The Wave Turbulence Approach to Gravitational Collapse in anti-de Sitter Space

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Over the last few decades there has been a great deal of research relating gravity to the dynamics of fluids. In the weakly turbulent regime fluids can be described using the wave turbulence formulation rather than full-blown Kolmogorov turbulence, which uses vortices. It is with these two ideas in mind that we construct a wave equation describing the gravitational collapse of a scalar field in anti-de Sitter (AdS) space as follows from Einstein's field equations. By developing this approximation we find a kinetic equation with truncated terms for increasingly "complicated" mode interactions. While this formulation was motivated by and has implications in the AdS/CFT correspondence we are more concerned with the implications of this wave turbulence formulation as it relates to gravitational waves in asymptotically flat spacetime. When there is negligible interaction between modes of the wave they act as simple harmonic oscillators, but for time scales on the order of $\sim 1/A^2$ where A is the amplitude of the wave the interactions become important and we simulate the energy cascade from one mode of the wave to the next using numerical techniques. The end result connects general relativity to a fluid description of the massless scalar field in a fully dynamical setting. Our ultimate goal is to determine if the wave turbulence approach to describing gravitational collapse of a massless scalar field in AdS is superior to methods that have been used in the past.

1 Introduction

1.1 Motivation I: Einstein and Navier-Stokes

The Einstein equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{1.1}$$

governs the long-distance behavior of essentially any gravitational system while the Navier-Stokes equation governs the hydrodynamics of most fluids. The connection between general relativity and hydrodynamics is perhaps most evident in that for every solution of the incompressible Navier-Stokes equation there is a "dual" solution in the vacuum Einstein equations. In other words, the near-horizon expansion in gravity is shown to be mathematically equivalent to the hydrodynamic expansion in fluid dynamics [1].

Out of these two universal theories arises an important area of interest in gravitational physics: the thermodynamical and hydrodynamical properties near black holes. A robust review of the topic is provided by [2, 3]; these monographs detail how entropy and quantum effects manifest near the event horizon. Friedel and Yokokura discuss the non-equilibrium thermodynamics of a constructed gravitational screen and find that the screen has the same thermodynamic properties of a viscous bubble [4]. Interestingly enough, the bubble's entropy production is analogous to propagating gravitational waves in the context of general relativity. One attempt in recent years to solve the quantum gravity problem is the anti de-Sitter / conformal field theory (AdS/CFT) correspondence, which Maldacena introduced in [5] (see his lecture notes [6]). AdS/CFT unites the gravity theories in AdS to gauge theories describing quantum phenomena and strings. Rangamani discusses the hydrodynamic description of strongly coupled conformal field theories [7] and highlights the emergence of an effective Navier-Stokes equation, namely

$$\partial_t \vec{u} + (u \cdot \nabla)\vec{u} - \nu\nabla^2\vec{u} = -\nabla w + \vec{g}, \tag{1.2}$$

where $\vec{u}$ is the flow velocity of the fluid. (G.K. Batchelor's textbook on fluid dynamics serves as a good introduction to the topic.) The non-linearity of (1.2) hints at a turbulent system. With this in mind we propose a turbulence equation can be written that can describe the gravitational collapse of a scalar field with the properties of a fluid in AdS.

1.2 Motivation II: Collapse and turbulence as it relates to gravity

In 2016 Rica formulated the longtime evolution of spacetime fluctuations using the language of wave turbulence [8]. The Einstein equations in vacuum imply the Ricci tensor ($R_{\mu\nu}$ in (1.1)) goes to zero, and Rica uses the harmonic gauge $g_{\mu\nu}\Gamma^{\lambda}_{\mu\nu} = 0$. In this weak field limit the metric can be written as $g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}$ where $\eta_{\mu\nu}$ is the Minkowski metric, ϵ is a small parameter, and $h_{\mu\nu}$ is the metric that encapsulates the gravitational wave. By using a normal mode expansion [9] Rica gets an equation that relates the perturbed metric $h_{\mu\nu}$ to the Fourier amplitudes A^s_σ (where σ labels the two polarization states of gravitational waves) and the dispersion relation $\omega_k = ck$. Furthermore, the author extrapolates a second-order differential equation describing the wave amplitudes (assuming to first order the Ricci tensor is zero), and the consequent wave equation. These are listed as equations (1.3),(1.4), and (1.5).

$$h_{\mu\nu} = \sum_{s,\sigma} \int \frac{1}{\sqrt{\omega_k}} A^s_{\vec{k},\sigma}(t) e^{(s)}_{\mu\nu}(\vec{k},\sigma) e^{i\vec{k}\cdot\vec{x}} d^3\vec{k}, \tag{1.3}$$

$$\ddot{A}^s_{\vec{k},\sigma}(t) + \omega_k^2 A^s_{\vec{k},\sigma}(t) = 0, \tag{1.4}$$

$$\dot{A}^s_{\vec{k},\sigma} = is\omega_k A^s_{\vec{k},\sigma} + \text{higher order terms.} \tag{1.5}$$

Through further manipulation the author arrives at an explicit description of the wave interaction amplitude that scales as the square root of the dispersion relation. This formulation is generally what we are trying to do; we start with Einstein's equations in AdS and recognize the gravitational collapse of a massless scalar field.

An approach to collapse of a scalar field in AdS was proposed in [10]; the authors adopt Eddington-Finkelstein coordinates and use perturbation theory to formulate collapse. They find a wave propagates radially inward to form a black brane in $d + 1$ dimensions. However, their motivation and conclusions are different from ours. Wave turbulence is not used, and the purpose is to probe the AdS/CFT correspondence. For spherically symmetric gravitational collapse in asymptotically AdS space the system is turbulent and can be described using a power law [11]. In this paper the authors find that there is a weakly nonlinear time followed by a devolution of the power spectrum of the Ricci scalar with frequency ω as ω^{-s} with $s \approx 1.7 \pm 0.1$. Our hope is to formulate a wave equation that describes collapse in this weakly turbulent regime.

The motivation to use spectral methods and nonlinear dynamical systems is elaborated on in [12]; using spectral methods allows us to analyze the evolution of individual modes of the wave before a singularity forms. This is evident in our formulation of the wave equation as we approximate the Einstein equations Φ and Π (introduced in section 3.1) as an appropriate series where the temporal basis functions are the unknown modes. The rescaling of the spatial dimension allows for smoothness at the origin and spatial infinity, in addition to a finite value of the mass-energy. The surprising result was that only a small perturbation of the initial scalar field distribution will lead to collapse, and is in part the motivation for this investigation into the mechanism with which collapse occurs; it has recently been suggested that the instability of asymptotically AdS spacetimes proceed through a turbulent mechanism [13, 14].

In [15] the authors find that for a small initial amplitude of the scalar field the field oscillates in a way consistent with geodesics in AdS, and for larger oscillations black holes will form. They also analyzed the process of thermalization numerically and found that it occurs rapidly for a variety of initial scalar field amplitude perturbations and resulting black hole masses, given that the amplitude is large enough. Black hole formation is interpreted on the field theory side as thermalization and the results of [15] indicate that thermalization is occurring as fast as possible while still being compatible with causality. Jałmużna et al. extend the claims made [15] as they found this weakly turbulent instability in AdS in all dimensions $d+1$ for $d \geq 3$ [16].

1.3 What is wave turbulence?

Wave turbulence pertains to aspects of turbulence that can be captured by the wave approximation. Wave turbulence is generally defined as out-of-equilibrium statistical mechanics of random non-linear waves. There have been several monographs written on the subject [17–20]. The textbook written by Nazarenko provides an excellent description of wave turbulence and current unsolved problems in physics to which it is applicable. Turbulence is best described as an energy flux through scales, and the provided energy spectrum is

$$E^{(3D)}(\vec{k}) = \frac{1}{2}\int_{\mathbb{R}^3} \langle \vec{u}(\vec{x}) \cdot \vec{u}(\vec{x}+\vec{r})\rangle e^{-i\vec{k}\cdot\vec{r}} \frac{d\vec{r}}{(2\pi)^3}. \tag{1.6}$$

If we assume isotropic turbulence this reduces to a more simple form, where we can integrate the spectrum over the 3D $\vec{k}$ space,

$$\frac{1}{2}\langle u^2\rangle = \int_{\mathbb{R}^3} E^{(3D)}(\vec{k})d\vec{k} = \int_0^\infty E^{(1D)}(k)dk. \tag{1.7}$$

A distinct property of wave turbulence is that the primary ingredient is a propagating wave rather than a hydrodynamic vortex. To illustrate how difficult the analytic description of turbulence can be, a quote from Werner Heisenberg: "When I meet God, I am going to ask him two questions: why relativity and why turbulence? I really

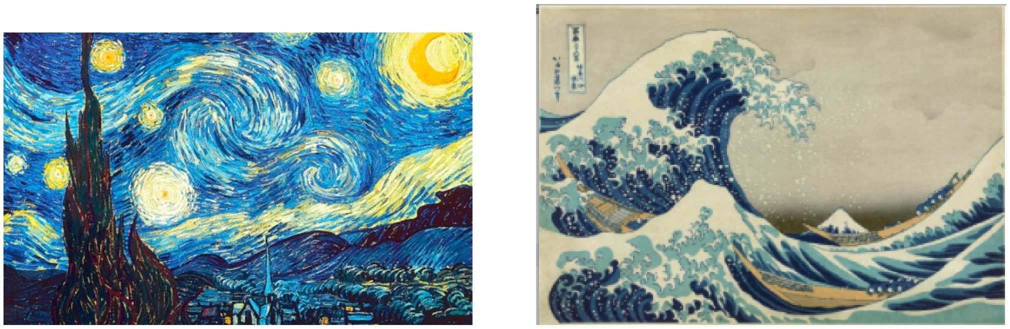


Figure 1. Van Gogh's *De Sterrennacht*, which depicts turbulent flow with remarkable accuracy. These particular eddies were later found to be consistent with Kolmogorov's formulation of turbulence (see [21] for more details.) Hokusai's *The Great Wave off Kanagawa* drove physicists to wonder if the turbulent flow indicated a tsunami or a rogue wave (it turned out to be the latter, see [24].)

believe he will have an answer for the first. [21]" Russian mathematician Andrey Kolmogorov found that vortices strongly interact only when their spatial extensions are of the same order, giving rise of the energy cascade from large eddies to smaller ones [22].

Nazarenko's book only provides a passing mention of collapse being analyzed using wave turbulence but it is in the context of the nonlinear Schrödinger model rather than gravity (see [23] for an application of this equation to Bose-Einstein condensates).

Shortly after the Nazarenko book was published Bizoń, Rostworowski, and Jałmużna wrote several papers on the weakly turbulent instability of AdS space and the collapse of a scalar field in AdS in higher dimensions. The equations needed to start our formulation (described in section 2.1) are from [13], where the authors determine that AdS is unstable under arbitrarily small generic perturbations. Further investigation provides evidence that the turbulence is too weak to produce a naked singularity and is therefore in agreement with the cosmic censorship hypothesis. Other investigations into collapse in AdS have been successful as well. Craps et. al. studied the collapse of a scalar field in AdS by using a resummation method identical to the renormalization group that fixes ultraviolet divergencies in perturbative quantum field theory; see [25, 26] for more details. Using what is called the two-time perturbative formalism the authors of [27] solve the problem of gravitational collapse in AdS by finding a class of quasiperiodic solutions that accurately describe the cascade of energy between modes.

1.4 Why wave turbulence precisely?

The steady-state assumption in turbulent flows is achieved in gravitational collapse through a separation of scales. Given the connections between Einstein equation and the Navier-Stokes equation that has been evolving for over three decades [28] it is necessary to pursue a description of collapse from the perspective of turbulent fluid dynamics. In particular, we have indications that the phenomena we are witnessing in gravitational collapse is more along the lines of wave turbulence [17, 18]. Given that waves are simpler than vortices mathematically and the relevant energy scales in this thesis it is a reasonable conclusion that we should use wave turbulence in our formulation.

1.5 What is AdS space and why we chose it

Bengtsson provides an excellent overview of AdS space [30], which is considerably different from the familiar Minkowski space and therefore deserves some introduction. AdS_n is written as the quadric

$$X_1^2 + \cdots + X_n^2 + X_{n+1}^2 - U^2 - V^2 = -1 \tag{1.8}$$

embedded in a flat $n+1$ dimensional space with the following metric

$$ds^2 = dX_1^2 + \cdots + dX_{n-1}^2 - dU^2 - dV^2. \tag{1.9}$$

$X_1^2 + \cdots + X_n^2 - X_{n+1}^2$ is an n-dimensional one sheeted hyperboloid embedded in a $n+1$-space. AdS_2, for example, is a two dimensional hyperboloid of one sheet embedded in a three-dimensional Minkowski space, which is shown in figure 2. This metric lends itself to the usage of hyperbolic geometry which then sheds light on the conformal boundary of the space (which is what we are interested in). Due to the negative curvature of the space we can solve Einstein's equations

$$R_{\alpha\beta} = \lambda g_{\alpha\beta} \tag{1.10}$$

with a negative cosmological constant. Using "sausage" coordinates [30], similar to the more familiar spherical coordinates with a temporal dimension tacked on,

$$d\hat{s}^2 = -dt^2 + \frac{4}{(1+\rho^2)^2}(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2). \tag{1.11}$$

This unphysical spacetime metric is a conformal subset related to the Einstein static universe. For asymptotically AdS space the Weyl tensor, which is in a sense a measure of the tidal forces felt at any point in a manifold, vanishes at infinity.

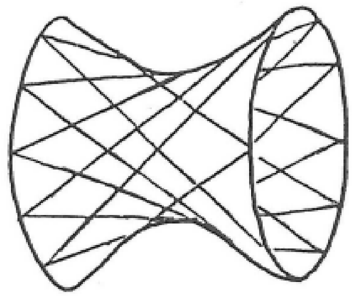


Figure 2. A visualization of null geodesics in AdS_2 courtesy of [30] and Christopher Wren.

Rangamani's lecture notes on the AdS/CFT correspondence also elaborate on asymptotically AdS spaces for any number of dimensions. The line element for a Schwarzschild black hole in AdS_{d+1} is

$$ds^2 = -r^2 f(br)dt^2 + \frac{dr^2}{r^2 f(br)} + r^2 \delta_{ij} dx^i dx^j, \tag{1.12}$$

$$\text{with} \quad f(r) = 1 - \frac{1}{r^d}, \tag{1.13}$$

where b is inversely proportional to the temperature of the black hole. This gives rise to a stress tensor that is an ideal conformal fluid stress tensor that corresponds to global thermal equilibrium.

1.6 Gravitational waves in asymptotically flat space

Albert Einstein's theory of general relativity revolutionized our understanding of the universe and served as a marked improvement on Newtonian mechanics. General relativity explained unsolved problems (the precession of the perihelion of Mercury) and predicted unforeseen phenomena (bending of starlight around the Sun) [31]. There were other parts of the theory that would have to wait to be confirmed by future generations of physicists like the existence of black holes. In September of 2015 the LIGO collaboration confirmed directly one of the more peculiar relics of Einstein's general relativity, gravitational waves, and in the process provided the strongest confirmation yet of GR. A good explanation of the formulation for propagation of gravitational waves is provided by [32]. In the weak field limit we can approximate the metric as Minkowski plus a small correction:

$$g_{ab} = \eta_{ab} + \epsilon h_{ab}. \tag{1.14}$$

We must introduce a variable ψ_{ab} and rewrite the Ricci scalar in terms of this new variable:

$$\psi_{ab} \equiv h_{ab} - \frac{1}{2}\eta_{ab} h, \tag{1.15}$$

$$R = \frac{1}{2}\epsilon(2\psi^{cd}{}_{,cd} - \Box h). \tag{1.16}$$

When subjected to the d'Alembert operator the perturbing metric (which encapsulates the gravitational wave) must be zero in this linearized regime. Using the Einstein gauge (where the field equations read $\Box\psi_{ab} = 0$), we find that the equation of motion leads to the following wave equation:

$$\Box\, h_{ab} = 0 \tag{1.17}$$

$$\rightarrow h_{ab} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & h_{22} & h_{23} \\ 0 & 0 & h_{23} & -h_{22} \end{bmatrix}. \tag{1.18}$$

This can be decomposed into + and $\times$ polarizations (mentioned in section 1.2); gravitational waves are typically in a superposition of these two polarization states as they propagate in a direction parallel to the stretching and squeezing of spacetime. The Rica paper [8] illustrates that the language of wave turbulence can be used to discern the behavior of gravitational waves in asymptotically flat space.

2 Collapse in Asymptotically $(d+1)$-dimensional AdS Spacetimes

2.1 The effective action, length element, and Einstein field equations

In this formulation we consider the dynamics of a massless scalar field φ in $d+1$ dimensions that is described by the following effective action,

$$S = \int d^{d+1}x\, \sqrt{-g}\Big(\frac{1}{16\pi G}(R - \Lambda) - \frac{1}{2}(\partial\varphi)^2\Big), \tag{2.1}$$

where G is Newton's constant and R is the Ricci scalar. The cosmological constant Λ is negative, as is the case for anti de Sitter space. If we set the cosmological constant equal to zero the metric would reduce to flat Minkowski space, and if its positive then we get a metric that describes de Sitter space. We are concerned with spherically symmetric configurations and assume the following ansatz for the metric:

$$ds^2 = \sec^2\left(\frac{x}{l}\right)\left(- Ae^{-2\delta}dt^2 + A^{-1}dx^2 + l^2\sin^2\left(\frac{x}{l}\right)d\Omega^2_{d-1}\right), \tag{2.2}$$

where $d\Omega$ is the differential displacement along the unit $d-1$ sphere.

By introducing auxiliary variables $\Phi \equiv \varphi'$ and $\Pi \equiv A^{-1}e^{\delta}\dot{\varphi}$ (where primes denote spatial derivatives and dots denote temporal ones) we can rewrite Einstein's field equations in the following way:

$$A' = \frac{1+2\sin^2(x)}{\sin(x)\cos(x)}(1-A) - A\sin(x)\cos(x)(\Phi^2+\Pi^2), \tag{2.3}$$

$$\delta' = -\sin(x)\cos(x)(\Phi^2+\Pi^2), \tag{2.4}$$

$$\dot{\Phi} = (Ae^{-\delta}\Pi)', \tag{2.5}$$

$$\dot{\Pi} = \frac{1}{\tan^2 x}\left(Ae^{-\delta}\Phi\tan^2 x\right)'. \tag{2.6}$$

Each of these four equations is dependent on t and x and $l^2 = -\frac{d(d-1)}{2\Lambda}$. Note that this constrains x to the range $0 < x < \frac{\pi}{2}$. The spherically symmetric solutions in vacuum can be achieved by setting the scalar field φ equal to zero and fixing the Einstein vacuum equations A and δ equal to 1 and 0, respectively. A and δ constrain the scalar field, while the equation for Π is a re-derivation of the Klein-Gordon equation $g^{\mu\nu}\nabla_\mu(\partial_\nu\varphi) = 0$. Π is like momentum so the right hand side of (2.6) acts as a force term.

2.2 The field equations and natural mass function

Here we introduce the ADM formalism to general relativity ([33] uses asymptotically flat spacetime in its formulation but the general principles are still applicable). There is a generalization of the ADM mass function $m(x,t)$ for AdS,

$$1 - \frac{2m}{r^{d-2}} + \frac{r^2}{l^2} = g^{\alpha\beta}\ \partial_\alpha r\ \partial_\beta r, \tag{2.7}$$

where r is a standard spherical coordinate related to the spatial AdS coordinate, x, in the following way:

$$r = l\tan(x/l). \tag{2.8}$$

This formalism is applicable to spacetimes that asymptotically approach a well-defined metric tensor (AdS in our investigation, Minkowski in others.) In this formulation spacetime is a family of spacelike surfaces and on each slice there exists a metric tensor describing the spatial coordinates and conjugate momenta. The ADM mass is all of the mass enclosed in the space and is therefore related to gravity at spatial infinity:

$$\lim_{x\to\frac{\pi}{2}} m(x,t) = M_{ADM}. \tag{2.9}$$

This formalism is a Hamiltonian formulation, rather than the familiar Lagrangian formulation, of general relativity. In numerical relativity it is more prudent to use the Hamiltonian formulation because there is a set of first order, rather than the more complicated second order, differential equations to be solved.

3 A Kinetic Equation for AdS Gravitational Collapse

The AdS limit of the metric (2.2) corresponds to $A = 1$ and $\delta = 0$. To find a scalar field φ that satisfies the Einstein equations (2.3), (2.4), (2.5), and (2.6) we can write the scalar field as a superposition of eigenmodes [13].

$$\varphi(t,x) = \sum_{j=0}^{\infty} a_j \cos(\omega_j t + \beta_j) e_j(x), \tag{3.1}$$

$$\ddot{\varphi} = \sum_{j=0}^{\infty} -\omega_j^2 a_j \cos(\omega_j t + \beta_j) e_j(x). \tag{3.2}$$

To first order the Sturm-Liouville operator $\hat{L}$ will provide an explicit form of the eigenmodes. The first order approximation of the scalar field is denoted by φ_1.

$$\hat{L}\varphi_1 = \sum_{j=0}^{\infty} a_j \cos(\omega_j t + \beta_j)\Bigg(\frac{-1}{\tan^2(x)}\partial_x\Big(\tan^2(x)\frac{de_j(x)}{dx}\Big)\Bigg), \tag{3.3}$$

$$\hat{L}\varphi_1 = \sum_{j=0}^{\infty} a_j \cos(\omega_j t + \beta_j)\Bigg(-\frac{d^2 e_j(x)}{dx^2} - 2\frac{\sec^2(x)}{\tan(x)}\frac{de_j(x)}{dx}\Bigg), \tag{3.4}$$

$$0 = \sum_{j=0}^{\infty} a_j \cos(\omega_j t + \beta_j)\Bigg(-\frac{d^2 e_j(x)}{dx^2} - 2\frac{\sec^2(x)}{\tan(x)}\frac{de_j(x)}{dx} - \omega_j^2 e_j(x)\Bigg), \tag{3.5}$$

$$\to 0 = \frac{d^2 e_j(x)}{dx^2} + 2\frac{\sec^2(x)}{\tan(x)}\frac{de_j(x)}{dx} + \omega_j^2 e_j(x). \tag{3.6}$$

Using a computer algebra program we confirm the form of the equations in [13] that show the eigenvalues and eigenfunctions (”oscillons”) of $\hat{L}$ are

$$\omega_j^2 = (3+2j)^2, \tag{3.7}$$

$$e_j(x) = d_j \cos^3(x)\ {}_2F_1\Big(-j, 3+j, \frac{3}{2}; \sin^2 x\Big), \tag{3.8}$$

$$d_j = \sqrt{\frac{16(j+1)(j+2)}{\pi}}. \tag{3.9}$$

The inner product in this particular Hilbert space between two oscillons of different modes is $\int_0^{\pi/2} e_j(x)e_k(x)\tan^2 x dx$. The operator $\hat{L}$ is Hermitian so there is an orthonormality condition involving the normalized oscillons available that will be of the utmost importance later,

$$\int_0^{\pi/2} e_j(x)e_k(x)\tan^2 x dx = \delta_{jk}, \tag{3.10}$$

where δ_{jk} is the familiar Kronecker delta. While difficult to prove using the explicit form of the oscillons it is a relatively easy matter to confirm this condition using a computer algebra program and testing the value for different modes.

3.1 Deriving more useful forms of A and δ

A more useful form of δ can be recovered through direct integration:

$$
\begin{aligned}
d\delta(t,x) &= -\sin(x)\cos(x)(\Phi^2+\Pi^2)dx, && (3.11)\\
dx &\equiv dy, && (3.12)\\
\rightarrow d\delta &= -\sin(y)\cos(y)(\Phi^2+\Pi^2)dy, && (3.13)\\
\delta(t,x) &= \delta_1 - \int_0^x \sin(y)\cos(y)(\Phi^2+\Pi^2)dy. && (3.14)
\end{aligned}
$$

A more useful form of A can be found using an integrating factor:

$$
\begin{aligned}
A' + p(x)A &= q(x), && (3.15)\\
I(x) &\equiv e^{\int p(x)dx}, && (3.16)\\
\frac{d}{dx}(I(x)A) &= I(x)A' + I(x)p(x)A = I(x)q(x), && (3.17)\\
\rightarrow A &= \frac{1}{I(x)}\left(C_1 + \int I(x)q(x)dx\right). && (3.18)
\end{aligned}
$$

Let us simplify this a bit:

$$
\begin{aligned}
\int_{y_0}^x \frac{-1}{\sin(y)\cos(y)}dy &= \int_{y_0}^x \frac{-1}{2\sin(2y)}d(2y), && (3.19)\\
&= \ln(\cot(x)) - \ln(\cot(y_0)), && (3.20)\\
\exp\left(\int_{y_0}^x \frac{-1}{\sin(y)\cos(y)}dy\right) &= \frac{\cot x}{\cot y_0}, \int_{y_0}^x \frac{-2\sin^2(y)}{\sin(y)\cos(y)}dy = 2\log(\cos(y))\Big|_{y_0}^x. && (3.21)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\exp\left(-\int_{y_0}^x \frac{1+2\sin^2(y)}{\sin(y)\cos(y)}dy\right) &= \frac{\cot(x)\cos^2(x)}{\cot(y_0)\cos^2(y_0)}, && (3.22)\\
\frac{\exp\left(-\int_{y_0}^z \frac{1+2\sin^2(y)}{\sin(y)\cos(y)}dy\right)}{\cos z \sin z} &= \frac{\sec^4(z)}{\cot(y_0)\cos^2(y_0)}. && (3.23)
\end{aligned}
$$

Now we have

$$
\begin{aligned}
A = \frac{1}{\cot^2(y_0)\cos^4(y_0)} \cos^2(x)\cot(x)\exp\left(-\int_{y_0}^x (\Phi^2+\Pi^2)\sin(y)\cos(y)dy\right) \quad (3.24)\\
\times\left(C_1 + \int_{z_0}^x \exp\left(-\int_{y_0}^x (\Phi^2+\Pi^2)\sin(y)\cos(y)dy\right)\sec^4(z)(1+2\sin^2(z)dz)\right).
\end{aligned}
$$

Combining terms (and assuming boundary conditions $\delta_1 = C_1 = 0$, $\frac{1}{\cot^2(y_0)\cos^4(y_0)} = 1$), we recover a new form of $Ae^{-\delta}$:

$$\nu(z) \equiv \sec^4 z(1 + 2\sin^2 z), \tag{3.25}$$
$$Ae^{-\delta} = \cos^2(x)\cot x \int_{z_0}^{x} \exp\left(\int_{y_0}^{z} \sin(y)\cos(y)(\Phi^2 + \Pi^2)dy\right)\nu(z)dz. \tag{3.26}$$

3.2 Deriving more useful forms of Π and Φ

A central goal in the formulation of wave turbulence is finding the kinetic equation that governs the flow of the turbulent fluid (which is the scalar field φ in this case). To do so to we have to rewrite Φ and Π,

$$\dot{\Phi} = \frac{d}{dt}(\varphi') = \frac{d}{dt}(Ae^{-\delta}\Pi)', \tag{3.27}$$
$$\dot{\Pi} = \frac{1}{\tan^2(x)}\left(Ae^{-\delta}\Phi\tan^2(x)\right)', \tag{3.28}$$

in terms of nonlinear dynamics. We can write both of these equations as an infinite series (see section 1.2):

$$\Pi = \sum_{j=0}^{\infty} a_j(t)e_j(x), \tag{3.29}$$
$$\Phi = \sum_{j=0}^{\infty} b_j(t)g_j(x). \tag{3.30}$$

The set of g_j eigenfunctions form an orthonormal basis with the following properties:

$$g_j \equiv \frac{e_j'}{3+2j}, \tag{3.31}$$
$$\frac{d}{dx}\left(\frac{1}{\tan^2(x)}\frac{d}{dx}(\tan^2(x)g_j(x))\right) + (3+2j)^2 g_j = 0. \tag{3.32}$$

This e_j term is like a potential term and its spatial derivative, g_j, is like a force term. We want to find the time derivatives of the temporal functions used in the infinite series. For $b_j(t)$,

$$\int_0^{\pi/2} (Ae^{-\delta}\Pi)' g_j(x) \tan^2(x) dx = \int_0^{\pi/2} g_j \dot{\Phi} \tan^2(x) dx, \tag{3.33}$$

$$= \sum_{k=0}^{\infty} \dot{b_k}(t) \int_0^{\pi/2} \tan^2(x) g_j(x) \Big(\sum_{k=0}^{\infty} g_k(x) \Big) dx. \tag{3.34}$$

Remembering the orthonormality condition from earlier,

$$\delta_{jk} \sum_{k=0}^{\infty} \dot{b}_k(t) = \int_0^{\pi/2} (Ae^{-\delta}\Pi)' g_j dx, \tag{3.35}$$

$$\to \dot{b}_j(t) = \int_0^{\pi/2} (Ae^{-\delta}\Pi)' g_j(x) \tan^2(x) dx. \tag{3.36}$$

For $a_j(t)$ we can do something similar:

$$\int_0^{\pi/2} (Ae^{-\delta}\Phi \tan^2 x)' e_j(x) dx = \int_0^{\pi/2} \Big(\sum_{k=0}^{\infty} \dot{a_k}(t) e_k(x) \Big) e_j(x) \tan^2 x \ dx, \tag{3.37}$$

$$= \Big(\sum_{k=0}^{\infty} \dot{a_k}(t) \Big) \delta_{jk}, \tag{3.38}$$

$$\to \dot{a_j}(t) = \int_0^{\pi/2} (Ae^{-\delta}\Phi \tan^2 x)' e_j(x) dx. \tag{3.39}$$

3.3 Moving toward a kinetic equation

By integrating over all AdS we ensure independence from the spatial coordinate for a and b. These are more useful forms of each eigenfunction that when summed over all modes make up Π and Φ and their first derivatives. We introduce a complex wave equation that incorporates the temporal functions and is therefore only dependent on time.

$$B_j^s \equiv \frac{a_j + isb_j}{\sqrt{3+2j}}. \tag{3.40}$$

It is important to note that the spectral parameter s can only have the values $\pm$ 1. Differentiating with respect to time we get

$$\dot{B_j^s} = \frac{1}{\sqrt{3+2j}} \int_0^{\pi/2} \tan^2 x \Big((Ae^{-\delta}\Phi)' e_j(x) + is(Ae^{-\delta}\Pi)' g_j(x) \Big) dx. \tag{3.41}$$

This form of the equation is not particularly useful so we need to rearrange the integrand so there are no derivative terms (i.e. Φ' or Π').

$$\int_0^{\pi/2} (Ae^{-\delta}\Phi\tan^2(x))' e_j dx = e_j Ae^{-\delta}\Phi\tan^2 x\Big|_0^{\pi/2} - \int_0^{\pi/2} Ae^{-\delta}\tan^2 x\Phi e_j' dx, \tag{3.42}$$

$$\int_0^{\pi/2} is(Ae^{-\delta}\Pi)' g_j \tan^2 x dx = is\left[g_j\tan^2 x Ae^{-\delta}\Pi\Big|_0^{\pi/2} - \int_0^{\pi/2}(Ae^{-\delta})(g_j\tan^2 x)' dx\right], \tag{3.43}$$

$$(g_j\tan^2 x)' = -(3+2j)e_j\tan^2(x). \tag{3.44}$$

After plugging in the evaluations of these integrals the time derivative of the wave can be written in the following way:

$$\dot{B}_j^s = -\sqrt{3+2j}\int_0^{\pi/2} Ae^{-\delta}\tan^2 x(\Phi g_j - is\Pi e_j)dx. \tag{3.45}$$

We need a way to write Π and Φ as a sum over terms that are a product of an oscillon and our new wave equation.

$$\Pi = \sum_{j=0}^{\infty} a_j(t)e_j(x), \tag{3.46}$$

$$\sqrt{3+2j}B_j^s = a_j + isb_j, \tag{3.47}$$

$$\sum_s \sqrt{3+2j}B_j^s = (a_j + ib_j) + (a_j - ib_j), \tag{3.48}$$

$$\Pi = \frac{1}{2}\sum_{j=0}^{\infty}\sum_s \sqrt{3+2j}B_j^s e_j(x), \tag{3.49}$$

$$\Phi = \sum_{j=0}^{\infty} b_j(t) g_j(x), \tag{3.50}$$

$$\sqrt{3+2j}\, s B_j^s = s a_j + i s^2 b_j, \tag{3.51}$$

$$\sum_s \sqrt{3+2j}\, s B_j^s = 2 i b_j, \tag{3.52}$$

$$\Phi = -\frac{i}{2} \sum_{j=0}^{\infty} \sum_s s \sqrt{3+2j} B_j^s g_j(x). \tag{3.53}$$

We can write the new equation as a sum over modes rather than as an integral that involves Π and Φ (as it is written in equation (3.41)).

$$\dot{B}_s^j = s \frac{i}{2} \sum_{j_1=0}^{\infty} \sum_{s_1} \sqrt{3+2j} \sqrt{3+2j_1} \int_0^{\pi/2} A e^{-\delta} (e_j e_{j_1} + s s_1 g_j g_{j_1}) dx B_{j_1}^{s_1}. \tag{3.54}$$

We want to Taylor expand the exponential function inside $Ae^{-\delta}$, as $Ae^{-\delta}$ is nested inside the integral that determines $\dot{B}_s^j$. This is the method with which we truncate our wave equation.

$$Ae^{-\delta} = \cos^2(x) \cot x \int_{z_0}^{x} (1 + \int_{y_0}^{z} \sin(y)\cos(y)(\Phi^2 + \Pi^2) dy + \dots) \nu(z) dz. \tag{3.55}$$

A change in variables $\xi(y) \equiv (\Phi^2 + \Pi^2) \sin y \cos y$ shortens the form of the equation in such a way that we can neatly provide more orders of the expansion:

$$Ae^{-\delta} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^x \Big(\int_0^z \xi dy \Big)^n \nu(z) dz, \tag{3.56}$$

$$\begin{aligned} Ae^{-\delta} &= 1 + \cos^2 x \cot x \int_0^x \Big(\int_0^z \xi dy \Big) \nu(z) dz, \\ &+ \frac{1}{2} \cos^2 x \cot x \int_0^x \Big(\int_0^z \xi dy \Big)^2 \nu(z) dz + \dots. \end{aligned} \tag{3.57}$$

To make this computation easier let's introduce a new function F that will make equation (3.54) a little more palatable:

$$
\begin{aligned}
F^{s_1 s_2}_{j_1 j_2}(y) &= \frac{1}{4}\sqrt{3+2j_1}\sqrt{3+2j_2}(e_{j_1}e_{j_2} - s_1 s_2 g_{j_1} g_{j_2}) \sin y \cos y, && (3.58)\\
\Pi^2 &= \frac{1}{4}\sum_{j_1 j_2}\sum_{s_1 s_2} \sqrt{3+2j_1}\sqrt{3+2j_2} B^{s_1}_{j_1} B^{s_2}_{j_2} e_{j_1} e_{j_2}, && (3.59)\\
\Phi^2 &= -\frac{1}{4}\sum_{j_1 j_2}\sum_{s_1 s_2} s_1 s_2 \sqrt{3+2j_1}\sqrt{3+2j_2} B^{s_1}_{j_1} B^{s_2}_{j_2} g_{j_1} g_{j_2}, && (3.60)\\
\xi(y) &= F^{s_1 s_2}_{j_1 j_2} B^{s_1}_{j_1} B^{s_2}_{j_2}. && (3.61)
\end{aligned}
$$

At this point we need to introduce a new index $m \equiv 2n$,

$$
\begin{aligned}
Ae^{-\delta} &= 1 + \sum_{n=1}^{N} \frac{1}{n!} \sum\sum \int_0^x \cos^2 x \cot x && (3.62)\\
&\times \Big(\Big(\prod_{m=2}^{M} \int_0^z F^{j_{m-1} j_m}_{s_{m-1} s_m} dy \Big) \nu(z) dz \Big) \Big(\prod_{m=2}^{M} B^{j_{m-1}}_{s_{m-1}} B^{j_m}_{s_m} \Big) dx,
\end{aligned}
$$

where the index m is positive even integers greater than 2 and $M = 2N$. For $N = 2$,

$$
\begin{aligned}
Ae^{-\delta} &= 1 + \sum_{j_1 j_2}\sum_{s_1 s_2} \int_0^x \cos^2 x \cot x \Big(\int_0^z F^{s_1 s_2}_{j_1 j_2}(y) dy \Big) \nu(z) dz \; B^{s_1}_{j_1} B^{s_2}_{j_2}, && (3.63)\\
&+ \frac{1}{2} \sum_{j_1 j_2 j_3 j_4} \sum_{s_1 s_2 s_3 s_4} \int_0^x \cos^2 x \cot x \\
&\times \Big(\int_0^z F^{s_1 s_2}_{j_1 j_2}(y) dy \Big) \Big(\int_0^z F^{s_3 s_4}_{j_3 j_4}(y) dy \Big) \nu(z) dz \; B^{s_1}_{j_1} B^{s_2}_{j_2} B^{s_3}_{j_3} B^{s_4}_{j_4}.
\end{aligned}
$$

3.3.1 First order of the kinetic equation

To first order $Ae^{-\delta} = 1$ so we can proceed with finding the first term in the kinetic equation:

$$
s\frac{i}{2} \sum_{j_1}\sum_{s_1} \sqrt{3+2j}\sqrt{3+2j_1} \int_0^{\pi/2} (1) \tan^2 x (e_j e_{j_1} + s s_1 g_j g_{j_1}) dx B^{s_1}_{j_1}. \qquad (3.64)
$$

Using the orthonormality condition involving the inner product of oscillons, this can be simplified.

$$= is \sum_{j_1} \sum_{s_1} \frac{1}{2} \sqrt{3+2j}\sqrt{3+2j_1}(\delta_{jj_1} + ss_1\delta_{jj_1}) B_{j_1}^{s_1}, \tag{3.65}$$

$$= is(3+2j)B_s^j. \tag{3.66}$$

3.3.2 The final kinetic equation extended to Nth order

We now introduce what is called the resonance manifold, or a coefficient that describes how the modes of our wave interact with each other.

$$\begin{aligned}
\Gamma_{jj_1\ldots j_{m-1}}^{-ss_1\ldots s_{m-1}} &\equiv \frac{1}{n!} \int_0^{\pi/2} F_{jj_1}^{-ss_1}(x), \qquad (3.67)\\
&\times \left(\int_0^x \nu(z) dz \Big(\prod_{m=4}^{M} \int_0^z F_{j_{m-2}j_{m-3}}^{s_{m-2}s_{m-3}}(y) dy \Big) dx \right),\\
F_{jj_1}^{-ss_1}(x) &= \frac{1}{4}\sqrt{3+2j}\sqrt{3+2j_1}(e_j e_{j_1} + ss_1 g_j g_{j_1}) \sin x \cos x, \qquad (3.68)\\
\tan^2 x \cos^2 x \cot x &= \sin x \cos x. \qquad (3.69)
\end{aligned}$$

The higher order terms of the kinetic equation then take on the following form:

$$\begin{aligned}
&= is \sum \sum \sqrt{3+2j_1}\sqrt{3+2j} \qquad (3.70)\\
&\times \frac{1}{2\times 4} \int_0^{\pi/2} \Big(\int_0^x \Big(\prod_{m=4}^{M} \int_0^z F_{j_{m-3}j_{m-2}}^{s_{m-3}s_{m-2}}(y) dy \Big) \nu(z) dz \ \Big)\\
&\times \sin x \cos x (e_j e_{j_1} + ss_1 g_j g_{j_1}) dx B_{j_1}^{s_1} \ldots B_{s_{m-1}}^{j_{m-1}}.
\end{aligned}$$

The product index is even integers greater than 4 and M is the order desired multiplied by 2. It is apparent that a pattern continues for higher order terms such that there for an nth order expansion there are $2n-1$ B's and wave interaction coefficients with $2n-1$ covariant and contravariant modes. This is because for each higher order truncation of $Ae^{-\delta}$ we introduce 2 new B's and one new $(\int F)$. The $(n!)$ term cancels out the one introduced by Γ. The explicit form after $N = 1$ is

$$i\Big(\frac{n!}{8}\Big)s \sum_{j_1\ldots j_{m-1}} \sum_{s_1\ldots s_{m-1}} \Gamma_{jj_1\ldots j_{m-1}}^{-ss_1\ldots s_{m-1}} B_{j_1}^{s_1} \ldots B_{j_{m-1}}^{s_{m-1}}. \tag{3.71}$$

Therefore, the kinetic equation to Nth order is

$$\dot{B}_j^s = is(3+2j)B_s^j + \frac{1}{8}\sum_{n=2}^{N} i(n!)s \sum_{j_1\cdots j_{m-1}} \sum_{s_1\cdots s_{m-1}} \Gamma_{jj_1\cdots j_{m-1}}^{-ss_1\ldots s_{m-1}} B_{j_1}^{s_1}\ldots B_{j_{m-1}}^{s_{m-1}}. \tag{3.72}$$

For $N=4$ (higher order expansions will be difficult to analyze numerically),

$$\begin{aligned}
\dot{B}_j^s &= is(3+2j)B_s^j + \frac{is}{4}\sum_{j_1j_2j_3}\sum_{s_1s_2s_3} \Gamma_{jj_1j_2j_3}^{-ss_1s_2s_3} B_{j_1}^{s_1}B_{j_2}^{s_2}B_{j_3}^{s_3} \qquad (3.73)\\
&+ \frac{i3s}{4}\sum_{j_1j_2j_3j_4j_5}\sum_{s_1s_2s_3s_4s_5} \Gamma_{jj_1j_2j_3j_4j_5}^{-ss_1s_2s_3s_4s_5} B_{j_1}^{s_1}B_{j_2}^{s_2}B_{j_3}^{s_3}B_{j_4}^{s_4}B_{j_5}^{s_5}\\
&+ i3s\sum_{j_1j_2j_3j_4j_5j_6j_7}\sum_{s_1s_2s_3s_4s_5s_6s_7} \Gamma_{jj_1j_2j_3j_4j_5j_6j_7}^{-ss_1s_2s_3s_4s_5s_6s_7} B_{j_1}^{s_1}B_{j_2}^{s_2}B_{j_3}^{s_3}B_{j_4}^{s_4}B_{j_5}^{s_5}B_{j_6}^{s_6}B_{j_7}^{s_7}.
\end{aligned}$$

In future work we will be using the resonance manifold for the first three non-free orders of the expansion, which can be written explicitly in the following way:

$$\begin{aligned}
\Gamma_{jj_1j_2j_3}^{-ss_1s_2s_3} &= \frac{1}{2}\int_0^{\pi/2} F_{jj_1}^{-ss_1}(x)\Big(\int_0^x \nu(z)dz\Big)\Big(\int_0^z F_{j_2j_3}^{s_2s_3}(y)dy\Big)dx, \qquad (3.74)\\
\Gamma_{jj_1j_2j_3j_4j_5}^{-ss_1s_2s_3s_4s_5} &= \frac{1}{6}\int_0^{\pi/2} F_{jj_1}^{-ss_1}(x)\Big(\int_0^x \nu(z)dz\Big)\Big(\int_0^z F_{j_2j_3}^{s_2s_3}(y)dy\Big) \qquad (3.75)\\
&\times\Big(\int_0^z F_{j_4j_5}^{s_4s_5}(y)dy\Big)dx,\\
\Gamma_{jj_1j_2j_3j_4j_5j_6j_7}^{-ss_1s_2s_3s_4s_5s_6s_7} &= \frac{1}{24}\int_0^{\pi/2} F_{jj_1}^{-ss_1}(x)\Big(\int_0^x \nu(z)dz\Big)\Big(\int_0^z F_{j_2j_3}^{s_2s_3}(y)dy\Big) \qquad (3.76)\\
&\times\Big(\int_0^z F_{j_4j_5}^{s_4s_5}(y)dy\Big)\Big(\int_0^z F_{j_6j_7}^{s_6s_7}(y)dy\Big)dx.
\end{aligned}$$

In practice summing to N orders is not feasible because the wave is only weakly turbulent and higher orders of the resonance manifold encapsulate unaccessible modes. We now have developed a kinetic equation that can be used to evaluate the evolution of the wave that proceeds through mode interactions. In the next section we elaborate on the numerical techniques that can be used.

4 Numerical Simulations

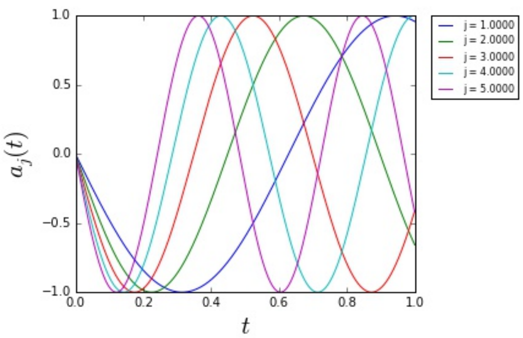
4.1 Simulations of the kinetic equation: the free limit

The free limit simply means that all wave interaction amplitudes are equal to zero and

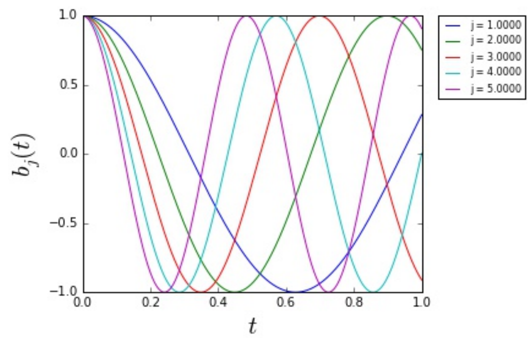
$$\dot{B}_s^j = is(3+2j)B_s^j, \tag{4.1}$$

see figures 3 and 4 for plots of the temporal functions and B_s^j in this limit.

Figure 3. Free limit temporal eigenfunctions, first few modes.



(a) $a_j(t)$ in the free limit.



(b) $b_j(t)$ in the free limit.

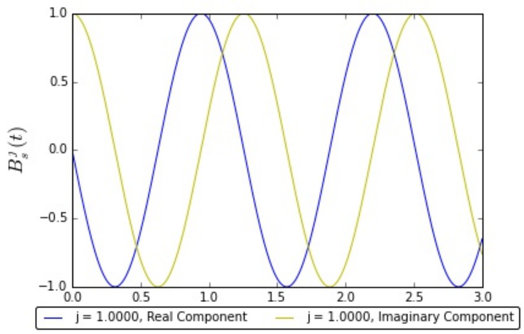
4.2 Simulations of the kinetic equation: truncation

The first step in this process is to find the wave interaction amplitudes. Given that s is constrained to two values the resonance manifold for given order N will have $16N^4$ terms. First we plot $e_j(x)$ and $g_j(x)$ (whose analytic form was computed using Mathematica) for the first few modes in figure 5. Next we must compute the function F given a certain set of modes (see equation (3.58)) and the plot of different F's for $N = 3$ is shown in figure 6. Our goal is to find the resonance manifold using (3.74), (3.75), (3.76); further investigation will deal with possibly higher modes but for now we will limit ourselves to $N = 2$. The end goal of this research, but beyond the scope of this thesis, is to simulate our kinetic equation using the array of wave interaction amplitudes being generated. Using Richardson's integration routine we found the 256 wave interaction amplitudes associated with $N = 2$. While most were about 0, 20 were between -25 and 15, and 4 were on the order of 10^6. It makes sense that most were very small; modes of the wave only interact with similarly energetic modes (as is typical of turbulent systems). Each of these four Γ's occurred when $j_2, j_3 = 2$. See figure 7 for more details.

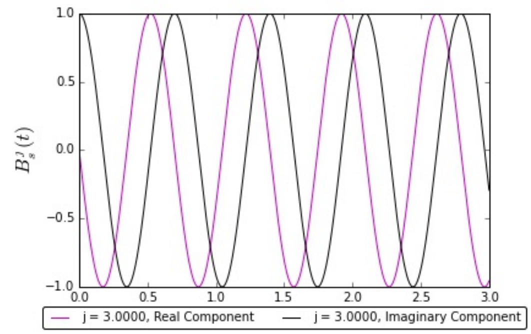
5 Discussion

In this thesis we started with the tangible idea that there is a connection between gravity and the behavior of fluids. Using the wave turbulence formulation in the regime

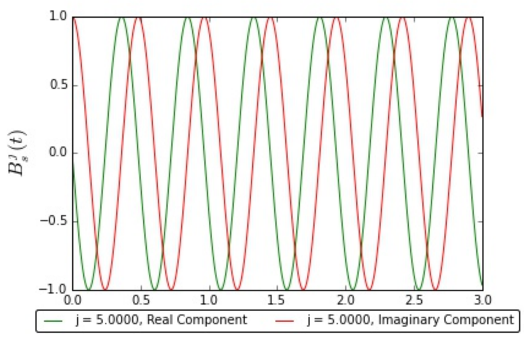
Figure 4. B_j^s with no interaction between modes for four different modes.



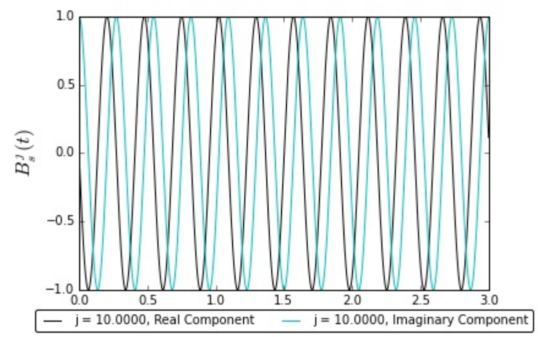
(a) $j = 1$



(b) $j = 3$

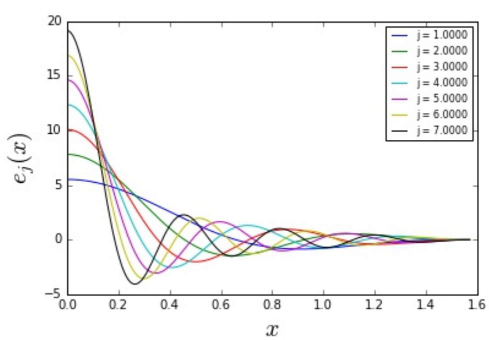


(c) $j = 5$

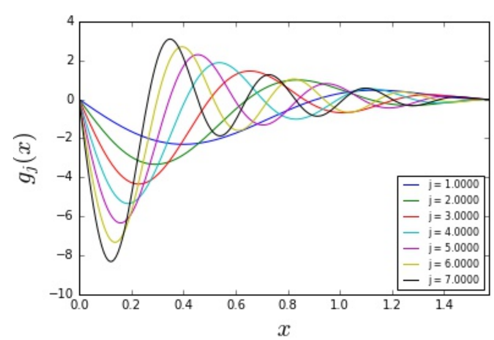


(d) $j = 10$

Figure 5. First few oscillon modes ($j = 1 \dots 7$).



(a) $e_j(x)$ for the first 7 modes.



(b) $g_j(x)$ for the first 7 modes.

described in section 2 we were able to formally construct a kinetic wave equation truncated to any desired order but a statistical foundation is still lacking. After a time where the free limit is an appropriate approximation the resonance manifold becomes

Figure 6. $F(s_1, s_2, j_1, j_2, x)$ from $j = 1$ to $j = 3$.

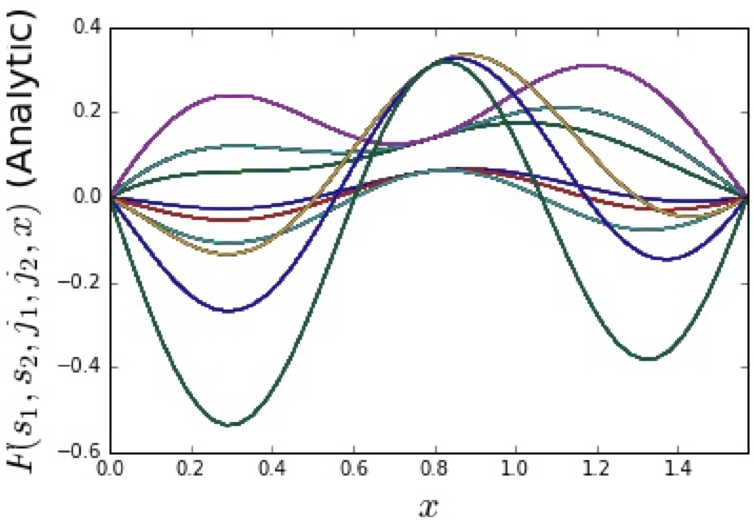
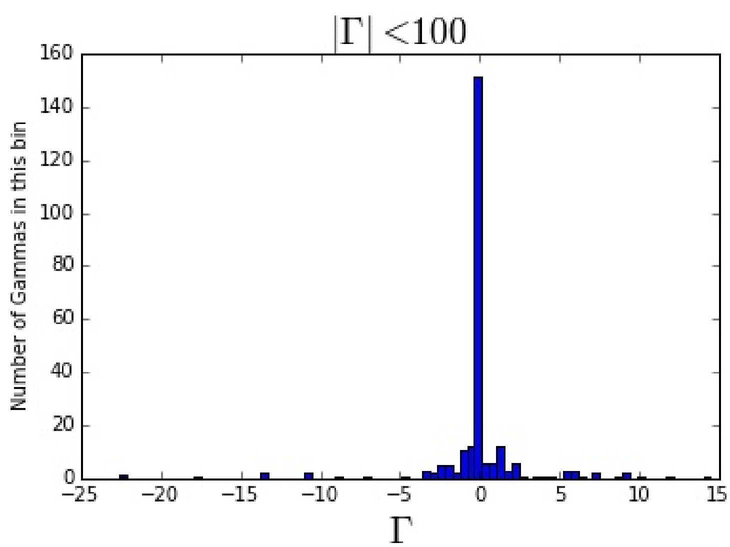


Figure 7. While unsurprisingly most wave interaction amplitudes were zero there were 4 out of 256 that were on the order of a million (neglected in this histogram, might be attributable to numerical routine that was written to compute these values.) Turbulent systems are self-similar so modes of the wave will only interact with modes equipped with similar energy.



increasingly important as modes of the wave interact with each other. We lay the foundations for a program aimed at recasting gravity into fluid dynamics, ultimately paving the way to identifying the precise content of the Einstein field equations as a form of the Navier-Stokes equation.

6 Prospects for Future Research: Wave Turbulence → Numerical Collapse

Sections 2 through 4 lay the foundations for further research, namely simulating the collapse of a massless scalar field in AdS using the methods of wave turbulence. The existence of a Kolmogorov-Zakharov spectrum in a particular regime has been established by numerically studying gravitational collapse. Can this spectrum be reproduced in the framework of wave turbulence where the spectrum is viewed as a stationary solution of the wave equation? Will wave turbulence improve upon the two-time formalism (elaborated on in section 1.3)? Choptuik scaling is a universal property of many gravitational systems near the threshold of black hole formation. Can the Choptuik spacetimes be better understood in the context of wave turbulence? Given that the critical exponent has the same value in asymptotically flat space and in asymptotically AdS spaces, we expect the critical solution to be a solution of the kinetic equation with some particular property, most naturally discrete self-similarity. Is the Choptuik scaling of universality at the threshold of black hole formation a result of wave turbulence in the gravity equations? Recall that Choptuik scaling is a mechanism by which the threshold of black hole formation has been shown to be universal in asymptotically AdS spacetimes. We propose to cast the problem of gravitational collapse in asymptotically AdS spacetimes in the appropriate language of wave turbulence and to use these powerful methods to better describe the process.

A Appendix I: Properties of the metric using Maple

Recall from section 2.1 that A and δ are the Einstein field equations that constrain the scalar field. Using the tensor package available in Maple we were able to find the Ricci scalar, Ricci tensor, and nonzero Riemann tensor components for the spacetime endowed with the metric (2.2) in 2+1 dimensions. The explicit form of these equations are difficult to recover (see (3.63)) so we simply write them as functions of x and t. To run this program for $d-1$ dimensions we would replace the differential displacement along the 1-sphere ($d\theta^2$) with the with the appropriate $d\Omega^2_{d-1}$ term. Using the ADM formalism (see sections 6 and 7 of [33] for details) we could use the Ricci scalar to solve the Hamiltonian and recover the equations of motion through this spacetime. This calculation, however, is beyond the scope of this thesis. The resulting output from our Maple program is attached as an appendix.

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> #Metric for d=2

> $with(tensor):$

> $coord := [t, x, \theta]$

$$coord := [t, x, \theta] \tag{1}$$

> $g := array(symmetric, sparse, 1..3, 1..3):$

> $g[1,1] := \left(\sec\left(\frac{x}{l}\right)^2\right)\left(-A(x,t)\cdot e^{-2\,\delta(x,t)}\right); g[2,2] := \left(\frac{1}{A(x,t)}\right)\sec\left(\frac{x}{l}\right)^2$

$$g_{1,1} := \sec\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)^2$$

$$g_{2,2} := \frac{\sec\left(\frac{x}{l}\right)^2}{A(x,t)} \tag{2}$$

> $g[3,3] := \left(\sec\left(\frac{x}{l}\right)^2\right)\cdot l^2\cdot\left(\sin\left(\frac{x}{l}\right)^2\right)$

$$g_{3,3} := \sec\left(\frac{x}{l}\right)^2 l^2 \sin\left(\frac{x}{l}\right)^2 \tag{3}$$

> $metric := create([-1,-1], eval(g));$

$$metric := table\left(\left[compts\right.\right. \tag{4}$$

$$= \begin{bmatrix} \sec\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)^2 & 0 & 0 \\ 0 & \frac{\sec\left(\frac{x}{l}\right)^2}{A(x,t)} & 0 \\ 0 & 0 & \sec\left(\frac{x}{l}\right)^2 l^2 \sin\left(\frac{x}{l}\right)^2 \end{bmatrix}, index\_char = [$$

$$-1, -1]\,]\,])$$

```
> tensorsGR(coord, metric,'contra_metric','det_met','C1','C2','Rm','Rc','R','G','C') :
> displayGR(Ricciscalar, R);
```

The Ricci Scalar

$$
\begin{aligned}
R = {} & \frac{1}{2}\, \frac{1}{A(x,t)^2\, l^2 \left(\cos\left(\frac{x}{l}\right)^2 - 1\right)} \Big(-4 \cos\left(\frac{x}{l}\right)^4 \left(\frac{\partial^2}{\partial x^2} A(x,t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right) \big(-A(x, \\
& t)\, e^{-2\,\delta(x,t)}\big) \cos\left(\frac{x}{l}\right) \big(-A(x,t)\, e^{-2\,\delta(x,t)}\big)^2 \sec\left(\frac{x}{l}\right) \big(-A(x,t)\, e^{-2\,\delta(x,t)}\big) A(x, \\
& t)^3\, e^{-2\,\delta(x,t)}\, l^2 + 4 \cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial^2}{\partial x^2} A(x,t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right) \big(-A(x, \\
& t)\, e^{-2\,\delta(x,t)}\big) \cos\left(\frac{x}{l}\right) \big(-A(x,t)\, e^{-2\,\delta(x,t)}\big)^2 \sec\left(\frac{x}{l}\right) \big(-A(x,t)\, e^{-2\,\delta(x,t)}\big) A(x, \\
& t)^3\, e^{-2\,\delta(x,t)}\, l^2 + 2 \left(\frac{\partial^2}{\partial t^2} A(x,t)\right) \cos\left(\frac{x}{l}\right) \big(-A(x,t)\, e^{-2\,\delta(x,t)}\big)^2 A(x,t)\, l^2 \\
& + 4 \cos\left(\frac{x}{l}\right)^4 \left(\frac{\partial}{\partial x} A(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right) \big(-A(x,t)\, e^{-2\,\delta(x,t)}\big)^2 \cos\left(\frac{x}{l}\right) \big(-A(x, \\
& t)\, e^{-2\,\delta(x,t)}\big)^2 A(x,t)^3 \left(e^{-2\,\delta(x,t)}\right)^2 l^2 - 4 \cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial}{\partial x} A(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right) \big(\\
& -A(x,t)\, e^{-2\,\delta(x,t)}\big)^2 \cos\left(\frac{x}{l}\right) \big(-A(x,t)\, e^{-2\,\delta(x,t)}\big)^2 A(x,t)^3 \left(e^{-2\,\delta(x,t)}\right)^2 l^2 \\
& + 2 \left(\frac{\partial}{\partial t} A(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right) \big(-A(x,t)\, e^{-2\,\delta(x,t)}\big) \cos\left(\frac{x}{l}\right) \big(-A(x, \\
& t)\, e^{-2\,\delta(x,t)}\big)^4 \sec\left(\frac{x}{l}\right) \big(-A(x,t)\, e^{-2\,\delta(x,t)}\big) A(x,t)\, e^{-2\,\delta(x,t)}\, l^2 - 2 \cos\left(\frac{x}{l}\right)^2 \Big(\frac{\partial^2}{\partial t^2} A(x, \\
& t)\Big) \cos\left(\frac{x}{l}\right) \big(-A(x,t)\, e^{-2\,\delta(x,t)}\big)^2 A(x,t)\, l^2 - 4 \ln(e) \left(\frac{\partial}{\partial t} A(x,t)\right) \Big(\frac{\partial}{\partial t} \delta(x,
\end{aligned}
\tag{5}
$$

$$t)\Big) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right) \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)^4 \sec\left(\frac{x}{l}\right)\Big(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}\Big)\, A(x,t)^2\, e^{-2\,\delta(x,t)}\, l^2 + 16\ln(e)\cos\left(\frac{x}{l}\right)^4 \left(\frac{\partial}{\partial x} A(x,t)\right)\Big(\frac{\partial}{\partial x}\delta(x,$$

$$t)\Big) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)^2 \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)^4 \sec\left(\frac{x}{l}\right)\Big($$

$$-A(x,t)\, e^{-2\,\delta(x,t)}\Big)^2 A(x,t)^4 \left(e^{-2\,\delta(x,t)}\right)^2 l^2 - 16\ln(e)\cos\left(\frac{x}{l}\right)^2 \Big(\frac{\partial}{\partial x} A(x,$$

$$t)\Big)\left(\frac{\partial}{\partial x}\delta(x,t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)^2 \cos\left(\frac{x}{l}\right)\Big(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}\Big)^4 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)^2 A(x,t)^4 \left(e^{-2\,\delta(x,t)}\right)^2 l^2$$

$$-16\ln(e)\cos\left(\frac{x}{l}\right)^4 \mathrm{D}^{(2)}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)\left(\frac{\partial}{\partial x} A(x,t)\right)\Big(\frac{\partial}{\partial x}\delta(x,$$

$$t)\Big) \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)^2 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right) A(x,t)^4 \left(e^{-2\,\delta(x,t)}\right)^2 l^2$$

$$+20\ln(e)\cos\left(\frac{x}{l}\right)^4 \left(\frac{\partial}{\partial x} A(x,t)\right)\left(\frac{\partial}{\partial x}\delta(x,t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}\Big) \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)^2 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right) A(x,$$

$$t)^3\, e^{-2\,\delta(x,t)}\, l^2 + 16\ln(e)\cos\left(\frac{x}{l}\right)^2 \mathrm{D}^{(2)}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)\Big(\frac{\partial}{\partial x} A(x,$$

$$t)\Big)\left(\frac{\partial}{\partial x}\delta(x,t)\right) \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)^2 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right) A(x,$$

$$t)^4 \left(e^{-2\,\delta(x,t)}\right)^2 l^2 - 8\ln(e)\cos\left(\frac{x}{l}\right)^3 \sin\left(\frac{x}{l}\right)\left(\frac{\partial}{\partial x}\delta(x,t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}\Big) \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)^2 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right) A(x,$$

$$t)^4\, e^{-2\,\delta(x,t)}\, l + 4\ln(e)\cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial}{\partial t} A(x,t)\right)\left(\frac{\partial}{\partial t}\delta(x,t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}\Big) \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)^4 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right) A(x,$$

$$t)^2\, e^{-2\,\delta(x,t)}\, l^2 - 20\ln(e)\cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial}{\partial x} A(x,t)\right)\left(\frac{\partial}{\partial x}\delta(x,t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,$$

$$
\begin{aligned}
&t)\, e^{-2\,\delta(x,\,t)}\Big)\cos\left(\frac{x}{l}\right)\Big(-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)^2 \sec\left(\frac{x}{l}\right)\Big(-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)\, A(x,\\
&t)^3\, e^{-2\,\delta(x,\,t)}\, l^2 + 8\ln(e)\cos\left(\frac{x}{l}\right)^4 \left(\frac{\partial^2}{\partial x^2}\,\delta(x,t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,\\
&t)\, e^{-2\,\delta(x,\,t)}\Big)\cos\left(\frac{x}{l}\right)\Big(-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)^2 \sec\left(\frac{x}{l}\right)\Big(-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)\, A(x,\\
&t)^4\, e^{-2\,\delta(x,\,t)}\, l^2 - 8\ln(e)\cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial^2}{\partial x^2}\,\delta(x,t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,\\
&t)\, e^{-2\,\delta(x,\,t)}\Big)\cos\left(\frac{x}{l}\right)\Big(-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)^2 \sec\left(\frac{x}{l}\right)\Big(-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)\, A(x,\\
&t)^4\, e^{-2\,\delta(x,\,t)}\, l^2 - 16\ln(e)\cos\left(\frac{x}{l}\right)^4 \left(\frac{\partial}{\partial x}\,A(x,t)\right)\left(\frac{\partial}{\partial x}\,\delta(x,t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(\\
&-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)^2 \cos\left(\frac{x}{l}\right)\Big(-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)^2 A(x,t)^4 \left(e^{-2\,\delta(x,\,t)}\right)^2 l^2\\
&+ 16\ln(e)^2\cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial}{\partial x}\,\delta(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)\cos\left(\frac{x}{l}\right)\Big(\\
&-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)^2 \sec\left(\frac{x}{l}\right)\Big(-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)\, A(x,t)^4\, e^{-2\,\delta(x,\,t)}\, l^2\\
&+ 16\ln(e)\cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial}{\partial x}\,A(x,t)\right)\left(\frac{\partial}{\partial x}\,\delta(x,t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,\\
&t)\, e^{-2\,\delta(x,\,t)}\Big)^2 \cos\left(\frac{x}{l}\right)\Big(-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)^2 A(x,t)^4 \left(e^{-2\,\delta(x,\,t)}\right)^2 l^2\\
&+ 4\cos\left(\frac{x}{l}\right)^3 \sin\left(\frac{x}{l}\right)\left(\frac{\partial}{\partial x}\,A(x,t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)\cos\left(\frac{x}{l}\right)\Big(\\
&-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)^2 \sec\left(\frac{x}{l}\right)\Big(-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)\, A(x,t)^3\, e^{-2\,\delta(x,\,t)}\, l\\
&- 16\ln(e)^2\cos\left(\frac{x}{l}\right)^4 \left(\frac{\partial}{\partial x}\,\delta(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)^2 \cos\left(\frac{x}{l}\right)\Big(\\
&-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)^4 \sec\left(\frac{x}{l}\right)\Big(-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)^2 A(x,t)^5 \left(e^{-2\,\delta(x,\,t)}\right)^2 l^2\\
&+ 16\ln(e)^2\cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial}{\partial x}\,\delta(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)^2 \cos\left(\frac{x}{l}\right)\Big(\\
&-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)^4 \sec\left(\frac{x}{l}\right)\Big(-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)^2 A(x,t)^5 \left(e^{-2\,\delta(x,\,t)}\right)^2 l^2\\
&+ 16\ln(e)^2\cos\left(\frac{x}{l}\right)^4 \mathrm{D}^{(2)}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)\left(\frac{\partial}{\partial x}\,\delta(x,\right.
\end{aligned}
$$

$$t)\Big)^2 \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)^2 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right) A(x,t)^5 \left(e^{-2\,\delta(x,t)}\right)^2 l^2$$

$$-16 \ln(e)^2 \cos\left(\frac{x}{l}\right)^4 \left(\frac{\partial}{\partial x}\,\delta(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right) \cos\left(\frac{x}{l}\right)\Big($$

$$-A(x,t)\, e^{-2\,\delta(x,t)}\Big)^2 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right) A(x,t)^4\, e^{-2\,\delta(x,t)}\, l^2$$

$$-16 \ln(e)^2 \cos\left(\frac{x}{l}\right)^2 \mathrm{D}^{(2)}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right) \Big(\frac{\partial}{\partial x}\,\delta(x,$$

$$t)\Big)^2 \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)^2 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right) A(x,t)^5 \left(e^{-2\,\delta(x,t)}\right)^2 l^2$$

$$+3 \cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial}{\partial t}\, A(x,t)\right)^2 \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)^2 l^2 - 3 \Big(\frac{\partial}{\partial t}\, A(x,$$

$$t)\Big)^2 \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)^2 l^2 + 4 \cos\left(\frac{x}{l}\right)^4 \mathrm{D}^{(2)}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}\Big) \left(\frac{\partial}{\partial x}\, A(x,t)\right)^2 \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)^2 \sec\left(\frac{x}{l}\right)\Big(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}\Big) A(x,t)^3 \left(e^{-2\,\delta(x,t)}\right)^2 l^2 - 2 \cos\left(\frac{x}{l}\right)^4 \left(\frac{\partial}{\partial x}\, A(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}\Big) \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)^2 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right) A(x,$$

$$t)^2\, e^{-2\,\delta(x,t)}\, l^2 - 4 \cos\left(\frac{x}{l}\right)^2 \mathrm{D}^{(2)}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right) \Big(\frac{\partial}{\partial x}\, A(x,$$

$$t)\Big)^2 \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)^2 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right) A(x,t)^3 \left(e^{-2\,\delta(x,t)}\right)^2 l^2$$

$$-2 \cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial}{\partial t}\, A(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right) \cos\left(\frac{x}{l}\right)\Big(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}\Big)^4 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right) A(x,t)\, e^{-2\,\delta(x,t)}\, l^2$$

$$+2 \cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial}{\partial x}\, A(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right) \cos\left(\frac{x}{l}\right)\Big(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}\Big)^2 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right) A(x,t)^2\, e^{-2\,\delta(x,t)}\, l^2$$

$$+16 \ln(e)^2 \cos\left(\frac{x}{l}\right)^4 \left(\frac{\partial}{\partial x}\,\delta(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,t)}\right)^2 \cos\left(\frac{x}{l}\right)\Big($$

$$-A(x,t)\, e^{-2\,\delta(x,t)}\Big)^2 A(x,t)^5 \left(e^{-2\,\delta(x,t)}\right)^2 l^2 - 4 \cos\left(\frac{x}{l}\right)^4 \Big(\frac{\partial}{\partial x}\, A(x,$$

$$t)\Big)^2 \mathrm{D}\Big(\sec\Big(\frac{x}{l}\Big)\Big)\Big(-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^2 \cos\Big(\frac{x}{l}\Big)\Big(-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^4 \sec\Big(\frac{x}{l}\Big)\Big($$

$$-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^2 A(x,t)^3 \left(e^{-2\,\delta(x,t)}\right)^2 l^2 - 16\ln(e)^2\cos\Big(\frac{x}{l}\Big)^2\Big(\frac{\partial}{\partial x}\,\delta(x,$$

$$t)\Big)^2 \mathrm{D}\Big(\sec\Big(\frac{x}{l}\Big)\Big)\Big(-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^2 \cos\Big(\frac{x}{l}\Big)\Big(-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^2 A(x,$$

$$t)^5 \left(e^{-2\,\delta(x,t)}\right)^2 l^2 + 4\cos\Big(\frac{x}{l}\Big)^2\Big(\frac{\partial}{\partial x}\,A(x,t)\Big)^2 \mathrm{D}\Big(\sec\Big(\frac{x}{l}\Big)\Big)\Big(-A(x,$$

$$t)\,e^{-2\,\delta(x,t)}\Big)^2 \cos\Big(\frac{x}{l}\Big)\Big(-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^4 \sec\Big(\frac{x}{l}\Big)\Big(-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^2 A(x,$$

$$t)^3 \left(e^{-2\,\delta(x,t)}\right)^2 l^2 - 2\cos\Big(\frac{x}{l}\Big)\sin\Big(\frac{x}{l}\Big)\Big(\frac{\partial}{\partial x}\,A(x,t)\Big)A(x,t)^2\,l - 4\,A(x,t)^3$$

$$+ 4^{A(x,t)^3}\cos\Big(\frac{x}{l}\Big)^2\Big)$$

> *displayGR(Ricci, Rc);*

The Ricci tensor

non-zero components :

$$R11 = \frac{1}{4}\,\frac{1}{A(x,t)^2\,l\sin\Big(\frac{x}{l}\Big)}\Bigg(-16\ln(e)^2\cos\Big(\frac{x}{l}\Big)^2\sin\Big(\frac{x}{l}\Big)\Big(\frac{\partial}{\partial x}\,\delta(x,$$

$$t)\Big)^2 \mathrm{D}\Big(\sec\Big(\frac{x}{l}\Big)\Big)\Big(-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^2 \cos\Big(\frac{x}{l}\Big)\Big(-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^2 \sec\Big(\frac{x}{l}\Big)\Big($$

$$-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^2 A(x,t)^5 \left(e^{-2\,\delta(x,t)}\right)^2 l + 16\ln(e)\cos\Big(\frac{x}{l}\Big)^2\sin\Big(\frac{x}{l}\Big)\Big(\frac{\partial}{\partial x}\,A(x,$$

$$t)\Big)\Big(\frac{\partial}{\partial x}\,\delta(x,t)\Big)\mathrm{D}\Big(\sec\Big(\frac{x}{l}\Big)\Big)\Big(-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^2 \cos\Big(\frac{x}{l}\Big)\Big(-A(x,$$

$$t)\,e^{-2\,\delta(x,t)}\Big)^2 \sec\Big(\frac{x}{l}\Big)\Big(-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^2 A(x,t)^4 \left(e^{-2\,\delta(x,t)}\right)^2 l$$

$$+ 16\ln(e)^2\cos\Big(\frac{x}{l}\Big)^2\sin\Big(\frac{x}{l}\Big)\mathrm{D}^{(2)}\Big(\sec\Big(\frac{x}{l}\Big)\Big)\Big(-A(x,t)\,e^{-2\,\delta(x,t)}\Big)\Big(\frac{\partial}{\partial x}\,\delta(x,$$

$$t)\Big)^2 \sec\Big(\frac{x}{l}\Big)\Big(-A(x,t)\,e^{-2\,\delta(x,t)}\Big)A(x,t)^5 \left(e^{-2\,\delta(x,t)}\right)^2 l$$

$$+ 16\ln(e)^2\cos\Big(\frac{x}{l}\Big)^2\sin\Big(\frac{x}{l}\Big)\Big(\frac{\partial}{\partial x}\,\delta(x,t)\Big)^2 \mathrm{D}\Big(\sec\Big(\frac{x}{l}\Big)\Big)\Big(-A(x,$$

$$t)\,e^{-2\,\delta(x,t)}\Big)^2 A(x,t)^5 \left(e^{-2\,\delta(x,t)}\right)^2 l - 4\cos\left(\frac{x}{l}\right)^2 \sin\left(\frac{x}{l}\right) \left(\frac{\partial}{\partial x} A(x,$$

$$t)\Big)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)^2 \cos\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)^2 \sec\left(\frac{x}{l}\right)\Big($$

$$-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^2 A(x,t)^3 \left(e^{-2\,\delta(x,t)}\right)^2 l$$

$$-16\ln(e)^2 \cos\left(\frac{x}{l}\right)^2 \sin\left(\frac{x}{l}\right)\left(\frac{\partial}{\partial x}\delta(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,$$

$$t)\,e^{-2\,\delta(x,t)}\Big) \sec\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right) A(x,t)^4 e^{-2\,\delta(x,t)}\, l$$

$$-16\ln(e)\cos\left(\frac{x}{l}\right)^2 \sin\left(\frac{x}{l}\right) \mathrm{D}^{(2)}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)\Big(\frac{\partial}{\partial x} A(x,$$

$$t)\Big)\left(\frac{\partial}{\partial x}\delta(x,t)\right)\sec\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right) A(x,t)^4 \left(e^{-2\,\delta(x,t)}\right)^2 l$$

$$-16\ln(e)\cos\left(\frac{x}{l}\right)^2 \sin\left(\frac{x}{l}\right)\left(\frac{\partial}{\partial x} A(x,t)\right)\left(\frac{\partial}{\partial x}\delta(x,t)\right)\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,$$

$$t)\,e^{-2\,\delta(x,t)}\Big)^2 A(x,t)^4 \left(e^{-2\,\delta(x,t)}\right)^2 l + 8\ln(e)\cos\left(\frac{x}{l}\right)^2 \sin\left(\frac{x}{l}\right)\Big(\frac{\partial^2}{\partial x^2}\delta(x,$$

$$t)\Big)\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)\sec\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right) A(x,t)^4 e^{-2\,\delta(x,t)}\, l$$

$$+20\ln(e)\cos\left(\frac{x}{l}\right)^2 \sin\left(\frac{x}{l}\right)\left(\frac{\partial}{\partial x} A(x,t)\right)\left(\frac{\partial}{\partial x}\delta(x,t)\right)\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,$$

$$t)\,e^{-2\,\delta(x,t)}\Big)\sec\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right) A(x,t)^3 e^{-2\,\delta(x,t)}\, l$$

$$+4\cos\left(\frac{x}{l}\right)^2 \sin\left(\frac{x}{l}\right)\mathrm{D}^{(2)}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)\left(\frac{\partial}{\partial x} A(x,t)\right)^2 \sec\left(\frac{x}{l}\right)\Big($$

$$-A(x,t)\,e^{-2\,\delta(x,t)}\Big) A(x,t)^3 \left(e^{-2\,\delta(x,t)}\right)^2 l + 4\cos\left(\frac{x}{l}\right)^2 \sin\left(\frac{x}{l}\right)\Big(\frac{\partial}{\partial x} A(x,$$

$$t)\Big)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)^2 A(x,t)^3 \left(e^{-2\,\delta(x,t)}\right)^2 l$$

$$+8\ln(e)\cos\left(\frac{x}{l}\right)^3 \left(\frac{\partial}{\partial x}\delta(x,t)\right)\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)\sec\left(\frac{x}{l}\right)\Big(-A(x,$$

$$t)\,e^{-2\,\delta(x,t)}\Big) A(x,t)^4 e^{-2\,\delta(x,t)} + 4\ln(e)\sin\left(\frac{x}{l}\right)\left(\frac{\partial}{\partial t} A(x,t)\right)\Big(\frac{\partial}{\partial t}\delta(x,$$

$t)\Big)\,\mathrm{D}\Big(\sec\Big(\frac{x}{l}\Big)\Big)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)\cos\Big(\frac{x}{l}\Big)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)^2\sec\Big(\frac{x}{l}\Big)\big(-A(x,$

$t)\,e^{-2\,\delta(x,t)}\big)\,A(x,t)^2\,e^{-2\,\delta(x,t)}\,l-4\cos\Big(\frac{x}{l}\Big)^2\sin\Big(\frac{x}{l}\Big)\Big(\frac{\partial^2}{\partial x^2}A(x,t)\Big)\,\mathrm{D}\Big(\sec\Big(\frac{x}{l}\Big)\Big)\big($

$-A(x,t)\,e^{-2\,\delta(x,t)}\big)\sec\Big(\frac{x}{l}\Big)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)\,A(x,t)^3\,e^{-2\,\delta(x,t)}\,l$

$-2\cos\Big(\frac{x}{l}\Big)^2\sin\Big(\frac{x}{l}\Big)\Big(\frac{\partial}{\partial x}A(x,t)\Big)^2\,\mathrm{D}\Big(\sec\Big(\frac{x}{l}\Big)\Big)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)\sec\Big(\frac{x}{l}\Big)\big($

$-A(x,t)\,e^{-2\,\delta(x,t)}\big)\,A(x,t)^2\,e^{-2\,\delta(x,t)}\,l-4\cos\Big(\frac{x}{l}\Big)^3\Big(\frac{\partial}{\partial x}A(x,t)\Big)\,\mathrm{D}\Big(\sec\Big(\frac{x}{l}\Big)\Big)\big(-A(x,$

$t)\,e^{-2\,\delta(x,t)}\big)\sec\Big(\frac{x}{l}\Big)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)\,A(x,t)^3\,e^{-2\,\delta(x,t)}-2\sin\Big(\frac{x}{l}\Big)\Big(\frac{\partial}{\partial t}A(x,$

$t)\Big)^2\,\mathrm{D}\Big(\sec\Big(\frac{x}{l}\Big)\Big)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)\cos\Big(\frac{x}{l}\Big)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)^2\sec\Big(\frac{x}{l}\Big)\big(-A(x,$

$t)\,e^{-2\,\delta(x,t)}\big)\,A(x,t)\,e^{-2\,\delta(x,t)}\,l-2\sin\Big(\frac{x}{l}\Big)\Big(\frac{\partial^2}{\partial t^2}A(x,t)\Big)\,A(x,t)\,l$

$+3\sin\Big(\frac{x}{l}\Big)\Big(\frac{\partial}{\partial t}A(x,t)\Big)^2\,l\Big)$

$$R12=\frac{1}{2}\,\frac{\frac{\partial}{\partial t}A(x,t)}{l\cos\Big(\frac{x}{l}\Big)\sin\Big(\frac{x}{l}\Big)\,A(x,t)}$$

$R22=-\frac{1}{4}\,\frac{1}{A(x,t)^3\cos\Big(\frac{x}{l}\Big)^2\,l^2\sin\Big(\frac{x}{l}\Big)^2}\Big(-4\cos\Big(\frac{x}{l}\Big)^4\Big(\frac{\partial^2}{\partial x^2}A(x,t)\Big)\,\mathrm{D}\Big(\sec\Big(\frac{x}{l}\Big)\Big)\big($

$-A(x,t)\,e^{-2\,\delta(x,t)}\big)\cos\Big(\frac{x}{l}\Big)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)^2\sec\Big(\frac{x}{l}\Big)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)\,A(x,$

$t)^3\,e^{-2\,\delta(x,t)}\,l^2+4\cos\Big(\frac{x}{l}\Big)^2\Big(\frac{\partial^2}{\partial x^2}A(x,t)\Big)\,\mathrm{D}\Big(\sec\Big(\frac{x}{l}\Big)\Big)\big(-A(x,$

$t)\,e^{-2\,\delta(x,t)}\big)\cos\Big(\frac{x}{l}\Big)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)^2\sec\Big(\frac{x}{l}\Big)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)\,A(x,$

$t)^3\,e^{-2\,\delta(x,t)}\,l^2+2\Big(\frac{\partial^2}{\partial t^2}A(x,t)\Big)\cos\Big(\frac{x}{l}\Big)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)^2\,A(x,t)\,l^2$

$+4\cos\Big(\frac{x}{l}\Big)^4\Big(\frac{\partial}{\partial x}A(x,t)\Big)^2\,\mathrm{D}\Big(\sec\Big(\frac{x}{l}\Big)\Big)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)^2\cos\Big(\frac{x}{l}\Big)\big(-A(x,$

$$t)\, e^{-2\,\delta(x,t)}\big)^2 A(x,t)^3 \left(e^{-2\,\delta(x,t)}\right)^2 l^2 - 4\cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial}{\partial x} A(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\big($$

$$-A(x,t)\, e^{-2\,\delta(x,t)}\big)^2 \cos\left(\frac{x}{l}\right)\big(-A(x,t)\, e^{-2\,\delta(x,t)}\big)^2 A(x,t)^3 \left(e^{-2\,\delta(x,t)}\right)^2 l^2$$

$$+ 2\left(\frac{\partial}{\partial t} A(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\big(-A(x,t)\, e^{-2\,\delta(x,t)}\big)\cos\left(\frac{x}{l}\right)\big(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}\big)^4 \sec\left(\frac{x}{l}\right)\big(-A(x,t)\, e^{-2\,\delta(x,t)}\big) A(x,t)\, e^{-2\,\delta(x,t)}\, l^2 - 2\cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial^2}{\partial t^2} A(x,\right.$$

$$\left.t)\right)\cos\left(\frac{x}{l}\right)\big(-A(x,t)\, e^{-2\,\delta(x,t)}\big)^2 A(x,t)\, l^2 - 4\cos\left(\frac{x}{l}\right)\left(\frac{\partial}{\partial x} A(x,t)\right)\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\big($$

$$-A(x,t)\, e^{-2\,\delta(x,t)}\big)\cos\left(\frac{x}{l}\right)\big(-A(x,t)\, e^{-2\,\delta(x,t)}\big)^2 \sec\left(\frac{x}{l}\right)\big(-A(x,t)\, e^{-2\,\delta(x,t)}\big) A(x,$$

$$t)^3\, e^{-2\,\delta(x,t)}\, l \sin\left(\frac{x}{l}\right) - 4\ln(e)\left(\frac{\partial}{\partial t} A(x,t)\right)\left(\frac{\partial}{\partial t}\delta(x,t)\right)\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\big(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}\big)\cos\left(\frac{x}{l}\right)\big(-A(x,t)\, e^{-2\,\delta(x,t)}\big)^4 \sec\left(\frac{x}{l}\right)\big(-A(x,t)\, e^{-2\,\delta(x,t)}\big) A(x,$$

$$t)^2\, e^{-2\,\delta(x,t)}\, l^2 + 8\ln(e)\cos\left(\frac{x}{l}\right)\left(\frac{\partial}{\partial x}\delta(x,t)\right)\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\big(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}\big)\cos\left(\frac{x}{l}\right)\big(-A(x,t)\, e^{-2\,\delta(x,t)}\big)^2 \sec\left(\frac{x}{l}\right)\big(-A(x,t)\, e^{-2\,\delta(x,t)}\big) A(x,$$

$$t)^4\, e^{-2\,\delta(x,t)}\, l \sin\left(\frac{x}{l}\right) + 16\ln(e)\cos\left(\frac{x}{l}\right)^4 \left(\frac{\partial}{\partial x} A(x,t)\right)\left(\frac{\partial}{\partial x}\delta(x,\right.$$

$$\left.t)\right)\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\big(-A(x,t)\, e^{-2\,\delta(x,t)}\big)^2 \cos\left(\frac{x}{l}\right)\big(-A(x,t)\, e^{-2\,\delta(x,t)}\big)^4 \sec\left(\frac{x}{l}\right)\big($$

$$-A(x,t)\, e^{-2\,\delta(x,t)}\big)^2 A(x,t)^4 \left(e^{-2\,\delta(x,t)}\right)^2 l^2 - 16\ln(e)\cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial}{\partial x} A(x,\right.$$

$$\left.t)\right)\left(\frac{\partial}{\partial x}\delta(x,t)\right)\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\big(-A(x,t)\, e^{-2\,\delta(x,t)}\big)^2 \cos\left(\frac{x}{l}\right)\big(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}\big)^4 \sec\left(\frac{x}{l}\right)\big(-A(x,t)\, e^{-2\,\delta(x,t)}\big)^2 A(x,t)^4 \left(e^{-2\,\delta(x,t)}\right)^2 l^2$$

$$- 16\ln(e)\cos\left(\frac{x}{l}\right)^4 \mathrm{D}^{(2)}\left(\sec\left(\frac{x}{l}\right)\right)\big(-A(x,t)\, e^{-2\,\delta(x,t)}\big)\left(\frac{\partial}{\partial x} A(x,t)\right)\left(\frac{\partial}{\partial x}\delta(x,\right.$$

$$\left.t)\right)\cos\left(\frac{x}{l}\right)\big(-A(x,t)\, e^{-2\,\delta(x,t)}\big)^2 \sec\left(\frac{x}{l}\right)\big(-A(x,t)\, e^{-2\,\delta(x,t)}\big) A(x,t)^4 \left(e^{-2\,\delta(x,t)}\right)^2 l^2$$

$$+ 20 \ln(e) \cos\left(\frac{x}{l}\right)^4 \left(\frac{\partial}{\partial x} A(x, t)\right) \left(\frac{\partial}{\partial x} \delta(x, t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}) \cos\left(\frac{x}{l}\right)(-A(x, t)\, e^{-2\,\delta(x,t)})^2 \sec\left(\frac{x}{l}\right)(-A(x, t)\, e^{-2\,\delta(x,t)})\, A(x,$$

$$t)^3\, e^{-2\,\delta(x,t)}\, l^2 + 16 \ln(e) \cos\left(\frac{x}{l}\right)^2 \mathrm{D}^{(2)}\left(\sec\left(\frac{x}{l}\right)\right)(-A(x, t)\, e^{-2\,\delta(x,t)}) \left(\frac{\partial}{\partial x} A(x,$$

$$t)\right) \left(\frac{\partial}{\partial x} \delta(x, t)\right) \cos\left(\frac{x}{l}\right)(-A(x, t)\, e^{-2\,\delta(x,t)})^2 \sec\left(\frac{x}{l}\right)(-A(x, t)\, e^{-2\,\delta(x,t)})\, A(x,$$

$$t)^4\, (e^{-2\,\delta(x,t)})^2\, l^2 - 8 \ln(e) \cos\left(\frac{x}{l}\right)^3 \sin\left(\frac{x}{l}\right) \left(\frac{\partial}{\partial x} \delta(x, t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}) \cos\left(\frac{x}{l}\right)(-A(x, t)\, e^{-2\,\delta(x,t)})^2 \sec\left(\frac{x}{l}\right)(-A(x, t)\, e^{-2\,\delta(x,t)})\, A(x,$$

$$t)^4\, e^{-2\,\delta(x,t)}\, l + 4 \ln(e) \cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial}{\partial t} A(x, t)\right) \left(\frac{\partial}{\partial t} \delta(x, t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}) \cos\left(\frac{x}{l}\right)(-A(x, t)\, e^{-2\,\delta(x,t)})^4 \sec\left(\frac{x}{l}\right)(-A(x, t)\, e^{-2\,\delta(x,t)})\, A(x,$$

$$t)^2\, e^{-2\,\delta(x,t)}\, l^2 - 20 \ln(e) \cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial}{\partial x} A(x, t)\right) \left(\frac{\partial}{\partial x} \delta(x, t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}) \cos\left(\frac{x}{l}\right)(-A(x, t)\, e^{-2\,\delta(x,t)})^2 \sec\left(\frac{x}{l}\right)(-A(x, t)\, e^{-2\,\delta(x,t)})\, A(x,$$

$$t)^3\, e^{-2\,\delta(x,t)}\, l^2 + 8 \ln(e) \cos\left(\frac{x}{l}\right)^4 \left(\frac{\partial^2}{\partial x^2} \delta(x, t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}) \cos\left(\frac{x}{l}\right)(-A(x, t)\, e^{-2\,\delta(x,t)})^2 \sec\left(\frac{x}{l}\right)(-A(x, t)\, e^{-2\,\delta(x,t)})\, A(x,$$

$$t)^4\, e^{-2\,\delta(x,t)}\, l^2 - 8 \ln(e) \cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial^2}{\partial x^2} \delta(x, t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)(-A(x,$$

$$t)\, e^{-2\,\delta(x,t)}) \cos\left(\frac{x}{l}\right)(-A(x, t)\, e^{-2\,\delta(x,t)})^2 \sec\left(\frac{x}{l}\right)(-A(x, t)\, e^{-2\,\delta(x,t)})\, A(x,$$

$$t)^4\, e^{-2\,\delta(x,t)}\, l^2 - 16 \ln(e) \cos\left(\frac{x}{l}\right)^4 \left(\frac{\partial}{\partial x} A(x, t)\right) \left(\frac{\partial}{\partial x} \delta(x, t)\right) \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)($$

$$-A(x, t)\, e^{-2\,\delta(x,t)})^2 \cos\left(\frac{x}{l}\right)(-A(x, t)\, e^{-2\,\delta(x,t)})^2\, A(x, t)^4\, (e^{-2\,\delta(x,t)})^2\, l^2$$

$$+ 16 \ln(e)^2 \cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial}{\partial x} \delta(x, t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)(-A(x, t)\, e^{-2\,\delta(x,t)}) \cos\left(\frac{x}{l}\right)($$

$$-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^{2}\sec\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)A(x,t)^{4}\,e^{-2\,\delta(x,t)}\,l^{2}$$

$$+16\ln(e)\cos\left(\frac{x}{l}\right)^{2}\left(\frac{\partial}{\partial x}A(x,t)\right)\left(\frac{\partial}{\partial x}\delta(x,t)\right)\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,$$

$$t)\,e^{-2\,\delta(x,t)}\Big)^{2}\cos\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)^{2}A(x,t)^{4}\left(e^{-2\,\delta(x,t)}\right)^{2}l^{2}$$

$$+4\cos\left(\frac{x}{l}\right)^{3}\sin\left(\frac{x}{l}\right)\left(\frac{\partial}{\partial x}A(x,t)\right)\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)\cos\left(\frac{x}{l}\right)\Big($$

$$-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^{2}\sec\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)A(x,t)^{3}\,e^{-2\,\delta(x,t)}\,l$$

$$-16\ln(e)^{2}\cos\left(\frac{x}{l}\right)^{4}\left(\frac{\partial}{\partial x}\delta(x,t)\right)^{2}\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)^{2}\cos\left(\frac{x}{l}\right)\Big($$

$$-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^{4}\sec\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)^{2}A(x,t)^{5}\left(e^{-2\,\delta(x,t)}\right)^{2}l^{2}$$

$$+16\ln(e)^{2}\cos\left(\frac{x}{l}\right)^{2}\left(\frac{\partial}{\partial x}\delta(x,t)\right)^{2}\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)^{2}\cos\left(\frac{x}{l}\right)\Big($$

$$-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^{4}\sec\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)^{2}A(x,t)^{5}\left(e^{-2\,\delta(x,t)}\right)^{2}l^{2}$$

$$+16\ln(e)^{2}\cos\left(\frac{x}{l}\right)^{4}\mathrm{D}^{(2)}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)\Big(\frac{\partial}{\partial x}\delta(x,$$

$$t)\Big)^{2}\cos\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)^{2}\sec\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)A(x,t)^{5}\left(e^{-2\,\delta(x,t)}\right)^{2}l^{2}$$

$$-16\ln(e)^{2}\cos\left(\frac{x}{l}\right)^{4}\left(\frac{\partial}{\partial x}\delta(x,t)\right)^{2}\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)\cos\left(\frac{x}{l}\right)\Big($$

$$-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^{2}\sec\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)A(x,t)^{4}\,e^{-2\,\delta(x,t)}\,l^{2}$$

$$-16\ln(e)^{2}\cos\left(\frac{x}{l}\right)^{2}\mathrm{D}^{(2)}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)\Big(\frac{\partial}{\partial x}\delta(x,$$

$$t)\Big)^{2}\cos\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)^{2}\sec\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)A(x,t)^{5}\left(e^{-2\,\delta(x,t)}\right)^{2}l^{2}$$

$$+3\cos\left(\frac{x}{l}\right)^{2}\left(\frac{\partial}{\partial t}A(x,t)\right)^{2}\cos\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)^{2}l^{2}-3\,\Big(\frac{\partial}{\partial t}A(x,$$

$$t)\Big)^{2}\cos\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)^{2}l^{2}+4\cos\left(\frac{x}{l}\right)^{4}\mathrm{D}^{(2)}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,$$

$$t)\,e^{-2\,\delta(x,t)}\Big)\left(\frac{\partial}{\partial x}A(x,t)\right)^{2}\cos\left(\frac{x}{l}\right)\left(-A(x,t)\,e^{-2\,\delta(x,t)}\right)^{2}\sec\left(\frac{x}{l}\right)\Big(-A(x,$$

$$t)\, e^{-2\,\delta(x,\,t)}\Big)\, A(x,t)^3 \left(e^{-2\,\delta(x,\,t)}\right)^2 l^2 - 2\cos\left(\frac{x}{l}\right)^4 \left(\frac{\partial}{\partial x} A(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,$$

$$t)\, e^{-2\,\delta(x,\,t)}\Big) \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right)^2 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right) A(x,$$

$$t)^2\, e^{-2\,\delta(x,\,t)}\, l^2 - 4\cos\left(\frac{x}{l}\right)^2 \mathrm{D}^{(2)}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right)\Big(\frac{\partial}{\partial x} A(x,$$

$$t)\Big)^2 \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right)^2 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right) A(x,t)^3 \left(e^{-2\,\delta(x,\,t)}\right)^2 l^2$$

$$- 2\cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial}{\partial t} A(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right) \cos\left(\frac{x}{l}\right)\Big(-A(x,$$

$$t)\, e^{-2\,\delta(x,\,t)}\Big)^4 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right) A(x,t)\, e^{-2\,\delta(x,\,t)}\, l^2$$

$$+ 2\cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial}{\partial x} A(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right) \cos\left(\frac{x}{l}\right)\Big(-A(x,$$

$$t)\, e^{-2\,\delta(x,\,t)}\Big)^2 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right) A(x,t)^2\, e^{-2\,\delta(x,\,t)}\, l^2$$

$$+ 16\ln(e)^2 \cos\left(\frac{x}{l}\right)^4 \left(\frac{\partial}{\partial x} \delta(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right)^2 \cos\left(\frac{x}{l}\right)\Big($$

$$-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)^2 A(x,t)^5 \left(e^{-2\,\delta(x,\,t)}\right)^2 l^2 - 4\cos\left(\frac{x}{l}\right)^4 \Big(\frac{\partial}{\partial x} A(x,$$

$$t)\Big)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right)^2 \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right)^4 \sec\left(\frac{x}{l}\right)\Big($$

$$-A(x,t)\, e^{-2\,\delta(x,\,t)}\Big)^2 A(x,t)^3 \left(e^{-2\,\delta(x,\,t)}\right)^2 l^2 - 16\ln(e)^2 \cos\left(\frac{x}{l}\right)^2 \Big(\frac{\partial}{\partial x} \delta(x,$$

$$t)\Big)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right)^2 \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right)^2 A(x,$$

$$t)^5 \left(e^{-2\,\delta(x,\,t)}\right)^2 l^2 + 4\cos\left(\frac{x}{l}\right)^2 \left(\frac{\partial}{\partial x} A(x,t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,$$

$$t)\, e^{-2\,\delta(x,\,t)}\Big)^2 \cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right)^4 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right)^2 A(x,$$

$$t)^3 \left(e^{-2\,\delta(x,\,t)}\right)^2 l^2 - 2\cos\left(\frac{x}{l}\right)\sin\left(\frac{x}{l}\right)\left(\frac{\partial}{\partial x} A(x,t)\right) A(x,t)^2\, l - 4\, A(x,t)^3$$

$$+ 4^{A(x,t)^3} \cos\left(\frac{x}{l}\right)^2\Bigg)$$

$$R33 = \frac{1}{2}\, \frac{1}{\cos\left(\frac{x}{l}\right)^2} \Bigg(4\ln(e)\cos\left(\frac{x}{l}\right)\sin\left(\frac{x}{l}\right)\left(\frac{\partial}{\partial x}\delta(x,t)\right)\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,$$

$t)\, e^{-2\,\delta(x,\,t)}\big)\cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right)^2 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right) A(x,$

$t)^2\, e^{-2\,\delta(x,\,t)}\, l - 2\cos\left(\frac{x}{l}\right)\sin\left(\frac{x}{l}\right)\left(\frac{\partial}{\partial x}A(x,t)\right)\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\big(-A(x,$

$t)\, e^{-2\,\delta(x,\,t)}\big)\cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right)^2 \sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right) A(x,$

$t)\, e^{-2\,\delta(x,\,t)}\, l + \cos\left(\frac{x}{l}\right)\sin\left(\frac{x}{l}\right)\left(\frac{\partial}{\partial x}A(x,t)\right) l - 2\,A(x,t)\cos\left(\frac{x}{l}\right)^2 + 2\,A(x,t)\Big)$

$$\textit{character : [-1, -1]} \qquad \textbf{(6)}$$

> *displayGR*(*Riemann*, *Rm*);

The Riemann Tensor

non-zero components :

$R1212 = -\frac{1}{4}\,\frac{1}{A(x,t)^3\cos\left(\frac{x}{l}\right)^2 l}\Big(-16\ln(e)^2\cos\left(\frac{x}{l}\right)^2\left(\frac{\partial}{\partial x}\delta(x,t)\right)^2\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\big($

$-A(x,t)\, e^{-2\,\delta(x,\,t)}\big)^2\cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right)^2\sec\left(\frac{x}{l}\right)\big(-A(x,$

$t)\, e^{-2\,\delta(x,\,t)}\big)^2 A(x,t)^5\left(e^{-2\,\delta(x,\,t)}\right)^2 l + 16\ln(e)\cos\left(\frac{x}{l}\right)^2\left(\frac{\partial}{\partial x}A(x,t)\right)\Big(\frac{\partial}{\partial x}\delta(x,$

$t)\Big)\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right)^2\cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right)^2\sec\left(\frac{x}{l}\right)\big($

$-A(x,t)\, e^{-2\,\delta(x,\,t)}\big)^2 A(x,t)^4\left(e^{-2\,\delta(x,\,t)}\right)^2 l + 16\ln(e)^2\,\mathrm{D}^{(2)}\left(\sec\left(\frac{x}{l}\right)\right)\big(-A(x,$

$t)\, e^{-2\,\delta(x,\,t)}\big)\left(\frac{\partial}{\partial x}\delta(x,t)\right)^2\sec\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right) A(x,$

$t)^5\left(e^{-2\,\delta(x,\,t)}\right)^2\cos\left(\frac{x}{l}\right)^2 l + 16\ln(e)^2\left(\frac{\partial}{\partial x}\delta(x,t)\right)^2\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\big(-A(x,$

$t)\, e^{-2\,\delta(x,\,t)}\big)^2 A(x,t)^5\left(e^{-2\,\delta(x,\,t)}\right)^2\cos\left(\frac{x}{l}\right)^2 l - 4\cos\left(\frac{x}{l}\right)^2\Big(\frac{\partial}{\partial x}A(x,$

$t)\Big)^2\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right)^2\cos\left(\frac{x}{l}\right)\left(-A(x,t)\, e^{-2\,\delta(x,\,t)}\right)^2\sec\left(\frac{x}{l}\right)\big($

$-A(x,t)\, e^{-2\,\delta(x,\,t)}\big)^2 A(x,t)^3\left(e^{-2\,\delta(x,\,t)}\right)^2 l - 16\ln(e)^2\left(\frac{\partial}{\partial x}\delta(x,t)\right)^2\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\big($

$$-A(x,t)\,e^{-2\,\delta(x,t)}\big)\,\sec\!\left(\frac{x}{l}\right)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)\,A(x,t)^4\,e^{-2\,\delta(x,t)}\cos\!\left(\frac{x}{l}\right)^2 l$$

$$-16\ln(e)\,\mathrm{D}^{(2)}\!\left(\sec\!\left(\frac{x}{l}\right)\right)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)\left(\frac{\partial}{\partial x}A(x,t)\right)\left(\frac{\partial}{\partial x}\delta(x,t)\right)\sec\!\left(\frac{x}{l}\right)\big($$

$$-A(x,t)\,e^{-2\,\delta(x,t)}\big)\,A(x,t)^4\left(e^{-2\,\delta(x,t)}\right)^2\cos\!\left(\frac{x}{l}\right)^2 l-16\ln(e)\left(\frac{\partial}{\partial x}A(x,t)\right)\left(\frac{\partial}{\partial x}\delta(x,\right.$$

$$\left.t)\right)\mathrm{D}\!\left(\sec\!\left(\frac{x}{l}\right)\right)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)^2 A(x,t)^4\left(e^{-2\,\delta(x,t)}\right)^2\cos\!\left(\frac{x}{l}\right)^2 l$$

$$+8\ln(e)\left(\frac{\partial^2}{\partial x^2}\delta(x,t)\right)\mathrm{D}\!\left(\sec\!\left(\frac{x}{l}\right)\right)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)\sec\!\left(\frac{x}{l}\right)\big(-A(x,$$

$$t)\,e^{-2\,\delta(x,t)}\big)\,A(x,t)^4\,e^{-2\,\delta(x,t)}\cos\!\left(\frac{x}{l}\right)^2 l+20\ln(e)\cos\!\left(\frac{x}{l}\right)^2\left(\frac{\partial}{\partial x}A(x,\right.$$

$$\left.t)\right)\left(\frac{\partial}{\partial x}\delta(x,t)\right)\mathrm{D}\!\left(\sec\!\left(\frac{x}{l}\right)\right)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)\sec\!\left(\frac{x}{l}\right)\big(-A(x,$$

$$t)\,e^{-2\,\delta(x,t)}\big)\,A(x,t)^3\,e^{-2\,\delta(x,t)}\,l+4\,\mathrm{D}^{(2)}\!\left(\sec\!\left(\frac{x}{l}\right)\right)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)\left(\frac{\partial}{\partial x}A(x,\right.$$

$$\left.t)\right)^2\sec\!\left(\frac{x}{l}\right)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)\left(e^{-2\,\delta(x,t)}\right)^2\cos\!\left(\frac{x}{l}\right)^2 A(x,t)^3\,l+4\left(\frac{\partial}{\partial x}A(x,\right.$$

$$\left.t)\right)^2\mathrm{D}\!\left(\sec\!\left(\frac{x}{l}\right)\right)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)^2\left(e^{-2\,\delta(x,t)}\right)^2\cos\!\left(\frac{x}{l}\right)^2 A(x,t)^3\,l$$

$$-8\ln(e)\cos\!\left(\frac{x}{l}\right)\sin\!\left(\frac{x}{l}\right)\left(\frac{\partial}{\partial x}\delta(x,t)\right)\mathrm{D}\!\left(\sec\!\left(\frac{x}{l}\right)\right)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)\sec\!\left(\frac{x}{l}\right)\big($$

$$-A(x,t)\,e^{-2\,\delta(x,t)}\big)\,A(x,t)^4\,e^{-2\,\delta(x,t)}+4\ln(e)\left(\frac{\partial}{\partial t}A(x,t)\right)\left(\frac{\partial}{\partial t}\delta(x,\right.$$

$$\left.t)\right)\mathrm{D}\!\left(\sec\!\left(\frac{x}{l}\right)\right)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)\cos\!\left(\frac{x}{l}\right)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)^2\sec\!\left(\frac{x}{l}\right)\big(-A(x,$$

$$t)\,e^{-2\,\delta(x,t)}\big)\,A(x,t)^2\,e^{-2\,\delta(x,t)}\,l-4\left(\frac{\partial^2}{\partial x^2}A(x,t)\right)\mathrm{D}\!\left(\sec\!\left(\frac{x}{l}\right)\right)\big(-A(x,$$

$$t)\,e^{-2\,\delta(x,t)}\big)\sec\!\left(\frac{x}{l}\right)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)\,e^{-2\,\delta(x,t)}\cos\!\left(\frac{x}{l}\right)^2 A(x,t)^3\,l$$

$$-2\cos\!\left(\frac{x}{l}\right)^2\left(\frac{\partial}{\partial x}A(x,t)\right)^2\mathrm{D}\!\left(\sec\!\left(\frac{x}{l}\right)\right)\big(-A(x,t)\,e^{-2\,\delta(x,t)}\big)\sec\!\left(\frac{x}{l}\right)\big(-A(x,$$

$$t)\,e^{-2\,\delta(x,t)}\big)\,A(x,t)^2\,e^{-2\,\delta(x,t)}\,l+4\cos\!\left(\frac{x}{l}\right)\sin\!\left(\frac{x}{l}\right)\left(\frac{\partial}{\partial x}A(x,t)\right)\mathrm{D}\!\left(\sec\!\left(\frac{x}{l}\right)\right)\big($$

$$-A(x,t)\,e^{-2\,\delta(x,t)}\Big)\sec\left(\frac{x}{l}\right)\Big(-A(x,t)\,e^{-2\,\delta(x,t)}\Big)A(x,t)^3\,e^{-2\,\delta(x,t)}-2\left(\frac{\partial}{\partial t}A(x,$$

$$t)\right)^2 \mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,t)\,e^{-2\,\delta(x,t)}\Big)\cos\left(\frac{x}{l}\right)\Big(-A(x,t)\,e^{-2\,\delta(x,t)}\Big)^2\sec\left(\frac{x}{l}\right)\Big(-A(x,$$

$$t)\,e^{-2\,\delta(x,t)}\Big)A(x,t)\,e^{-2\,\delta(x,t)}\,l-2\left(\frac{\partial^2}{\partial t^2}A(x,t)\right)A(x,t)\,l+3\left(\frac{\partial}{\partial t}A(x,t)\right)^2 l\Bigg)$$

$$R1313=-\frac{1}{\cos\left(\frac{x}{l}\right)}\left(A(x,t)\,e^{-2\,\delta(x,t)}\left(2\ln(e)\left(\frac{\partial}{\partial x}\delta(x,t)\right)A(x,t)-\left(\frac{\partial}{\partial x}A(x,\right.\right.\right.$$

$$\left.\left.\left.t)\right)\right)\sec\left(\frac{x}{l}\right)\Big(-A(x,t)\,e^{-2\,\delta(x,t)}\Big)\mathrm{D}\left(\sec\left(\frac{x}{l}\right)\right)\Big(-A(x,t)\,e^{-2\,\delta(x,t)}\Big)\sin\left(\frac{x}{l}\right)l\right)$$

$$R1323=-\frac{1}{2}\,\frac{\left(\frac{\partial}{\partial t}A(x,t)\right)\sin\left(\frac{x}{l}\right)l}{A(x,t)\cos\left(\frac{x}{l}\right)^3}$$

$$R2323=\frac{1}{2}\,\frac{-\cos\left(\frac{x}{l}\right)\sin\left(\frac{x}{l}\right)\left(\frac{\partial}{\partial x}A(x,t)\right)l+2\,A(x,t)\cos\left(\frac{x}{l}\right)^2-2\,A(x,t)}{A(x,t)\cos\left(\frac{x}{l}\right)^4}$$

$$character : [-1, -1, -1, -1] \tag{7}$$

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