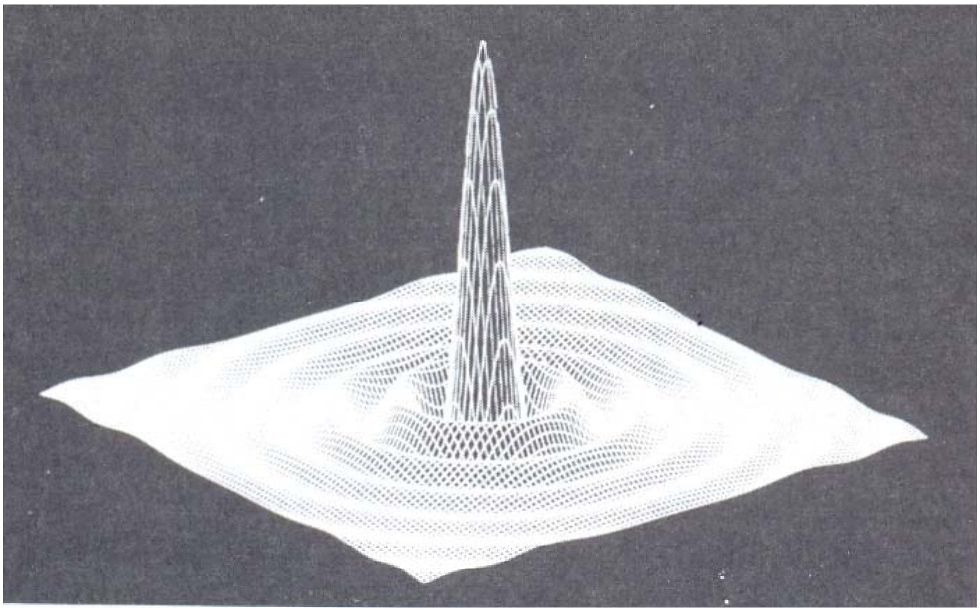
Astronomische Waarneemtechnieken (Astronomical Observing Techniques)

7th Lecture: 29 October 2008



Based on "Observational Astrophysics" (Springer) by P. Lena, F. Lebrun & F. Mignard, 2nd edition – Chapter 4.2 and on "Astronomical Optics" by Daniel J. Schroeder

Part I
Geometrical Optics

Part II
Diffraction Optics

Reminder: Coherent Radiation

A light source may exhibit temporal and spatial coherence. The coherence function Γ_{12} between two points (1,2) is the cross-correlation between their complex amplitudes:

$$\Gamma_{12}(\tau)=\langle E_1(t+\tau)E_2^*(t)\rangle$$

The normalized representation is called the degree of coherence:

$$\gamma_{12}(\tau)=\frac{\Gamma_{12}(\tau)}{\sqrt{I_1 I_2}}$$

which leads to an interference pattern* with an intensity distribution of:

$$I=I_1+I_2+2\sqrt{I_1 I_2}\,\mathrm{Re}[\gamma_{12}(\tau)]$$

where

$$|\gamma_{12}|=1 \quad \text{coherent}$$
$$|\gamma_{12}|=0 \quad \text{incoherent}$$
$$0<|\gamma_{12}|<1 \quad \text{partial coherence}$$

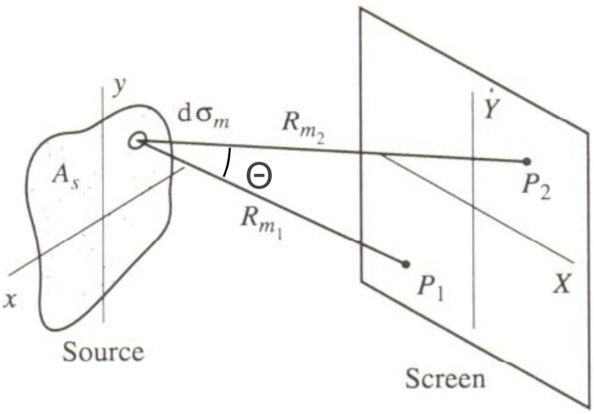
and the visibility $V=|\gamma_{12}(\tau)|$ for $I_1 = I_2$.

*e.g., from Young's double slit experiment

The Zernike-van Cittert Theorem (1)

Consider a monochromatic, extended, incoherent source A_s with intensity $I(x,y)$.

Consider further a surface element dσ (dσ <<λ), which illuminates two points P_1 and P_2 at distances R_1 and R_2 on a screen.



The quantity measuring the correlation of the electric fields between P_1 and P_2 (for any surface element dσ at distance r) is:

$$\langle V_1(t)V_2^*(t)\rangle=\int_{A_s} I(r)\frac{\exp[ik(R_1-R_2)]}{R_1 R_2}dr$$

The Zernike-van Cittert Theorem (2)

Generally, the degree of coherence is then given by the Zernike-van Cittert theorem:

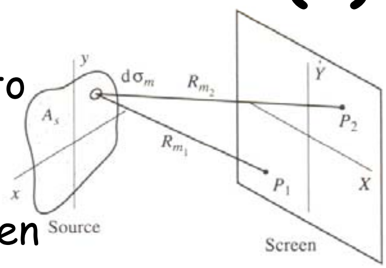
$$\gamma_{12}(0) = \frac{1}{\sqrt{\langle |V_1|^2 \rangle \langle |V_2|^2 \rangle}} \int_{source} I(r) \frac{\exp[ik(R_1 - R_2)]}{R_1 R_2} dr$$

In words, the general Zernike-van Cittert theorem describes the relation between the degree of coherence between two points on the screen and the intensity distribution across the illuminating source A_s.

Frits Zernike (1888-1966) : Dutch physicist and winner of the Nobel prize for physics in 1953 for his invention of the phase contrast microscope,

The Z-vC Theorem for Large Distances (3)

For large distances from source to screen (relative to the distance between P_1 and P_2 and the size of the source) we can use angular variables [x/R=α, y/R=β, Θ=(α,β), and $\Delta X = X_2 - X_1$] to describe the source as seen from the screen.



Then the general Z-vC theorem simplifies (Lena p. 118) to:

$$|\gamma_{12}(0)| = \left| \frac{\iint_{source} I(\theta) \exp\left[-\frac{i2\pi}{\lambda} (\alpha \Delta X + \beta \Delta Y) \right] d\theta}{\iint_{source} I(\theta) d\theta} \right|$$

For large distances, the modulus* of the degree of coherence $|\gamma_{12}|$ between two points is the modulus of the normalized Fourier transform of the source intensity distribution.

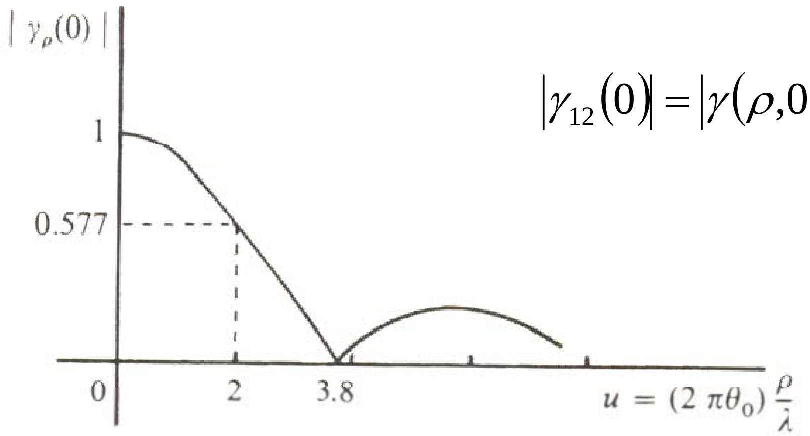
*absolute value of a complex number

The Z-vC Theorem for a Circular Source (4)

Now: calculate the complex degree of coherence for a circular source of radius r_0.
Let P_1 be at the center of the screen and P_2 at distance ρ where $\Theta = r_0/R$.

$$I(\theta) = \Pi\left(\frac{r}{2r_0}\right) = \Pi\left(\frac{\theta}{2\theta_0}\right)$$

Then the modulus of the degree of coherence for a circular source is:



$$|\gamma_{12}(0)| = |\gamma(\rho,0)| = \left|\frac{J_1\left(2\pi\theta_0 \frac{\rho}{\lambda}\right)}{\pi\theta_0 \frac{\rho}{\lambda}}\right| = \frac{2|J_1(u)|}{u}$$

Θ_0 = angular size of circular source
ρ = distance of P_2 from center
Λ = wavelength
J_1 = 1st order Bessel function

The Coherence Étendue

1. Consider a point source at infinity: $\Theta_0 = r_0/R \to 0$ and thus $|\gamma_{12}| \to 1$. In this case a plane wave illuminates the screen coherently.

2. Consider a source of finite size, which subtends a solid angle Ω. An circular area $S = \pi\rho^2$ of the screen corresponds to a beam étendue ε of:

$$\varepsilon = S\,\Omega = \pi\rho^2\,\pi\theta_0^2 \overset{u \equiv 2\pi\theta_o\rho/\lambda}{=} \frac{\lambda^2}{4}u^2$$

If we choose e.g., u = 2 so that $2 = 2\pi\theta_o\rho/\lambda \Rightarrow \rho = \frac{\lambda}{\pi\theta_o}$

we can calculate that: $\left|\gamma\left(\rho = \frac{\lambda}{\pi\theta_o}\right)\right| = \frac{2|J_1(2)|}{2} = 0.577$

which yields a degree of coherence greater than 50% (what we want!).

Hence, the beam remains coherent for an étendue $\varepsilon = \lambda^2$

Note: Beam Étendue, A×Ω, and Throughput

Étendue (frz.) = `extent'

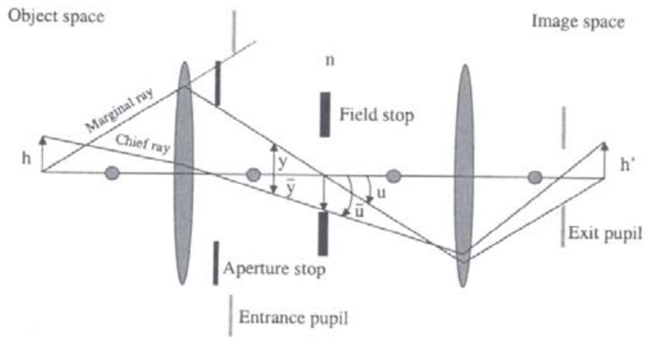
The geometrical étendue is the area A of the source times the solid angle Ω the system's entrance pupil subtends as seen from the source.

The étendue never increases in any optical system. A perfect optical system produces an image with the same étendue as the source.

The geometric étendue may be viewed as the maximum beam size the instrument can accept.

Hence, the étendue is also called acceptance, throughput, and the A·Ω product.

Here $A=h^2\pi$, and Ω is given by the angle of the marginal ray.



Note:

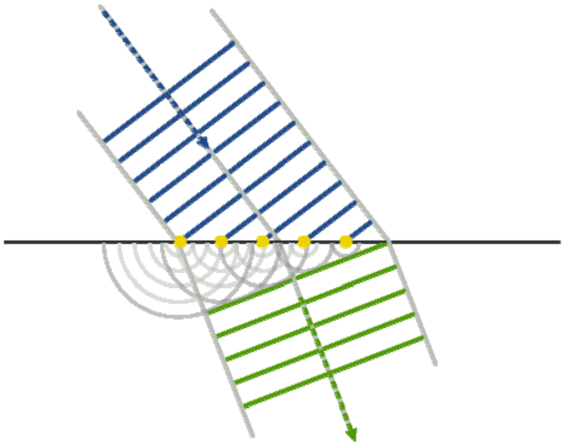
So far we have considered the coherence from a source at infinity.

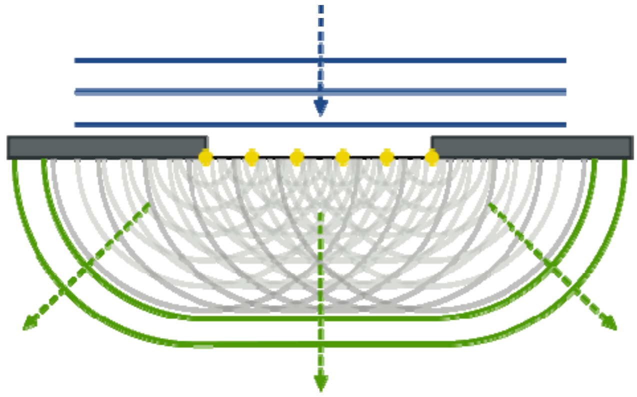
Now we will consider diffraction caused by a pupil "near infinity".

The Huygens-Fresnel Principle

Fermat's view: "A wavefront is a surface on which every point has the same OPD."

Huygens' view: "At a given time, each point on primary wavefront acts as a source of secondary spherical wavelets. These propagate with the same speed and frequency as the primary wave."





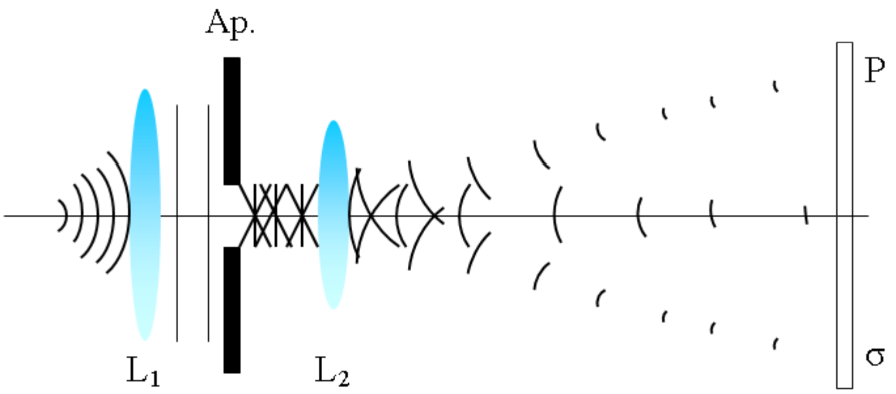
The Huygens-Fresnel principle was theoretically demonstrated by Kirchhoff (→ Fresnel-Kirchhoff diffraction integral)

Fresnel and Fraunhofer Diffraction

Fresnel diffraction = near-field diffraction

When a wave passes through an aperture and diffracts in the near field it causes the observed diffraction pattern to differ in size and shape for different distances.

For Fraunhofer diffraction at infinity (far-field) the wave becomes planar.



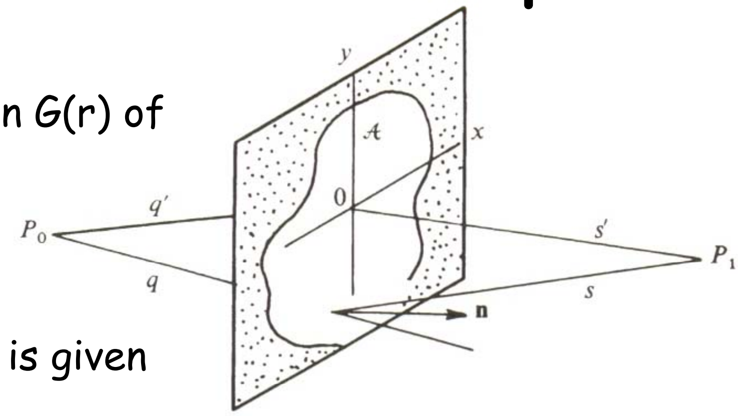
Fresnel: $F = \frac{r^2}{d \cdot \lambda} \geq 1$

Fraunhofer: $F = \frac{r^2}{d \cdot \lambda} << 1$

(where r = aperture size and d = distance to screen)

Fraunhofer Diffraction at a Pupil

Consider a circular pupil function G(r) of unity within A and zero outside.



Then the diffraction at infinity is given by:

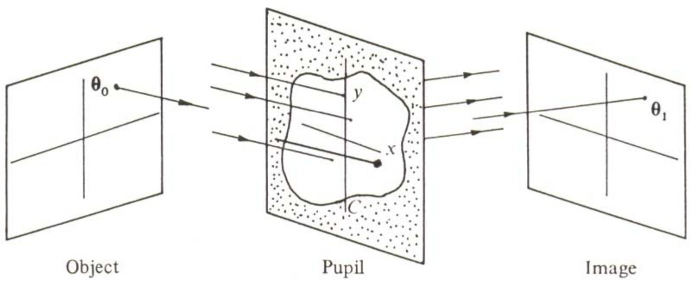
$$V_1(\theta_1,t)=\lambda\sqrt{\frac{E}{A}}\iint_{screenA} G\left(\tfrac{r}{\lambda}\right)e^{-i2\pi(\theta_1-\theta_0)\cdot\frac{r}{\lambda}}\,\frac{dr}{\lambda^2}$$

Theorem: *When a screen is illuminated by a source at infinity, the amplitude of the field diffracted in any direction is the Fourier transform of the pupil function characterizing the screen A.*

The conjugate variables are the angular direction and the reduced coordinates r/λ on the screen.

Relation between Object and Image

Let $V(\Theta_0)$ and $V(\Theta_1)$ be the complex field amplitutes of points in the object and image plane.
Let $K(\Theta_0;\Theta_1)$ be the transmission of the system (i.e., the complex amplitude per solid angle round Θ_1)



Then the image of an extended object is a linear superposition:

$$V(\theta_1)=\iint_{object} V_0(\theta_0)\,K(\theta_1-\theta_0)\,d\theta_0 = V_0(\theta_0) * K(\theta_0)$$

This is a convolution equation which can be conveniently addressed in Fourier space with $K(\theta)=\iint G(r)\exp\left[-i2\pi\frac{r}{\lambda}\theta\right]\frac{dr}{\lambda^2}$

and the spatial frequency ω: $\tilde{V}_0(\omega)=\iint_{object} V_0(\theta_0)\exp[-i2\pi\theta_0\omega]\,d\omega$

and similarly: $\tilde{V}(\omega)=FT\{V(\theta_1)\}$

Relation between Object and Image (2)

The convolution equation becomes $\tilde{V} = \tilde{V}_0 \tilde{K} = \tilde{V}_0 G$

In other words, the Fourier transform of the image equals the product of Fourier transform of the object and the pupil function.

Note that:
Amplitude of $\tilde{V}_0(\omega)$ ⇔ strength of frequency component ω in the image
Phase of $\tilde{V}_0(\omega)$ ⇔ position of this component ω in the image

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So far considered: Coherent sources
but more realistic in astronomy: Incoherent sources

Main difference: add intensities rather than amplitudes:

$$I(\theta_1) = \iint_{object} I_0(\theta_0) \left|K(\theta_1 - \theta_0)\right|^2 d\theta_0$$

to get $\tilde{I}(\omega) = \tilde{I}_0(\omega)\tilde{H}(\omega)$

where $\tilde{H}(\omega) = FT\left\{|K|^2\right\} = FT\left\{KK^*\right\} = G(\lambda\omega) * G^*(-\lambda\omega)$

The Modulation Transfer Function (MTF)

The equation $\tilde{V} = \tilde{V}_0 \tilde{K} = \tilde{V}_0 G$ can be interpreted as spatial linear filtering, which depends only on the pupil function G(r/λ)

For a centrally symmetric pupil the above *autoconvolution* is just the *autocorrelation*:

$$\tilde{H}(\omega) = G(\lambda\omega) * G^*(-\lambda\omega) = \iint_{pupil} G(\lambda\omega + r) G^*(r) \frac{dr}{\lambda^2}$$

and normalized to the pupil area (in the same reduced units of r/λ):

$$\tilde{T}(\omega) = \frac{\tilde{H}(\omega)}{\iint_{pupil} G(r) G^*(r) \frac{dr}{\lambda^2}}$$

The function $\tilde{T}(\omega)$ is called the (intensity) modulation transfer function (MTF).

The Point Spread Function (1)

The function $|K|^2 = H(\theta)$ (i.e., the Fourier transform of $\tilde{H}(\omega)$) is called the point spread function (PSF) of the system.

The PSF – if circular symmetric – is often described by the half power beam width (HPBW) in angular units, which characterizes the angular resolution of the image.

A word on filtering: all physical pupils have finite sizes → cut-off frequencies $\omega_c = \left(u_c^2 + v_c^2\right)^{1/2}$ must exist. The pupil will act as a low-pass filter on the spatial frequencies of the object I(Θ).

According to the Nyquist-Shannon sampling theorem I(Θ) shall be sampled with a rate of at least $\Delta\theta = 1/2\omega_c$

The Point Spread Function (2)

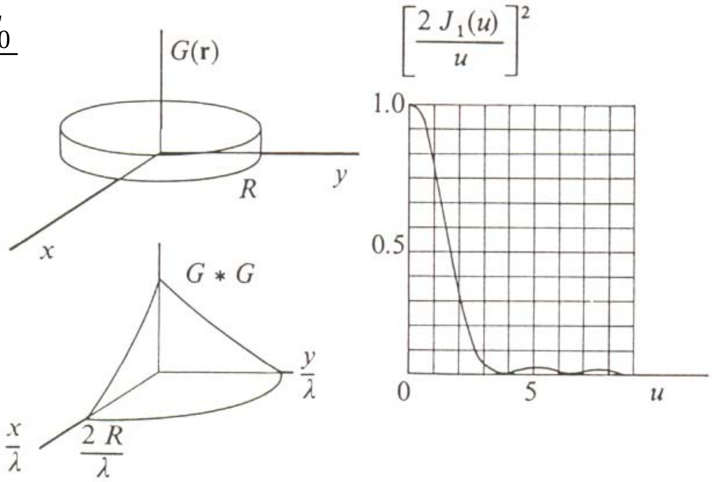
Example: consider circular pupil with pupil function: $G(r) = \Pi\left(\dfrac{r}{2r_0}\right)$

Then the autoconvolution is the autocorrelation,

and $$G(r) * G(r) = \pi\, r_0^2 \left[\frac{2}{\pi} \arccos\left(\frac{r}{2r_0}\right) - \frac{r}{r_0}\left(1 - \frac{r^2}{4r_0^2}\right)^{1/2}\right]$$

The MTF is $$\tilde{T}(\omega) = \frac{2}{\pi}\left[\arccos\left(\frac{\lambda\omega}{2r_0}\right) - \frac{\lambda\omega}{r_0}\left(1 - \frac{\lambda^2\omega^2}{4r_0^2}\right)^{1/2}\right]$$

with a cut-off frequency of $\omega_c = \dfrac{2r_0}{\lambda}$



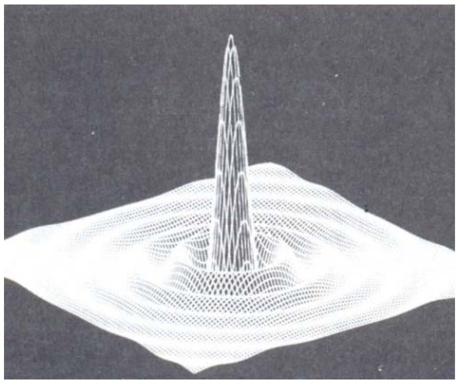
Right:

- the pupil function G(r)
- its autocorrelation G(r)*G(r)
- and its MTF.

The Point Spread Function (3)

When the circular pupil is illuminated by a point source $[I_0(\Theta) = \delta(\Theta)]$ then the resulting PSF can be described with a 1st order Bessel function by:

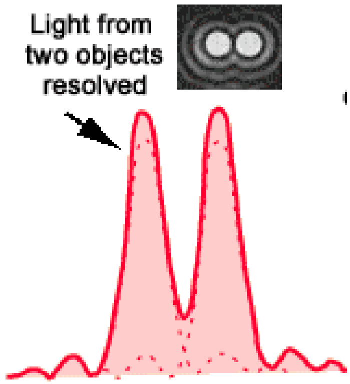
$$I_1(\theta) = \left(\frac{2J_1(2\pi\, r_0\theta/\lambda)}{2\pi\, r_0\theta/\lambda} \right)^2$$



This is also called the Airy function.

The radius of the first dark ring (minimum) is at:

$$r_1 = 1.22\lambda(F\#) \qquad \text{or} \qquad \alpha_1 = \frac{r_1}{f} = 1.22\frac{\lambda}{D}$$



Reminder: the Rayleigh criterion states that two sources can be resolved if the peak of the second source is no closer than the 1st dark Airy ring of the first source.

The PSF of a Real Telescope

Most "real" telescopes have a central obscuration, which modifies our simple pupil function $G(r) = \Pi(r/2r_0)$

The resulting PSF can be described by a modified function

$$I_1(\theta) = \frac{1}{(1-\varepsilon^2)^2} \left(\frac{2J_1(2\pi\, r_0\theta/\lambda)}{2\pi\, r_0\theta/\lambda} - \varepsilon^2 \frac{2J_1(2\pi\, r_0\varepsilon\theta/\lambda)}{2\pi\, r_0\varepsilon\theta/\lambda} \right)^2$$

where ε is the fraction of central obscuration to total pupil area.

Radii of Dark Rings in Airy Pattern[a,b]

ε	w_1	w_2	w_3
0.00	1.220	2.233	3.238
0.10	1.205	2.269	3.182
0.20	1.167	2.357	3.087
0.33	1.098	2.424	3.137
0.40	1.058	2.388	3.300
0.50	1.000	2.286	3.491
0.60	0.947	2.170	3.389

[a] Subscript on w is the number of the dark ring starting at the innermost ring.
[b] $w = v/\pi$.

Astronomical instruments sometimes use a phase mask to reduce the secondary lobes of the PSF (from diffraction at "hard edges". Phase masks introduce a position dependent phase change. This is called apodisation.

The Strehl Ratio

A convenient measure to assess the quality of an optical system is the Strehl ratio.

The Strehl ratio (SR) is the ratio of the observed *peak intensity* of the PSF compared to the theoretical maximum peak intensity of a point source seen with a perfect imaging system working at the diffraction limit.

Using the wavenumber $k=2\pi/\lambda$ and the RMS wavefront error ω one can calculate that:

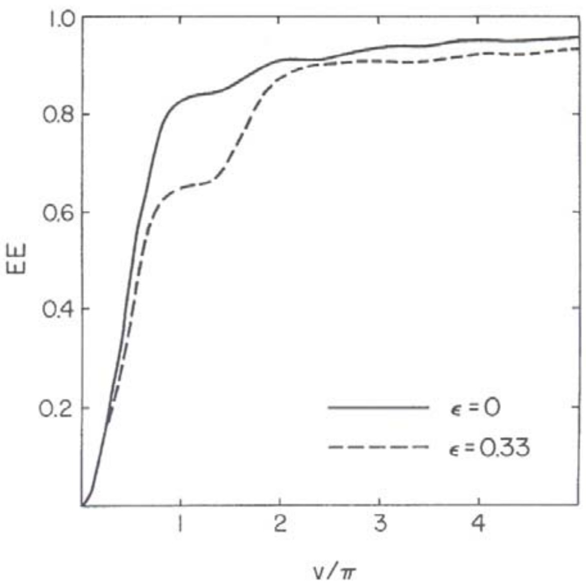
$$SR = e^{-k^2\omega^2} \approx 1 - k^2\omega^2$$

Commonly, a SR > 0.8 is considered diffraction-limited, which corresponds to an average wavefront error of about $\lambda/14$.

The Encircled Energy

In many practical applications (e.g., imaging of very faint sources) the main goal is the maximum concentration of light within a small area. The fraction of the total PSF intensity within a certain radius is given by the encircled energy (EE):

$$EE = \frac{1-\varepsilon^2}{2I_0} \int_0^{v_0} I(P)\, v\, dv$$



Encircled Energy Fraction within Airy Dark Rings[a]

ε	EE_1	EE_2	EE_3
0.00	0.838	0.910	0.938
0.10	0.818	0.906	0.925
0.20	0.764	0.900	0.908
0.33	0.654	0.898	0.904
0.40	0.584	0.885	0.903
0.50	0.479	0.829	0.901
0.60	0.372	0.717	0.873

[a] Subscript on EE is number of dark ring starting at innermost ring.