and co-authors discovered the first X-ray source beyond the solar system (Sco X-1, $L_x \sim 6\ 10^4\ L_{sun}$), following the launch of the USA Arobee rocket (Nobel prize in 2002).

- 1960s : BH becomes of astrophysical interest only after the discovery of the compact X-ray sources (1962), quasars (1963) and pulsars (1968). The compact object field is of growing of importance

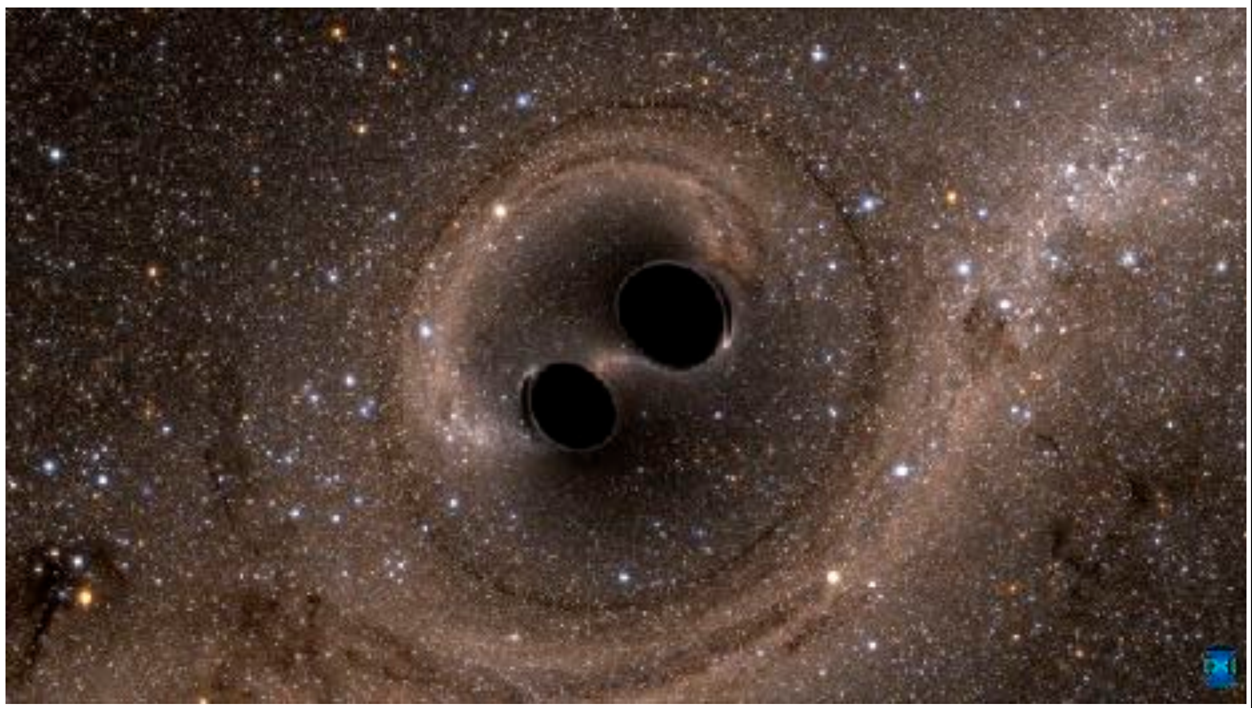
The first BH candidate

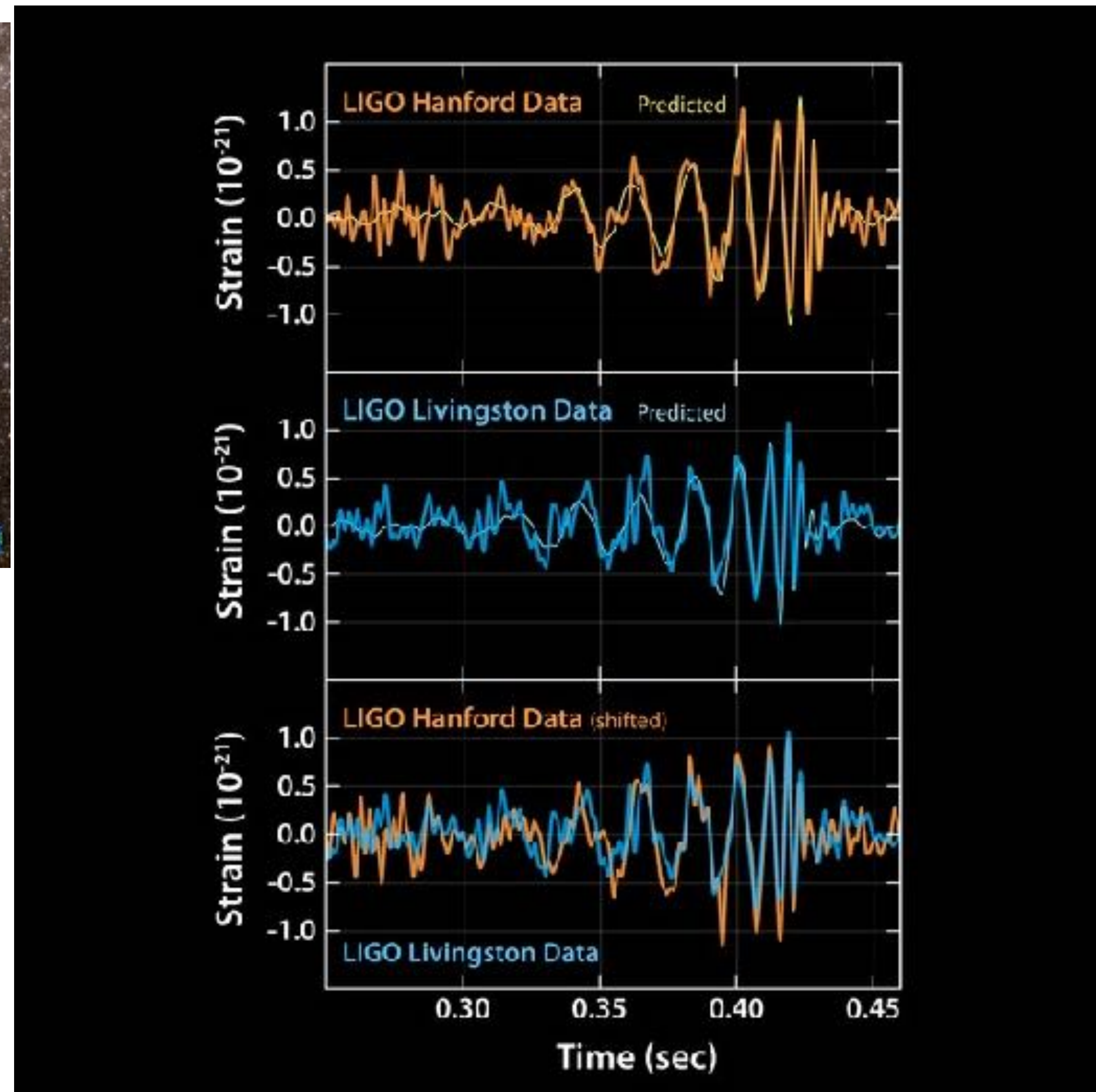
- 1970: Observation of the X-ray binary *Cygnus X-1*, first BH candidate: the compact object mass is at least 6 M_{sun}, that excludes a NS for any equation of state



The first detection of a BH in GW & of a *binary* black hole candidate

14 September 2015

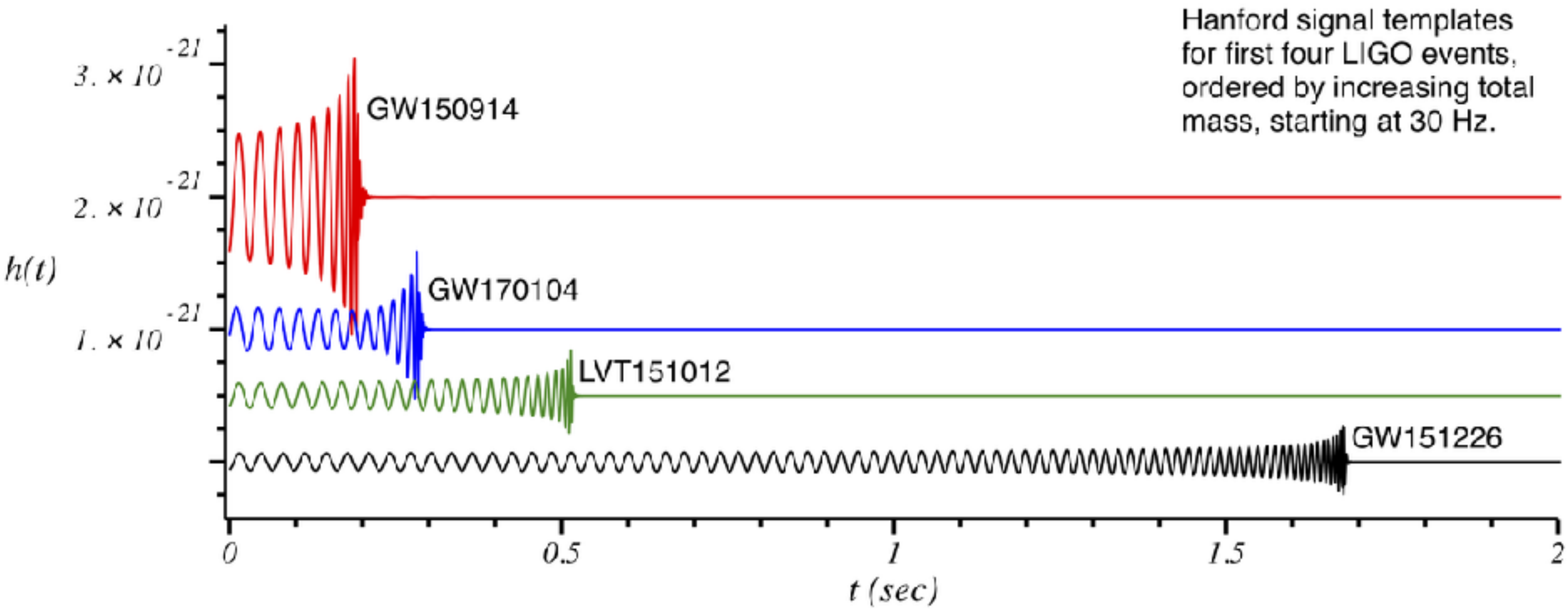




~30 solar masses

GW150914

Four confirmed detections since…



Note the mass dependence…

2017 Nobel prize in Physics

From left: Kip Thorne, Ron Drever and Robbie Vogt, the first director of the LIGO project, with a 40-meter prototype of the LIGO detectors at the California Institute of Technology in 1990.

"for decisive contributions to the LIGO detector and the observation of gravitational waves"

Nobelprize.org

Special mention to

Ron Drever

co-founder of LIGO

what do we study about black hole in isolation?

- They do not emit radiation in isolation, only quantic BHs emit Hawking's radiation. But for solar mass and supermassive BH is too dim
- Become of astrophysical interest only when liberate energy by accretion or GW
- It is difficult to discuss an "internal structure" for BHs
- Here we study the modification of space time around them

Minkowski space-time

$d\tau$ = proper time

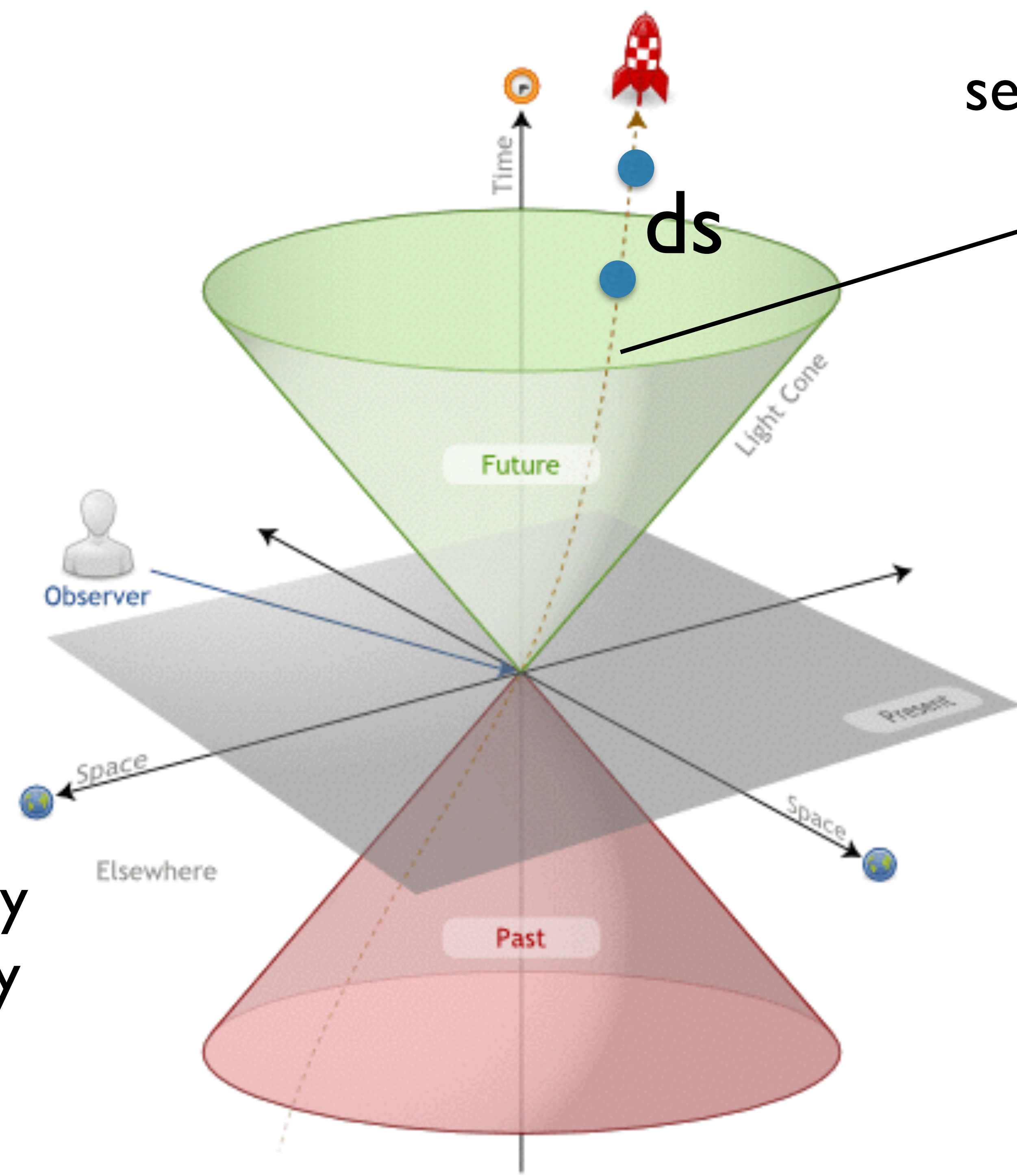
$d\tau > 0$ future

ds distance between two events

event:

(ct, r, θ, ϕ)

dt = time measure by observer at infinity

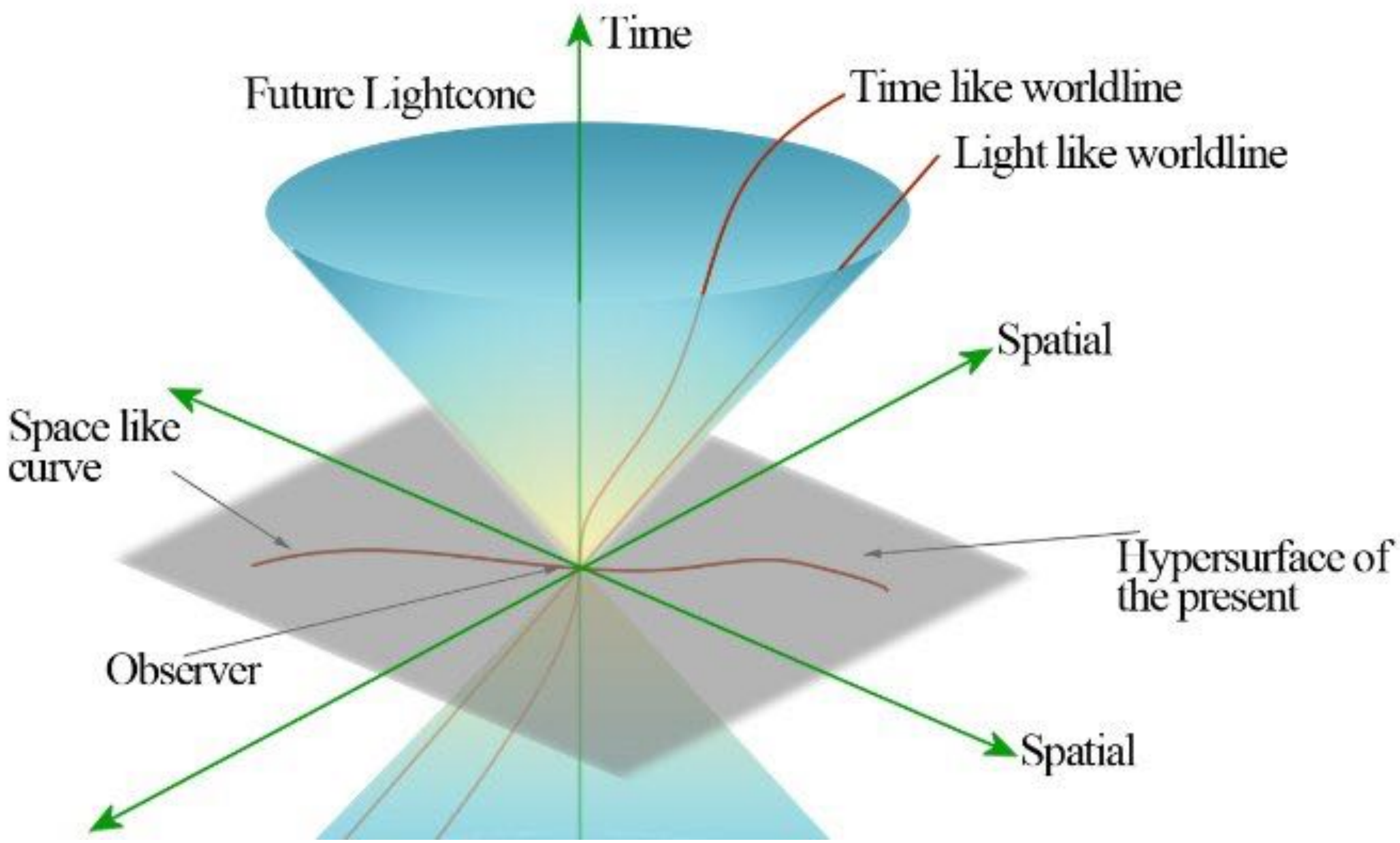


sequence of events for the rocket

events or sequence of events as measured by an observer at rest:

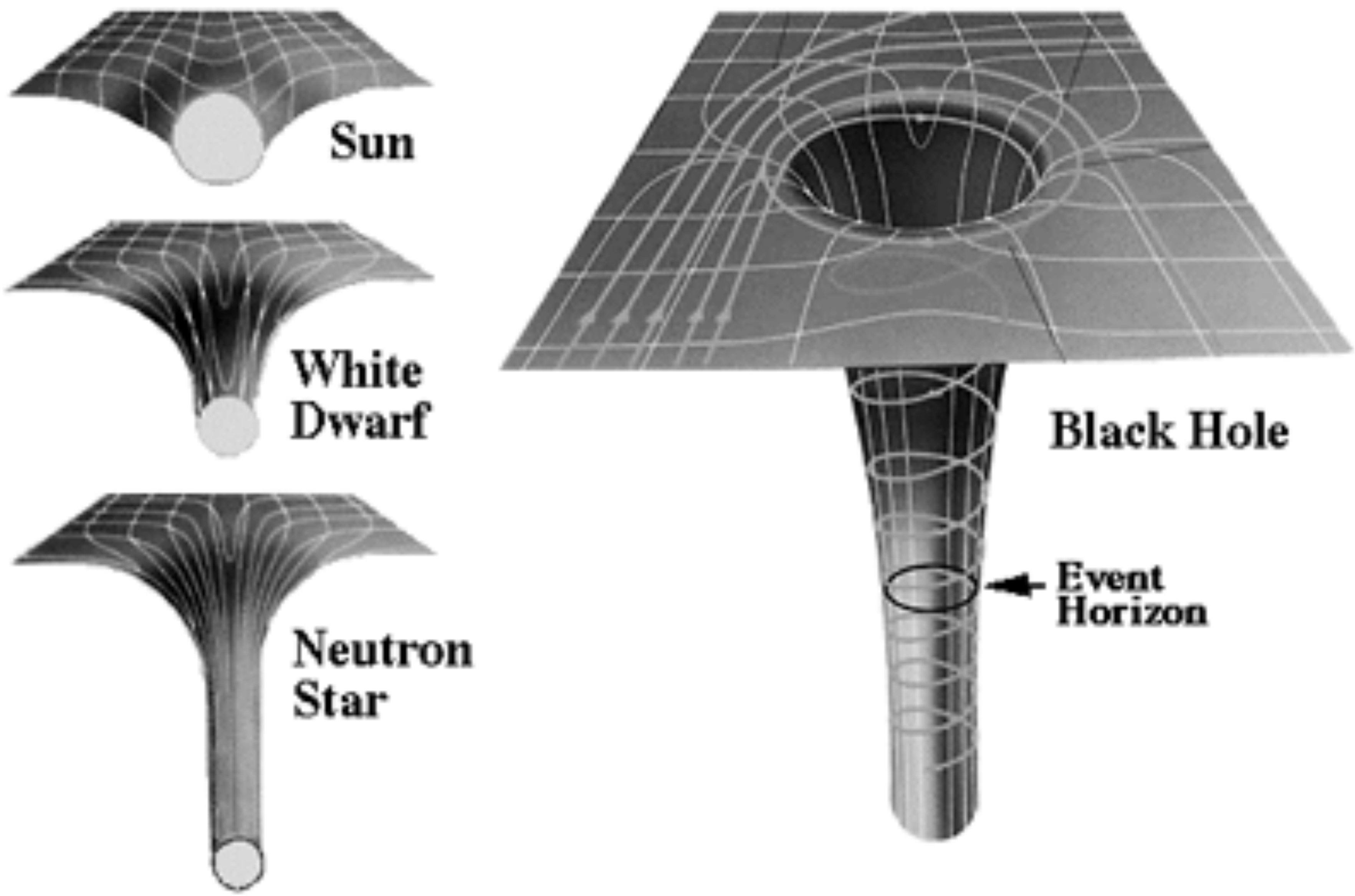
$$ds^2 = -c^2 d\tau^2 = -c^2 dt^2 + dr^2 + r^2(d\theta^2 + \sin\theta^2\, d\varphi^2)$$

Minkowski space-time



$$-c^2 d\tau^2 = -c^2 dt^2 + dr^2 + r^2(d\theta^2 + \sin\theta^2 \, d\varphi^2)$$

BH space-time



space-time modified by a spherical mass at rest

Schwarzchild metric

This metric describes the space time outside a static, spherical object of mass M (no spin or charge). In the case of a BH, it described the whole space-time, as M is assumed r=0.

$$-c^2 \mathrm{d}\tau^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 \mathrm{d}t^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2\right)$$

proper time

spherical coordinates,
only g_{tt} and g_{rr} are not = one

Schwarzchild radius:

$$R_S = \frac{2GM}{c^2}.$$

$$g_{tt} = -\left(1 - \frac{R_S}{r}\right) c^2,$$

$$g_{rr} = \left(1 - \frac{R_S}{r}\right)^{-1}.$$

$$r \gg R_s \rightarrow 1$$

Schwarzchild radius

3 properties (2 really…)

1) "Static" limit

let's take $\mathrm{d}r = \mathrm{d}\theta = \mathrm{d}\phi = 0$

$$\mathrm{d}s^2 = -c^2\left(1 - \frac{2GM}{rc^2}\right)\mathrm{d}t^2 = -c^2\mathrm{d}\tau^2.$$

Future = dτ >0 ==> dt >0 always otherwise paradoxes:

if a mass can go back in time, can have two different ages measured by its proper times at the same event (ct,r,θ,φ): impossible!

“Static” limit

let’s take $\mathrm{d}r = \mathrm{d}\theta = \mathrm{d}\phi = 0$

$$\mathrm{d}s^2 = -c^2\left(1 - \frac{2GM}{rc^2}\right)\mathrm{d}t^2 = -c^2\mathrm{d}\tau^2.$$

$$d\tau = +\left(1 - \left(\frac{R_\mathrm{s}}{r}\right)\right)^{1/2} dt > 0$$

For $r > R_s$ a solution is possible

for r<Rs no static solution is possible

2) R_s as dynamical boundary

One can also show (we not do it, but it is easy to see) that for $r < R_s$ not only $dr=0$ is impossible <u>but $dr < 0$ always</u>

⇒ mass that passes through Rs inevitably falls towards $r=0$

⇒ no Hydrostatic Equilibrium object with size $R < Rs$ is possible, it would collapse

3) R_s as event horizon

for $\quad dr = d\theta = d\phi = 0 \quad$ Einstein's dilation of time

$$\frac{dt}{d\tau} = 1 + z = \frac{1}{\sqrt{1 - \frac{2GM}{rc^2}}} > 1 \quad \text{for} \quad r > R_S \,.$$

for $r=R_s$ redshift is infinite, what does it means ?

That R_s divides two regions with no causal connection: signals cannot be exchanged between $r < R_s$ and $r > R_s$

dynamics around a BH

Newtonian mechanics

mass m orbiting a mass M

$$V_{\text{eff}} = \frac{l^2}{2mr^2} - \frac{GmM}{r}$$

$$V_{\text{eff}} = V'$$

$l \neq 0 \quad \frac{dV_{\text{eff}}}{dr} = 0$ circular orbit

$$l = 0 \rightarrow V_{\text{eff}} = -\frac{k}{r}$$

“free fall”

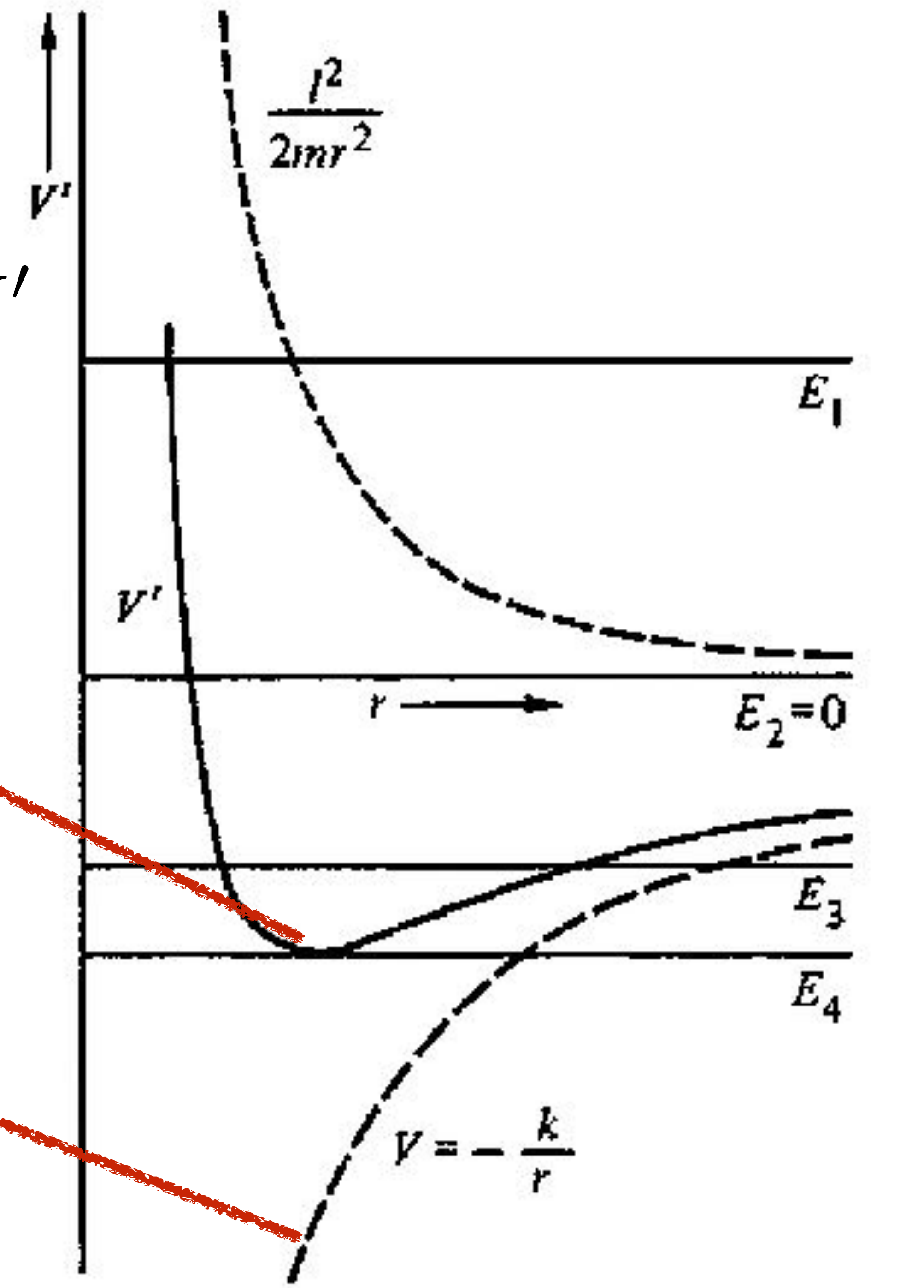
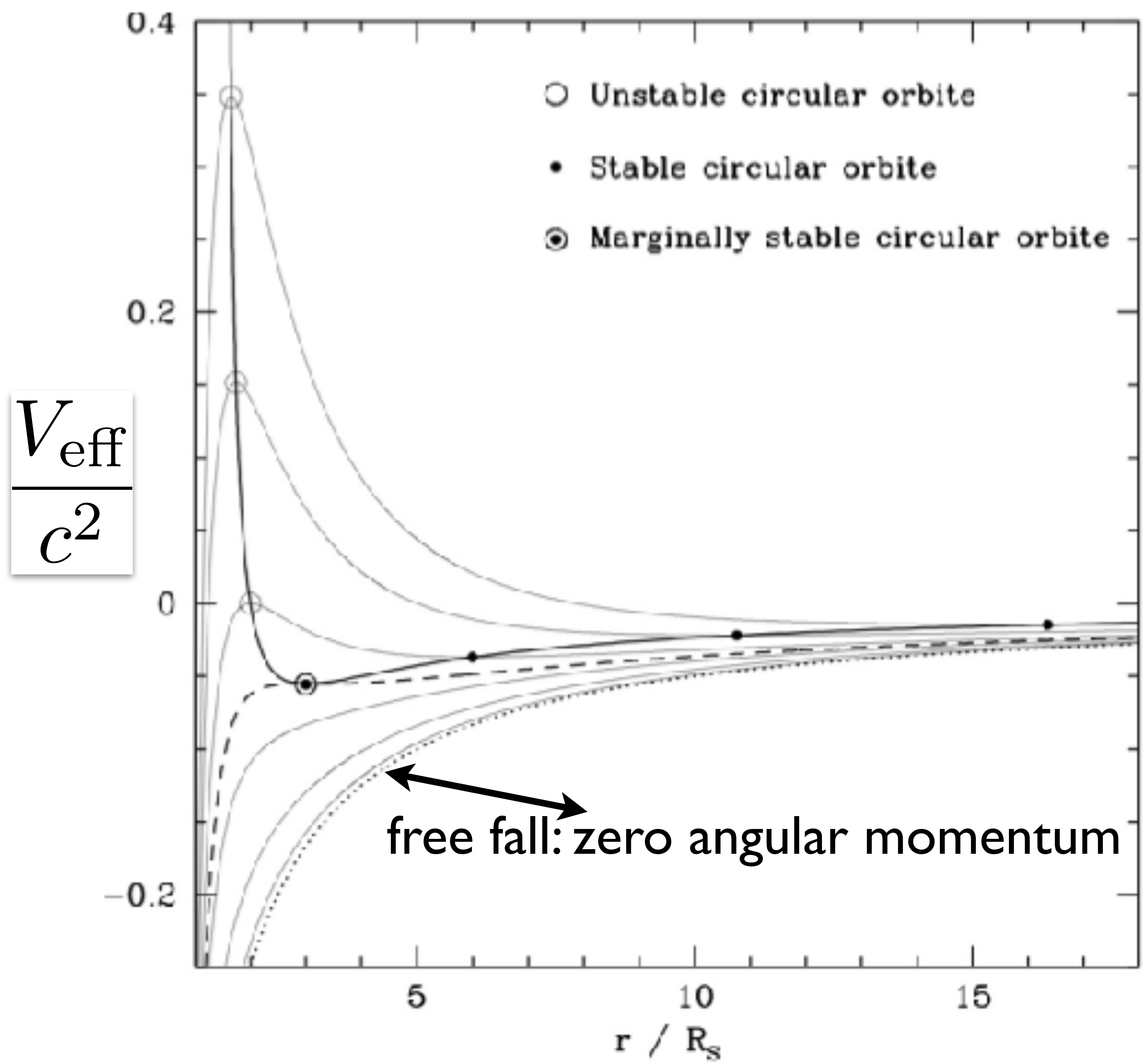


FIGURE 3.3 The equivalent one-dimensional potential for attract

Last stable circular orbit



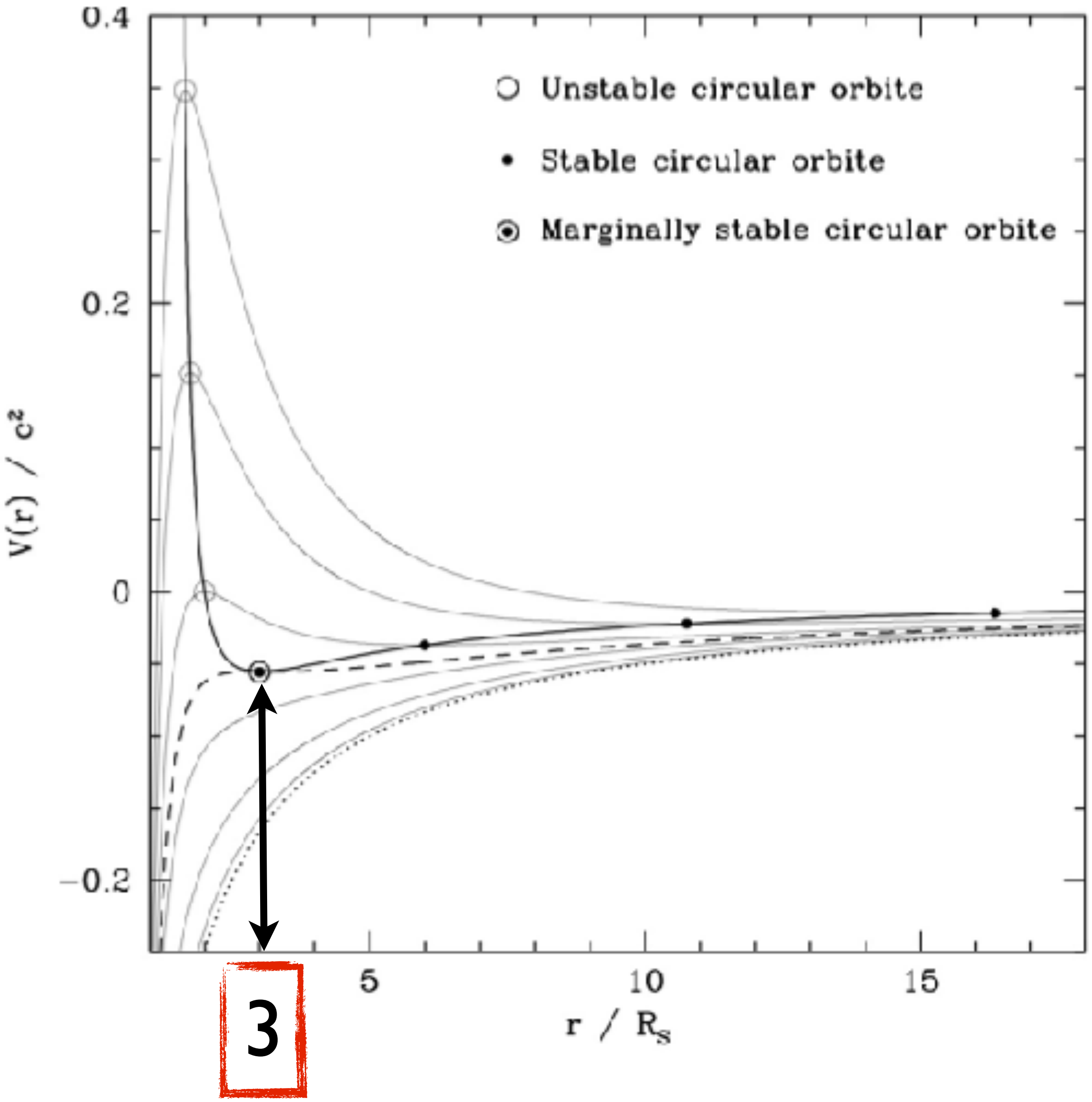
Relativistic *effective* potential V for a particle with mass m>0. Effective potential includes gravitational +rotational energy

Different curves for different specific angular momenta "l". From bottom to top, increasing angular momentum

Solid line is the line of points where circular orbit exists, where $dV_{eff}/dr = 0$

minima = stable and exist only for $|l| \geq \sqrt{3} R_S c$, therefore ...

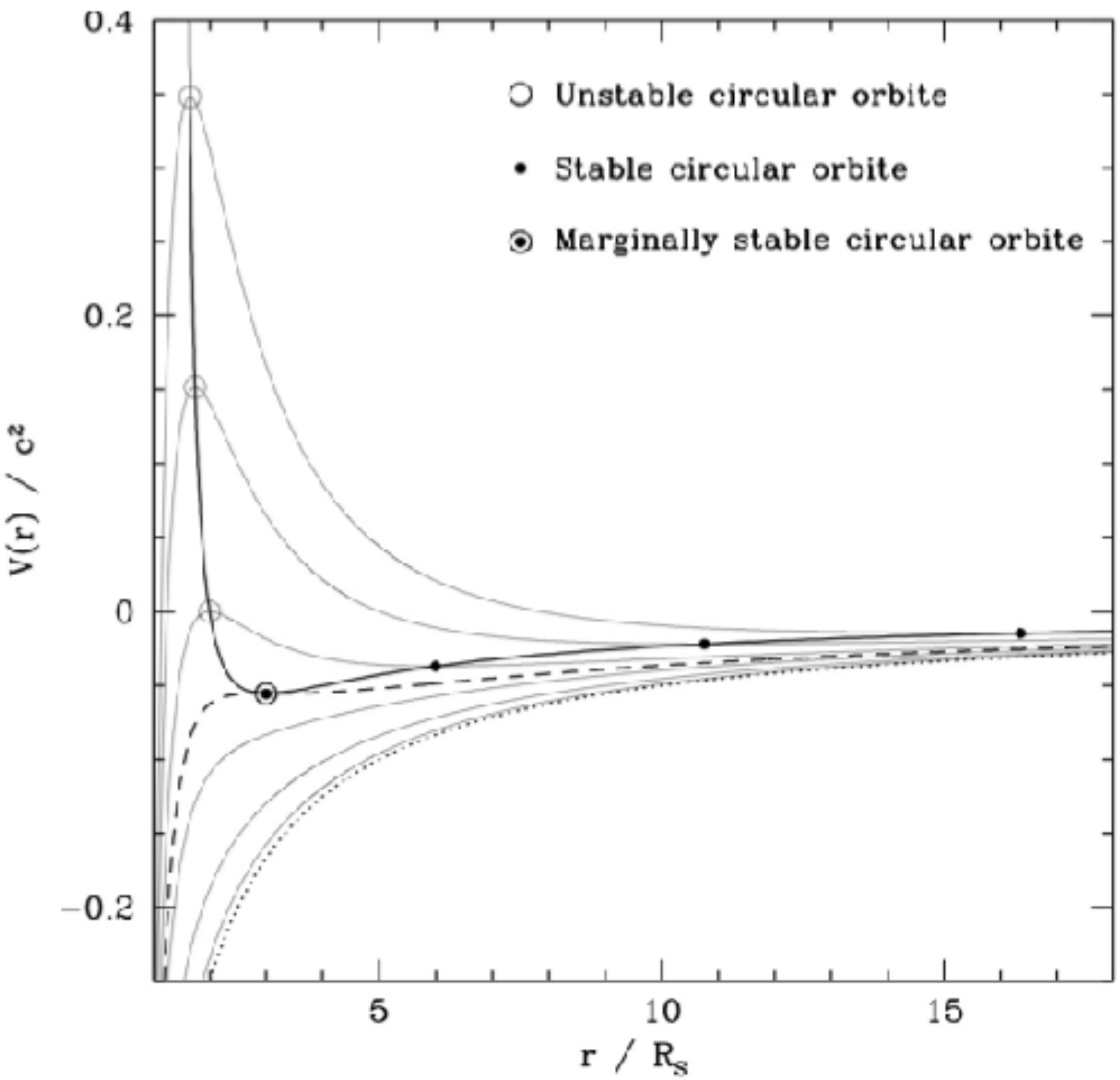
Last stable circular orbit



We find that the last stable circular orbit for a Schwarzschild BH has :

$$\boxed{R = 3R_S \;\; ; \;\; l = \sqrt{3}R_S c}$$

Accretion efficiency



$$\boxed{R = 3R_S \;\; ; \;\; l = \sqrt{3}R_S c}$$

No accretion disc exists < 3 R_s

On this orbit matter has an energy per unit mass, without rest mass energy of

$$e - c^2 = \left(2\sqrt{2}/3 - 1\right) \simeq -0.0572c^2$$

So the maximum energy that can be liberated per unit time by a non-rotating BH is $\boxed{L_{\rm acc} < 0.0572\,\dot{m}c^2}$

Kerr metric

A general solution of space-time around a BH depends on mass, spin and charge (Carter and Hawking showed it). Wheeler summarised it with "BHs have no hair" (no-hair theorem). This space-time is described by the Neuman-Kerr solution. We do not derive or use it...I will give you the main results for a Kerr (spinning) BH.

Event Horizon for a spinning (Kerr) BH

$$R_{\mathrm{H}} = \frac{R_{\mathrm{s}}}{2}\left(1 + \sqrt{(1 - a^2)}\right) = R_{\mathrm{g}}\left(1 + \sqrt{(1 - a^2)}\right)$$

the *gravitational* radius is $R_{\mathrm{g}} = \frac{GM}{c^2}$

the a-dimensional spin parameter $-1 \leq a = \frac{Jc}{GM^2} \leq 1$

J = BH angular momentum

$$R_{\rm H} = \frac{R_{\rm s}}{2}\left(1+\sqrt{(1-a^2)}\right) = R_{\rm g}\left(1+\sqrt{(}1-a^2)\right)$$

max:

$$|a| = |\frac{Jc}{GM^2}| = 1 \rightarrow |J| = \frac{GM^2}{c} \quad \rightarrow |J| = MR_{\rm g}c$$

BH rotates at c at $R_H=R_g$

- No spin :
  - |a|= 0
  - R_H = 2Rg : Schwarzschild radius

A black becomes smaller as it spins faster

Ergosphere

The static boundary does *not* coincide with R_H as for the non-spinning BH but defines a region above R_H: the <u>*ergosphere*</u>

$$R_{\rm H} \leq r \leq R_{\rm static}(\theta) = $$
$$= R_{\rm g}\left(1 + \sqrt{1 - a^2 \cos\theta^2}\right)$$

Note, for $a = 0 \rightarrow R_{\rm static} = R_{\rm s}$

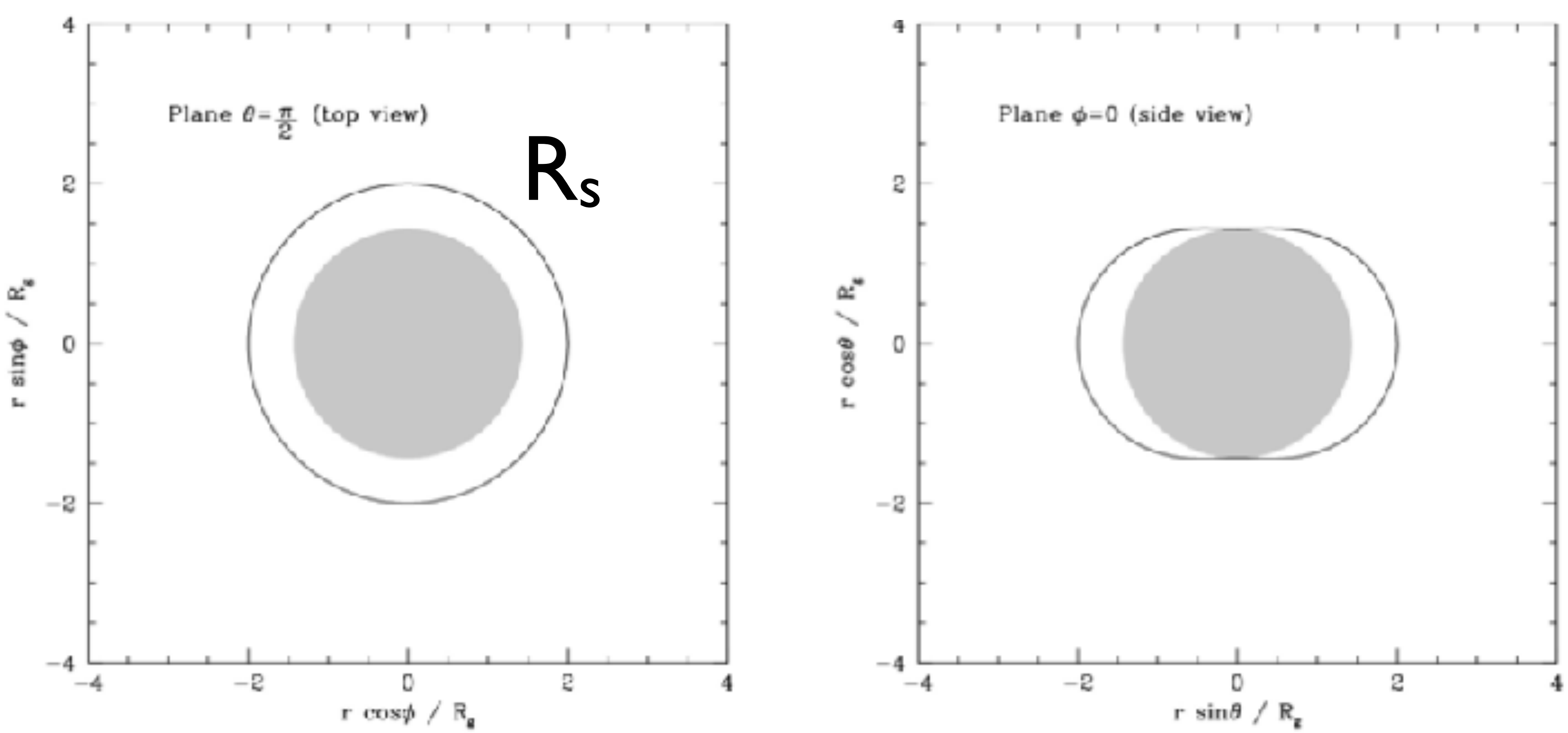


Figure: ergosphere for $a = 0.9$

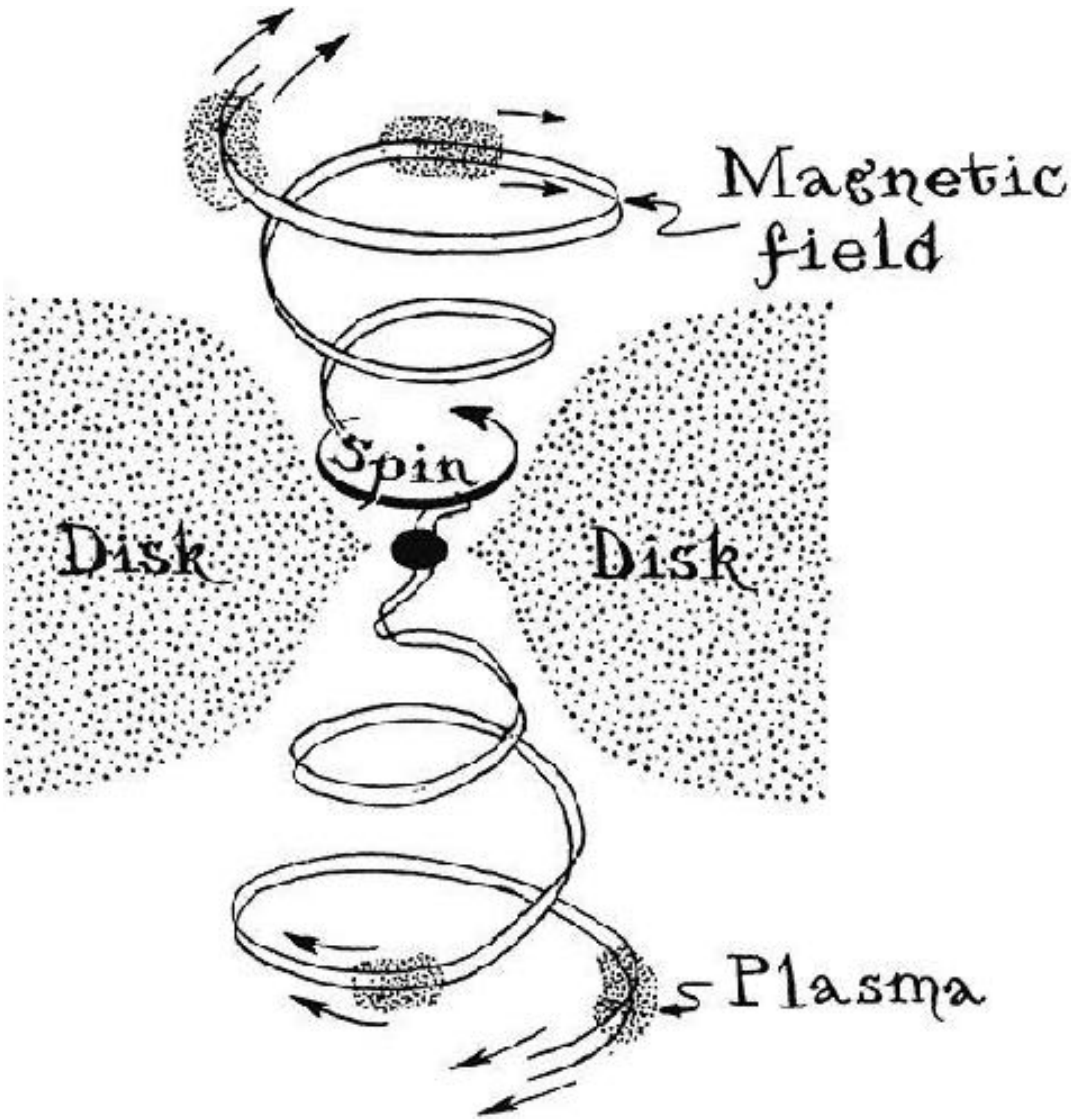
Ergosphere: meaning

Outside the ergosphere a particle can be at rest and move in all directions

Inside the ergosphere a particle cannot be at rest and it must rotate as dictated by the BH spin

can we see this effect?

The Blandford -Znajek exploits the presence of an ergosphere to mechanism extracts BH rotational enegy via magnetic field lines that are brought there by accretion. This is one of the leading mechanism to power jets



Accretion efficiency

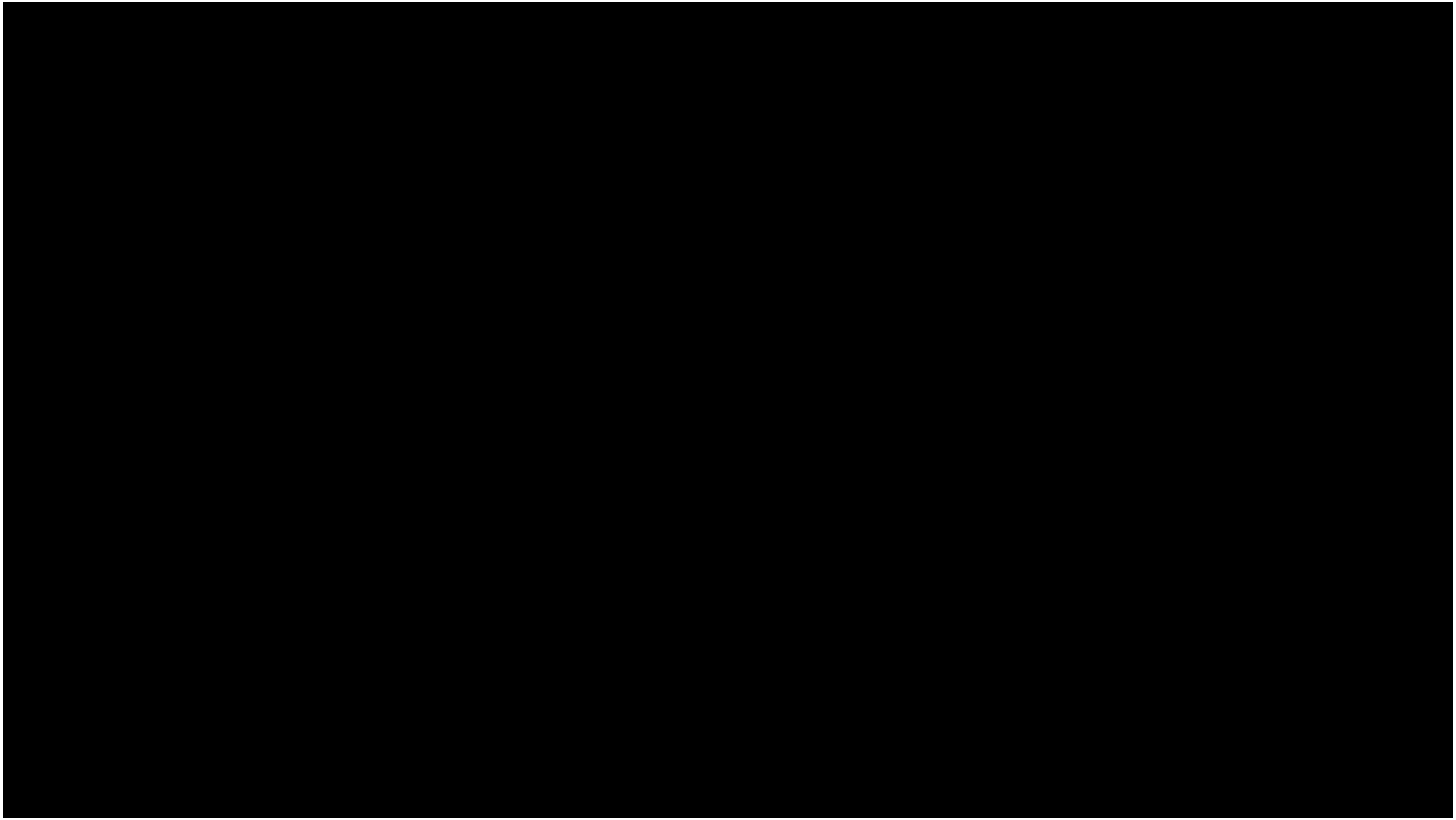
Qualitatively same argument as for a non-rotating black hole gives an efficiency of ~40% for a=1

Please, derive the following expression:

$$L_{\rm acc} = \frac{1}{2}\dot{m}c^2 \frac{R_{\rm s}}{R} \approx 0.06\ \dot{m}c^2\ \frac{3R_{\rm s}}{R}$$
$$\approx 0.36 \frac{R_{\rm g}}{R}$$

merging BH binaries and GWs

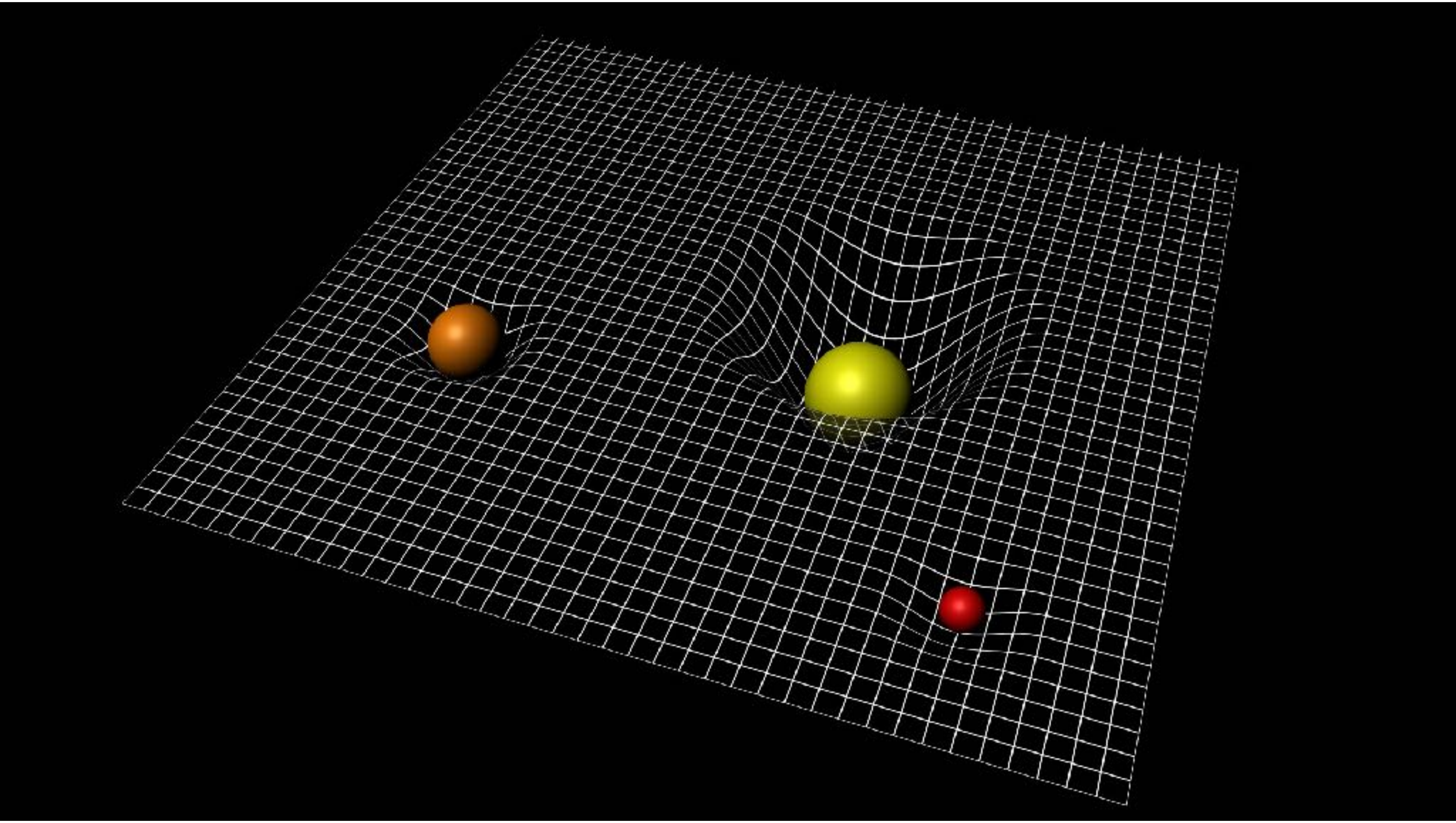
Let’s start with a basic non rigorous introduction to GW



you tube movie “the absurdity of detecting GW”

https://www.youtube.com/watch?v=iphcyNWFD10

From general relativity, gravity can be expressed as space-time curvature caused by the presence of mass

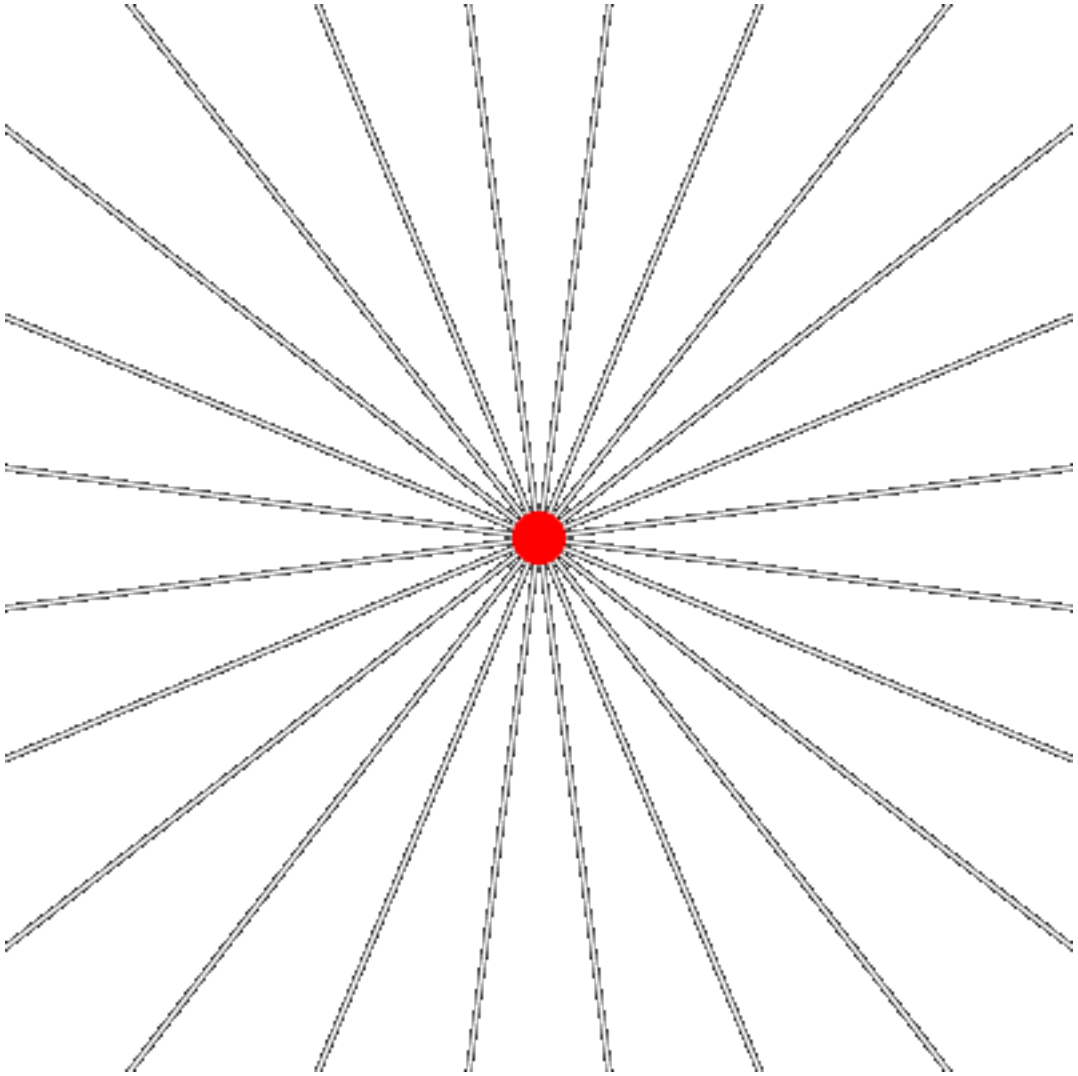


Analogous to electromagnetism:

- accelerated mass distributions will produce ripples in space-time.

Close in we have the field of a moving charge, and farther out we have the field of a stationary charge. Between these two regions is a spherical shell of stretched field lines connecting the two fields. This shell carries the information about the charge's sudden surge of acceleration: it expands at speed c, but has a constant thickness equal to cΔt, where Δt is the duration of the acceleration.
The stretched field lines in this shell are what we call electromagnetic radiation. Two properties are immediately obvious from the diagram:

- The fields in electromagnetic radiation are not radial, but transverse (i.e. perpendicular to the radius).
- Far from the source, the field lines of the radiation are much more tightly packed than the "backgound" of the stationary or uniformly moving source.



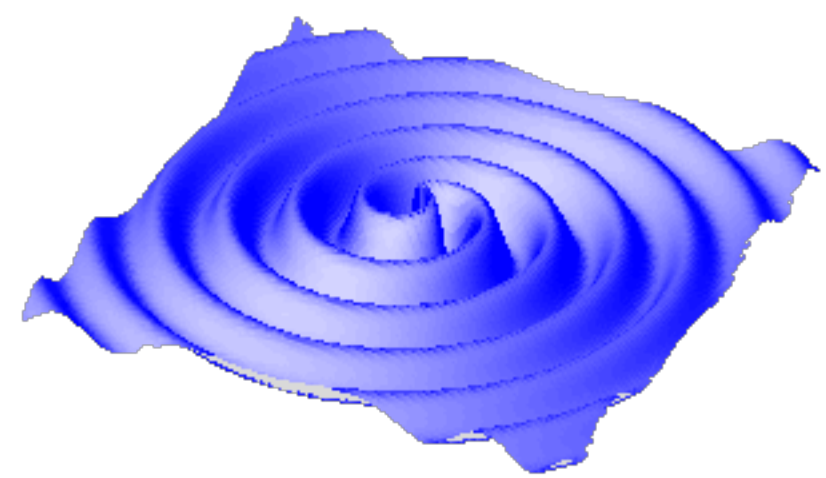
field of an accelerated charge

Analogous to electromagnetism:

- accelerations of mass distributions will produce ripples in space-time.

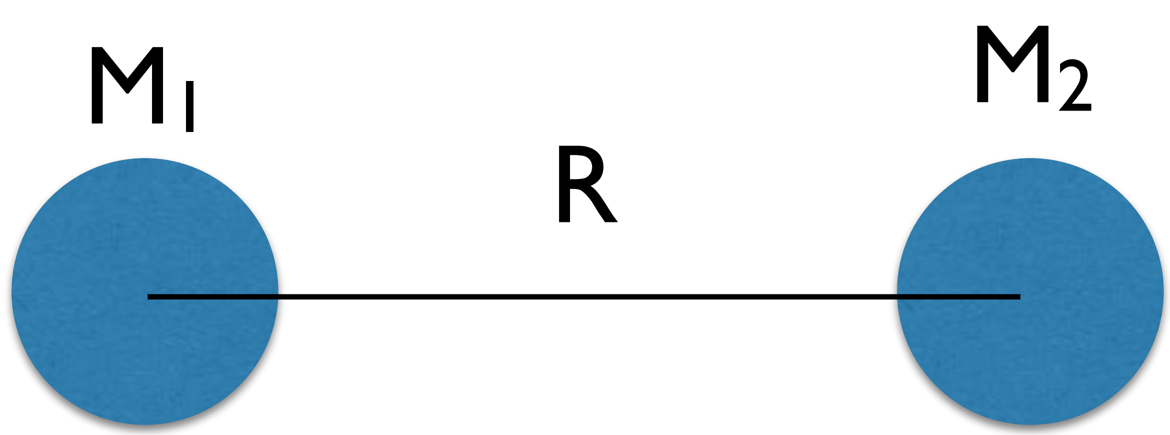
- These ripples are transverse gravitational waves

- They propagate at the speed of light



- They transport energy and angular momentum lost by the source

gravitational wave from a binary

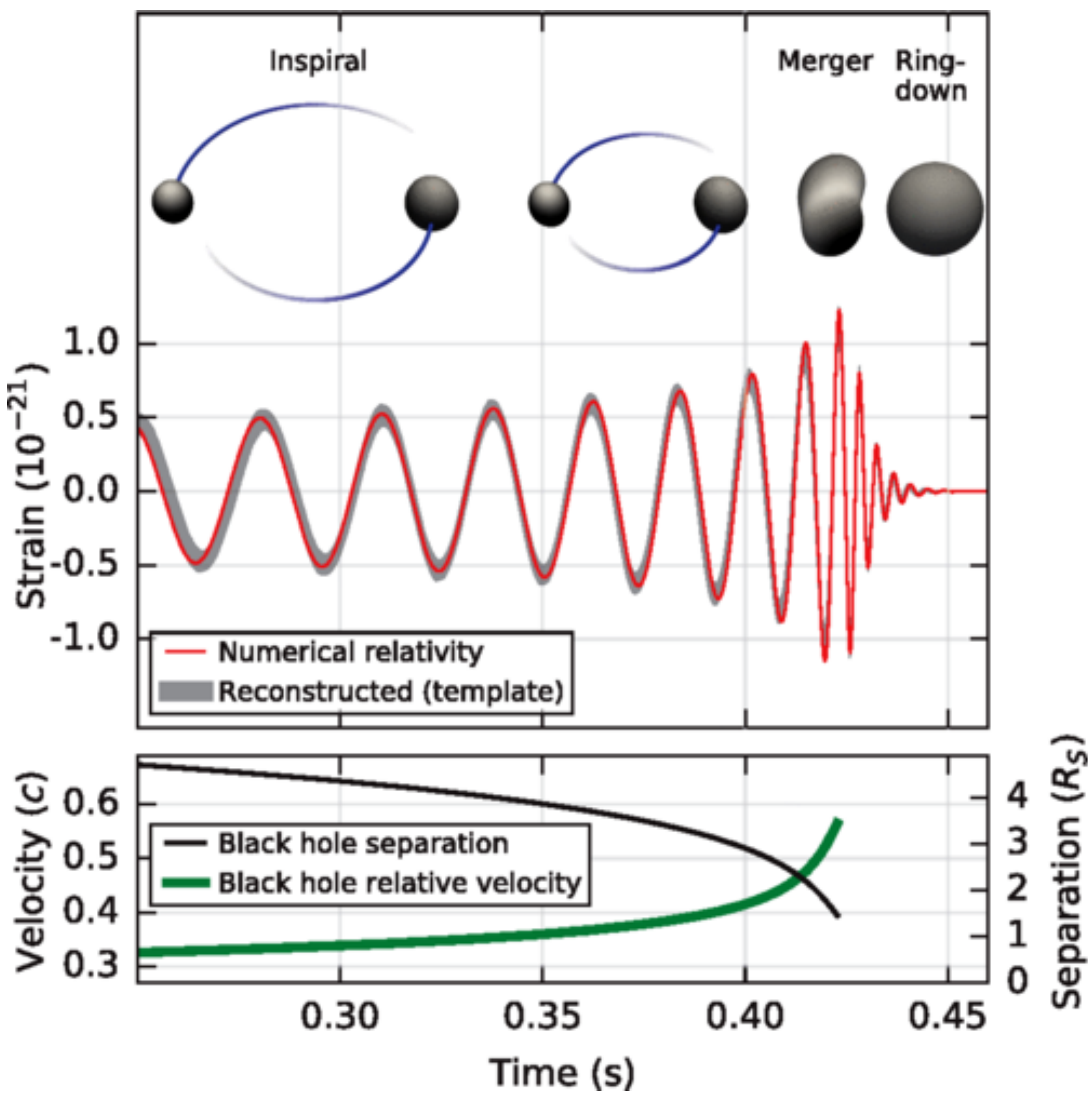


At a distance r from us

recall: $$P = 2\pi\sqrt{\frac{R^3}{G(M_1+M_2)}}$$ $$f_o = \frac{\omega}{2\pi} = 1/P$$

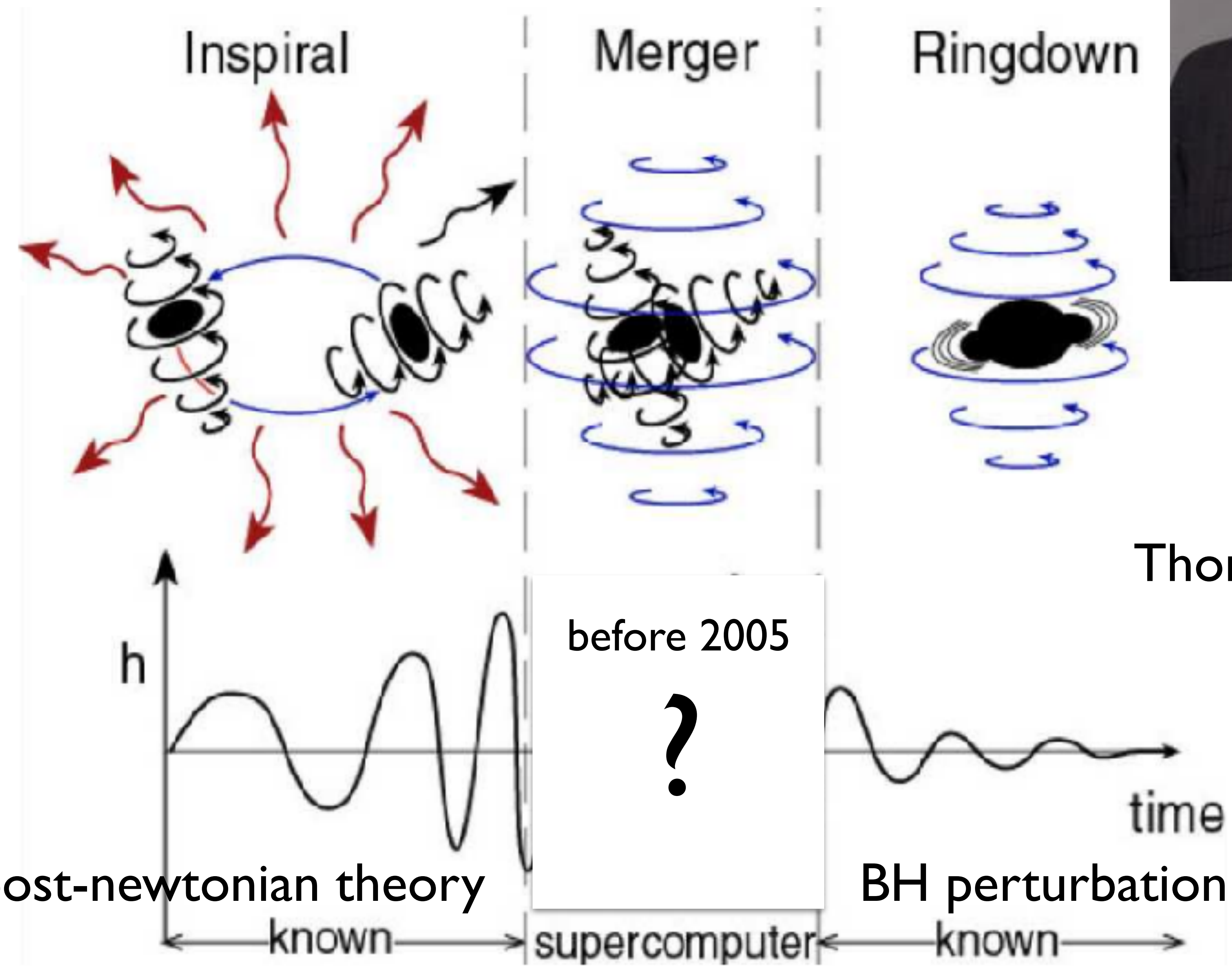
$$M = M_1 + M_2$$

the imprint on the wave of different phases



waveform calculation

F. Pretorious



Thorne's cartoon

before 2005

?

post-newtonian theory

BH perturbation theory

Pretorious 2005 numerical relativity breakthrough: first merger of equal mass BHs

wave properties, an analytical overview

GW amplitude:

The energy carried by any wave is proportional to its amplitude square h^2.
So conservation of energy dictates that the amplitude goes as

$$h \propto 1/r$$

Conservation of mass and momentum implies that (unlike electromagnetism) the lowest order in the mass distribution that can vary with time is the quadrupole: waves depend on the acceleration of the moment of inertia

$$h \sim \frac{G}{c^4}\frac{M\dot{R}^2}{r}$$

adimensional in geometric unit G=c=1

wave properties, an analytical overview

GW frequency:

The characteristic gravitational-wave frequency of a quasi-circular black-hole binary, produced by the dominant quadrupole component, is

$$f_{\rm GW} \approx 2 \times f_{\rm o}$$

GW Luminosity: $L \sim 4\pi r^2 f_{\rm GW}^2 h^2$

adimensional in geometric unit G=c=1

It can be shown that

$$L \sim L_{\rm GW} \left(\frac{GM}{c^2 R} \right)^5$$

remember compactness =1 for a BH!

with the scale factor equal to

$$L_{\rm GW} = c^5/G \approx 3.6 \times 10^{59} \ \mathrm{erg/s} \quad !!!!$$

GW Luminosity:

$$L_{\mathrm{GW}} = c^5/G \approx 3.6 \times 10^{59}\ \mathrm{erg/s} \quad !!!!$$

for comparison:

A Galaxy Milky -Way type: $L_{\mathrm{MW}} \approx 10^{44}\ \mathrm{erg/s}$

All galaxies in visible light in the visible universe emit:

$$L_{\mathrm{visible}} \approx 10^{56}\ \mathrm{erg/s}$$

Exercises

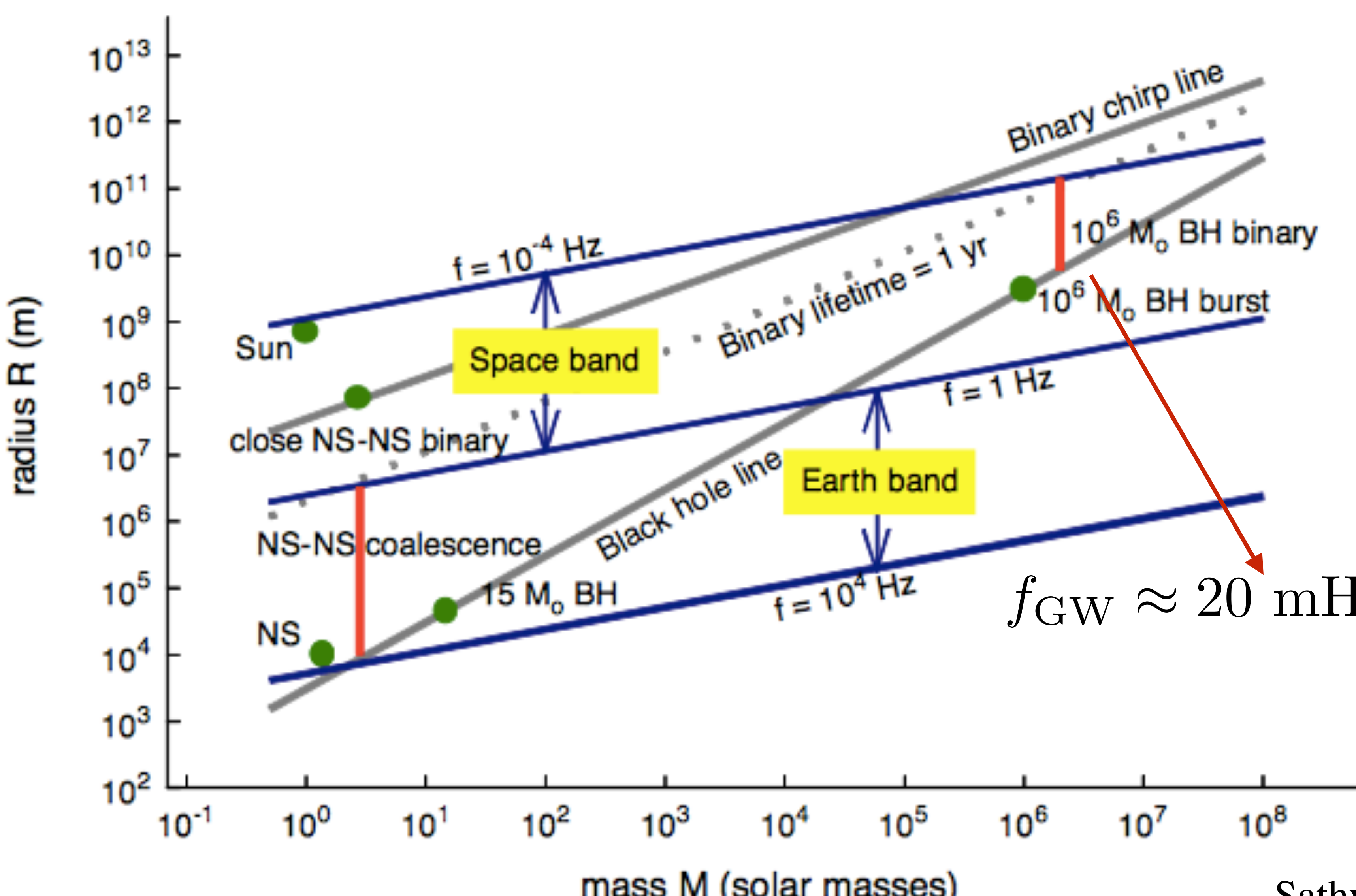
1a) calculate the frequency $f_{\rm GW}$ at merger for black holes as a function of mass M

1b) calculate the frequency $f_{\rm GW}$ for GW150914 i.e. for a 30+30 solar mass BH

Solutions

1a) $f_{\rm GW} \approx 2\times f_{\rm o} = \dfrac{2}{P}$ $\quad f_{\rm GW} \approx \dfrac{1}{\pi}\sqrt{\dfrac{GM}{R^3}}$

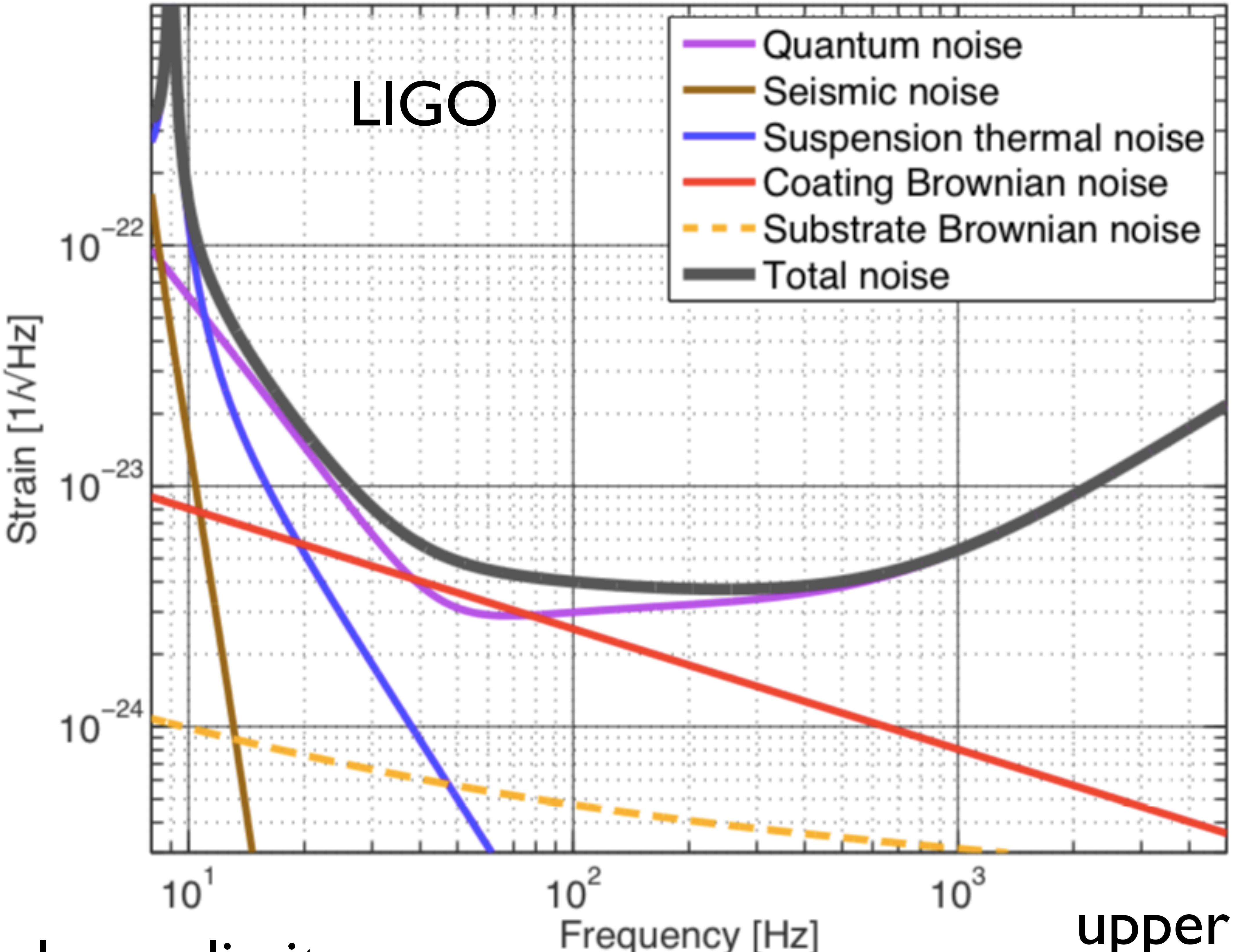
$$At \quad R = R_{\rm s} = 2GM/c^2 \quad f_{\rm GW} \approx 2 \text{ KHz} \left(\frac{20\ M_{\rm sun}}{M}\right)$$



$$f_{\rm GW} \approx 20 \text{ mHz} \left(\frac{2\times 10^6 M_{\rm sun}}{M}\right)$$

LISA !

Sathyaprakash and Schutz 2009



lower limit

upper limit

https://arxiv.org/pdf/1103.2728.pdf

Solutions

1b) $f_{\rm GW} \approx 2 \times f_{\rm o} = \dfrac{2}{P}$ $\qquad$ $f_{\rm GW} \approx \dfrac{1}{\pi}\sqrt{\dfrac{GM}{R^3}}$

$At \quad R = R_{\rm s} = 2GM/c^2$

$$f_{\rm GW} \approx 0.7 \text{ KHz} \left(\frac{60\ M_{\rm sun}}{M}\right)$$

$\boxed{\text{GW150914}}$

In the following let's use for $\boxed{\text{GW150914}}$

$$f_{\rm GW} = 1 \text{ KHz} \left(\frac{60\ M_{\rm sun}}{M}\right)$$

Exercises

1) calculate the frequency $f_{\rm GW}$ at merger for black holes as a function of their mass

2) derive the dependence of h (strain) on $f_{\rm GW}$ and M

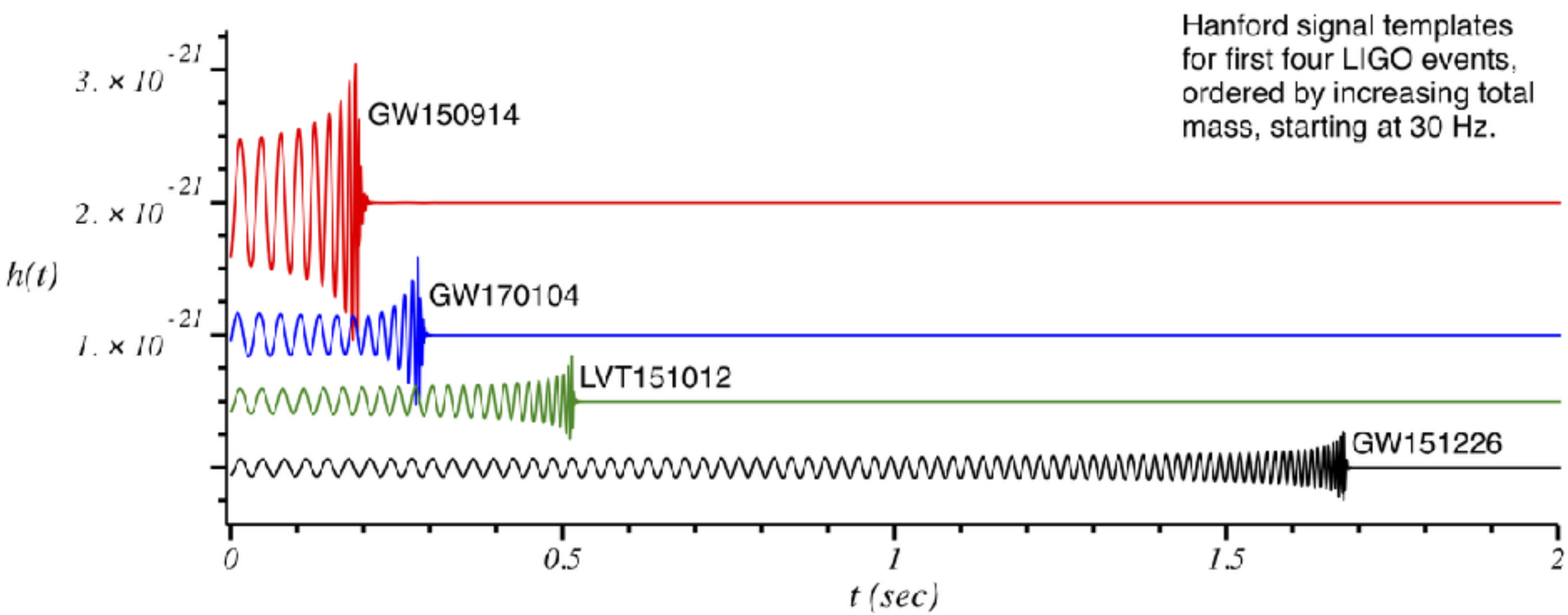
Solutions

2)

$$h \propto \frac{M\dot{R}^2}{r} \propto M f_{\rm GW}^2 R^2$$

$$f_{\rm GW} \propto \left(\frac{M}{R^3}\right)^{1/2}$$

$$\Rightarrow \quad h \sim \frac{G}{c^4} \frac{M^{5/3} f^{2/3}}{r}$$



Exercises

1) calculate the frequency f_{GW} at merger for black holes as a function of their mass

2) derive the dependence of h (strain) on f_{GW} and M

3) derive the a-dimensional strain magnitude h approximately at merger for GW150914:

NOTE: r = 410 Mpc ; M1= 35 Msun , M2= 30 Msun

Solutions

3) $h \sim \dfrac{G}{c^4} \dfrac{M^{5/3} f^{2/3}}{r}$

$$\mathrm{G} = \mathrm{c} = 1 \;;\; \mathrm{M_{sun}} = 1.5 \text{ km};\; 1 \text{ s} = 3 \times 10^5 \text{ km}$$

$$h \approx \frac{(65 \times 1.5 \text{ km})^{5/3} \left(\frac{10^3}{3 \times 10^5 \text{ km}}\right)^{2/3}}{3.6 \times 10^{22} \text{ km}}$$

$$h \approx 10^{-21}$$

3) $h \approx 10^{-21}$

LIGO had to measure dL/L ~ h

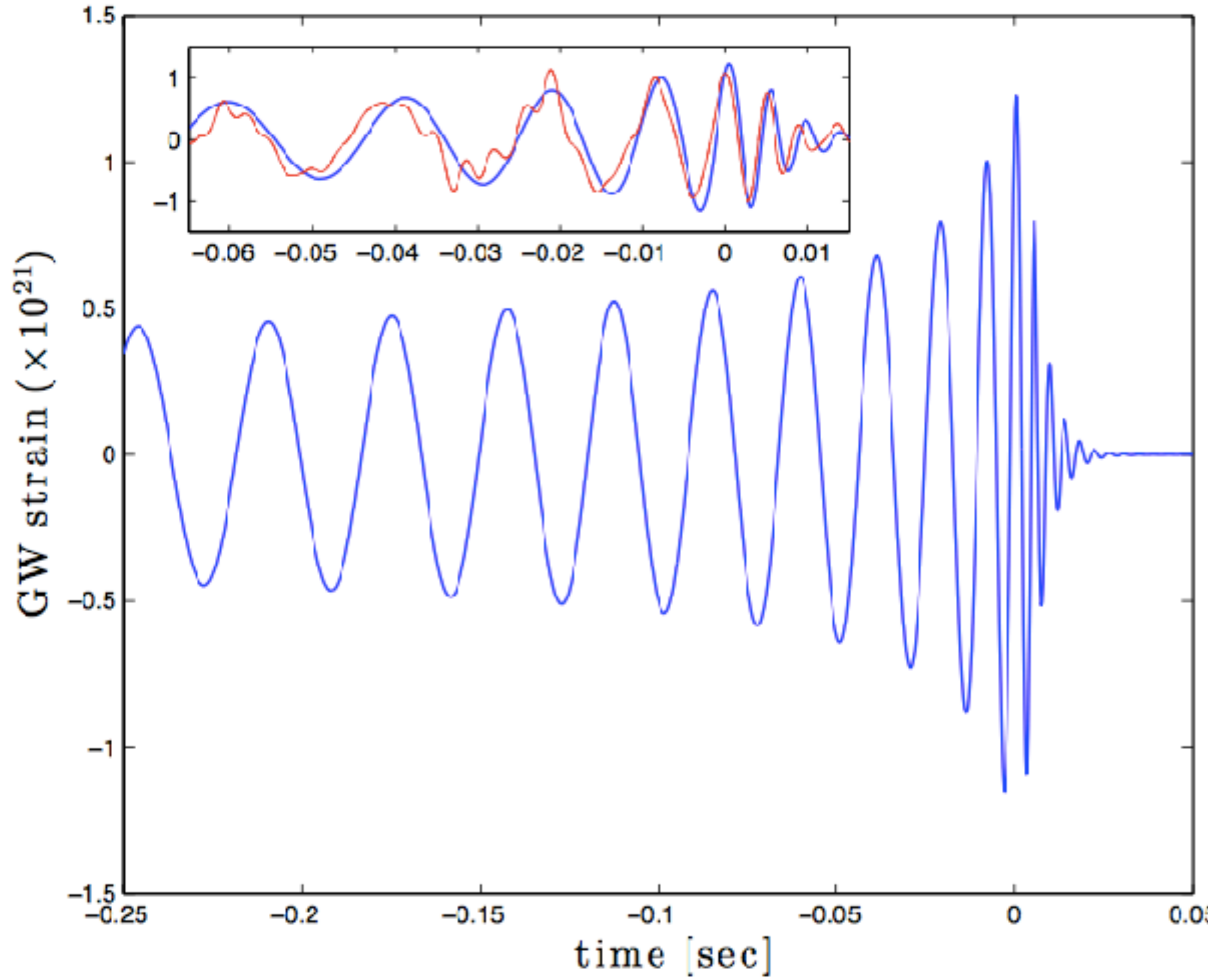


Figure 1: The GW strain h for GW150914 (multiplied by 10^{21}) as a function of the time t in seconds (from the numerical relativity waveform). The time is shifted so that maximum strain is at $t = 0$ s. The inset shows a section of the data in the main figure (in blue), together with the observed H1 data (in red), i.e., the data from the Hanford detector. The features of the L1 data, the data from the Livingston detector, are similar to those of the H1 data.

3) $h \approx 10^{-21}$

NOTE: LIGO and VIRGO
need to measure dL/L ~ h !

Quoting from LIGO website:

LIGO is designed to detect a change in distance between its mirrors *1/10,000th the width of a proton!* This is equivalent to measuring the distance to the nearest star to an accuracy smaller than the width of a human hair!

remember video ?!

Exercises

1) calculate the frequency $f_{\rm GW}$ at merger for black holes as a function of their mass

2) derive the dependence of h (strain) on $f_{\rm GW}$ and M

3) derive the a-dimensional strain magnitude h approximately at merger for GW150914:

NOTE: r = 410 Mpc ; M1= 35 Msun , M2= 30 Msun

4) derive the dependence of the rate of change of frequency $\dot{f}$ on $f_{\rm GW}$ and M and found out why the inspiral phase (+merger) is called the "chirp" part

4) $\dot{f} \propto -\frac{M^{1/2}}{R^{5/2}} \dot{R}$ $\quad\quad \dot{R} =$ coalescence rate

$$L = -\dot{E}_{\mathrm{orbital}}$$

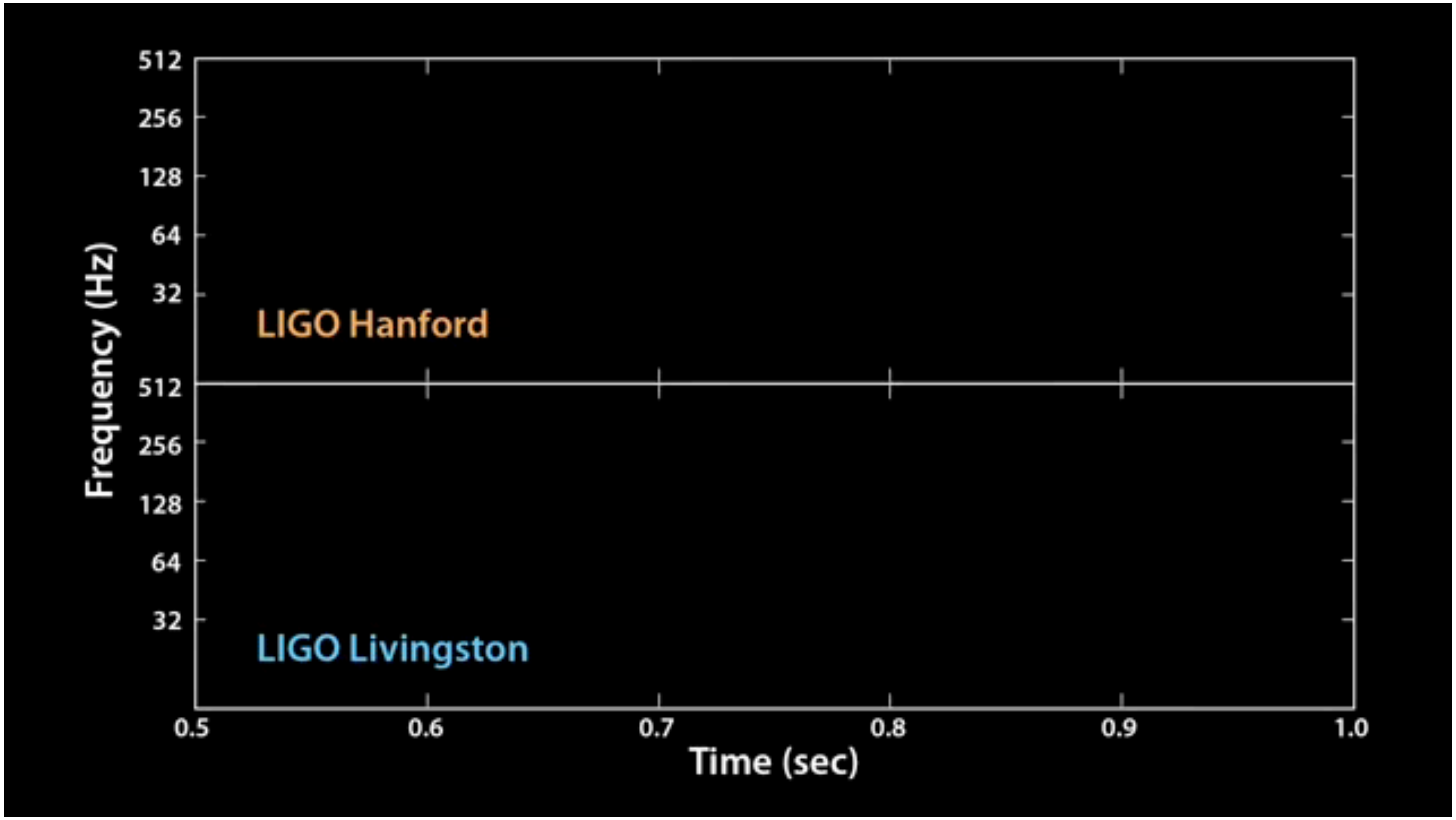
$$L \propto f^2 h^2 \qquad\qquad -\dot{E}_{\mathrm{orbital}} \propto - \left(\frac{M}{R}\right)^2 \dot{R}$$

$$\dot{R} \propto -(M/R)^3$$

$$\dot{f} \propto M^{5/3} f^{11/3}$$

a chirp!

the frequency very steeply increases with time



512
256
128
64
32
LIGO Hanford
512
256
128
64
32
LIGO Livingston
Frequency (Hz)
0.5
0.6
0.7
0.8
0.9
1.0
Time (sec)