Relevant Material for Lecture 6

"Galaxies: Structure, Dynamics, and Evolution"

This process is called **phase-mixing**. It is relevant in non-equilibrium situations.

Equilibrium models BT 4.4 1

In equilibrium models, we have by definition $\frac{\partial f}{\partial t} = 0$. Furthermore, we have seen from the CBE $df/dt = 0$

compare to integrals of motion $I[\vec{x}(t), \vec{v}(t)]$

$$\frac{d}{dt} I[\vec{x}(t), \vec{v}(t)] = 0$$

hence integrals of motion satisfy CBE !
the distribution function for an equilibrium model is an integral of motion !

as a consequence: Jeans theorem

- Any steady state solution of CBE depends on w only through integrals of motion, and any function of integrals yields steady state solution of CBE

first part obvious: f itself is an integral of motion
second part also obvious: since $dI/dt = 0$, we have $df(I)/dt = 0$.

A consequence of Jeans theorem:

- A galaxy can be constructed by adding up orbits. Along each orbit, the DF is constant.

this is another justification of Schwarzschild's method.

Spherical systems BT 4.4 2 (p 221+)

- use classical integrals of motion

classical integrals are Energy (E) and angular momentum $\vec{L}$

for exact spherical symmetry, DF can only depend on L^2, not on $\vec{L}$

implication: not individual orbits are used to build models, but sets of orbits !

Isotropic models

distribution function $f(E)$

Define

$$\psi = -\Phi + \Phi_0$$
$$\varepsilon = -E + \Phi_0 = \psi - \tfrac{1}{2}v^2$$

choose Φ_0 such that
$f > 0$ for $\varepsilon > 0$
$f = 0$ for $\varepsilon \leq 0$

The density is integral of distribution function over all velocities:

$$\rho(r) = \int f(\varepsilon) d\vec{v}$$

$$= \int_0^{v_{max}} f(\varepsilon) 4\pi v^2 dv$$

substitute $\varepsilon = \psi - \frac{1}{2}v^2$, $d\varepsilon = -vdv$. Since ε will run from 0 (for $v = v_{max}$) to ψ (for $v = 0$):

$$\rho(r) = 4\pi \int_0^{\psi} f(\varepsilon) \sqrt{2(\psi - \varepsilon)}\, d\varepsilon$$

We can make models by specifying $f(\varepsilon)$. We find a relation of the form $\rho(r) = F(\psi)$. We then have to find solutions of this equation, which also satisfy the Poisson equation:

$$4\pi G \rho(\vec{x}) = \vec{\nabla}^2 \Phi(\vec{x})$$

Solutions can be constructed. For example, assume $f(\epsilon) = \epsilon^{\gamma}$. These give densities of the form $\rho = cnst \times \psi^{(\gamma+1.5)}$. These models are analogous to gas polytropes.

What is "basic set of building blocks" for models of type $f(\varepsilon)$?

- fill in delta function $f(\varepsilon) = \delta(\varepsilon - \varepsilon_0)$

$$\rho = 4\sqrt{2}\pi\sqrt{\psi - \varepsilon_0}$$

maximum in the center, decreasing outwards, to zero at $\psi(r) = \varepsilon_0$.
These are NOT individual orbits, but combinations of orbits.

Distribution function $f(\varepsilon)$ from density BT p 236-237

As we saw on the previous pages, we can calculate the density of a model from the distribution by the integral equation:

$$\rho(r) = 4\pi \int_0^{\psi} f(\varepsilon)\sqrt{2(\psi - \varepsilon)}\, d\varepsilon$$

We now show how to derive f from ρ. Write ρ as $\rho(\psi)$. This is always possible when we know ρ. Then:

$$\frac{1}{2\pi\sqrt{2}} \frac{d\rho}{d\psi} = \int_0^{\psi} \frac{f(\varepsilon)\, d\varepsilon}{\sqrt{\psi - \varepsilon}}$$

This is of Abel form, with solution (Eddington):

$$f(\varepsilon) = \frac{1}{2\pi^2\sqrt{2}} \frac{d}{d\varepsilon} \left\{ \int_0^{\varepsilon} \frac{d\rho}{d\psi} \frac{d\psi}{\sqrt{\varepsilon - \psi}} \right\}$$

Thus, $f(\varepsilon) \geq 0$ if and only if $\{\ldots\}$ increases monotonically with ε. This is true for models that are more centrally concentrated than $\rho(r) = \sqrt{1 - r^2}$

Intermezzo: Abel integral equation

Let

$$f(x) = \int_0^x \frac{g(t)dt}{(x-t)^{1/2}}$$

then

$$g(t) = \frac{1}{\pi} \frac{d}{dt} \int_0^t \frac{f(x)dx}{(t-x)^{1/2}}$$

Proof:
substitute the first equation into the right hand side of the second:

$$F(t) = \frac{1}{\pi} \frac{d}{dt} \int_0^t \int_0^x \frac{g(s)}{(x-s)^{1/2}} \frac{1}{(t-x)^{1/2}} ds dx$$

interchange order of integration

$$F(t) = \frac{1}{\pi} \frac{d}{dt} \int_0^t \int_s^t \frac{g(s)}{(x-s)^{1/2}} \frac{1}{(t-x)^{1/2}} dx ds$$

Use

$$\int_s^t \frac{1}{(x-s)^{1/2}(t-x)^{1/2}} dx = \pi$$

Hence

$$F(t) = \frac{1}{\pi} \frac{d}{dt} \int_0^t g(s)\pi ds$$

It is easy to see that this results into $F(t) = g(t)$

Velocity moments

define velocity dispersions $\overline{v_r^2}$, $\overline{v_\theta^2}$, $\overline{v_\phi^2}$:

$$\overline{v_i^2} = \frac{1}{\rho}\int f(\vec{v}) v_i^2 d\vec{v}$$

If $f = f(\varepsilon)$, f is isotropic: at a given location $\vec{x}$, the distribution in velocities is the same in all directions:

$$f = f(\varepsilon) = f(\psi(r) - \tfrac{1}{2}v^2)$$

Hence $\overline{v_r^2} = \overline{v_\theta^2} = \overline{v_\phi^2} \equiv \sigma^2$
We can write for such systems

$$\sigma^2 = \frac{1}{3}\frac{1}{\rho}\int f(\vec{v})4\pi\ v^2 dv$$

$$= \frac{4\pi}{3\rho}\int f(\vec{v})(2(\psi-\varepsilon))^{3/2}d\varepsilon$$

Isotropic models: Examples: Plummer model BT p 223-225

Potential: $\psi = \dfrac{GM}{\sqrt{r^2+a^2}}$

Mass: $M(r) = \dfrac{Mr^3}{(r^2+a^2)^{3/2}}$

Density: $\rho(r) = \dfrac{3Ma^2}{4\pi(r^2+a^2)^{5/2}}$

Plummer model $\cong$ polytrope with index $n = 5$:

$$\rho = C\psi^5 \qquad \text{with} \qquad C = \frac{3a^2}{4\pi G^5 M^4}$$

Eddington formula:

$$f(\varepsilon) = \frac{64C}{7\sqrt{2}\pi^2}\epsilon^{7/2} \geq 0$$

velocity dispersion: $\overline{v_r^{22}} = \frac{1}{6}\psi = \dfrac{GM}{6\sqrt{r^2+a^2}}$

circular velocity: $v_c^2 = \dfrac{GM(r)}{r} = \dfrac{GMr^2}{(r^2+a^2)^{3/2}}$

Velocity Moments and the Jeans equations
BT 4.2 p 195-198

We usually don't observe the motions of individual stars, but we can observe the average motions, and the spread in velocities (the velocity dispersion). Here we derive equations for the densities, average velocities, and dispersions.

We can derive these WITHOUT taking into account the full distribution function. Assume a population of objects with density ν and distribution function f in a potential Φ. Notice that ν is not necessarily the same as ρ, which is the total matter density. Integrate distribution function $f(\vec{x},\vec{v})$ over velocities. This gives the velocity moments:

$$\nu(\vec{x}) = \int f(\vec{x},\vec{v})\, d^3\vec{v}$$

$$\nu\, \overline{v_i}\, (\vec{x}) = \int v_i f(\vec{x},\vec{v})\, d^3\vec{v}$$

$$\nu\, \overline{v_i v_j}\, (\vec{x}) = \int v_i v_j f(\vec{x},\vec{v})\, d^3\vec{v}$$

where $\overline{v_i}$ is the mean velocity, etc.

Define the velocity dispersions σ^2_{ij} by:

$$\sigma^2_{ij} = \overline{v_i v_j} - \overline{v_i}\ \overline{v_j}$$

Integrate CBE over velocities. This results in:

$$\int \frac{\partial f}{\partial t} d\vec{v} + \sum_{i=1}^{3} \int v_i \frac{\partial f}{\partial x_i} d\vec{v} - \sum_{i=1}^{3} \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} d\vec{v} = 0$$

The last term is zero by straight integration over dv_i

The second term can be simplified by moving the derivative outside the integral:

$$\frac{\partial \nu}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu\, \overline{v_i} = 0$$

This is a continuity equation for the mean streaming motion $\overline{\vec{v}}$ of the stars in configuration space

Integrate CBE times v_j over velocities

$$\frac{\partial}{\partial t} \int f v_j d\vec{v} + \sum_{i=1}^{3} \int v_i v_j \frac{\partial f}{\partial x_i} d\vec{v} - \sum_{i=1}^{3} \frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} d\vec{v} = 0$$

The last term can be simplified. Do the partial integration over dv_i and use the fact that f vanishes for large v:

$$\int v_j \frac{\partial f}{\partial v_i} dv_i = [v_j f] - \int \frac{\partial v_j}{\partial v_i} f dv_i = 0 - \int \delta_{ij} f dv_i,$$

where $\delta_{ij} = 1$ for $i = j$ and 0 for $i \neq j$. Hence

$$\int v_j \frac{\partial f}{\partial v_i} d\vec{v} = \int -\delta_{ij} f d\vec{v} = -\delta_{ij}\nu$$

Hence we obtain

$$\frac{\partial}{\partial t}\int f v_j d\vec{v} + \sum_{i=1}^{3}\int v_i v_j \frac{\partial f}{\partial x_i} d\vec{v} + \sum_{i=1}^{3}\frac{\partial \Phi}{\partial x_i}\delta_{ij}\nu = 0$$

or

$$\frac{\partial(\nu\,\overline{v_j}\,)}{\partial t} + \sum_{i=1}^{3}\frac{\partial}{\partial x_i}(\nu\,\overline{v_i v_j}\,) + \nu\frac{\partial \Phi}{\partial x_j} = 0 \qquad (j = 1, 2, 3)$$

(BT eq 4-24a)

Multiply continuity equation by $\overline{v_j}$, and subtract from the last equation:

$$\nu\frac{\partial\,\overline{v_j}}{\partial t} - \sum_{i=1}^{3}\overline{v_j}\,\frac{\partial \nu\,\overline{v_i}}{\partial x_i} + \sum_{i=1}^{3}\frac{\partial \nu\,\overline{v_i v_j}}{\partial x_i} = -\nu\frac{\partial \Phi}{\partial x_j}$$

This can be rewritten as

$$\nu\frac{\partial\,\overline{v_j}}{\partial t} + \sum_{i=1}^{3}\nu\,\overline{v_i}\,\frac{\partial\,\overline{v_j}}{\partial x_i} = -\nu\frac{\partial \Phi}{\partial x_j} - \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\nu\sigma_{ij}^2$$

(BT eq 4-27)

These are the Jeans equations.

- Almost the same as Euler equations for fluid, but instead of $\partial p/\partial x_j$ we have the summation over the stress tensor $\partial\nu\sigma_{ij}^2\partial x_i$. For a stationary model the left terms disappear completely, and the velocity dispersion tensor counter-acts gravity, just like for a star made of gas. Note that the pressure in a galaxy is anisotropic ! But notice: no equation of state for our "gas" in a galaxy, in contrast to stars !

- Generally 3 equations for 6 unknowns: many solutions!

- Caveat: solutions are not guaranteed to be physical, since no check that $f \geq 0$

Velocity ellipsoid

The tensor σ^2_{ij} is symmetric $\Rightarrow$ it is diagonal in locally orthogonal coordinates $(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$:

$$\begin{pmatrix} \tilde{\sigma}_{11} & 0 & 0 \\ 0 & \tilde{\sigma}_{22} & 0 \\ 0 & 0 & \tilde{\sigma}_{33} \end{pmatrix}$$

The ellipsoid with semi-axes $\tilde{\sigma}_{11}$, $\tilde{\sigma}_{22}$, and $\tilde{\sigma}_{33}$, oriented along the local axes $\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{x}_3$, is called the velocity ellipsoid. It is sometimes used to describe the local velocity distribution

Jeans equations for spherical models

BT 4.2 d) page 203-209

Assume a coordinate system (r, θ, ϕ). We assume the system is invariant under rotations about the center. Hence we have

$$\begin{aligned} \overline{v_r} &= \overline{v_\theta} = \overline{v_\phi} = 0 \\ \overline{v_r v_\theta} &= \overline{v_r v_\phi} = \overline{v_\theta v_\phi} = 0 \\ \overline{v_\theta^2} &= \overline{v_\phi^2} \end{aligned}$$

so that velocity ellipsoid is everywhere aligned with (r, θ, ϕ) coordinates.

Now the Jeans equations reduce to

$$\frac{d(\nu\,\overline{v_r^2}\,)}{dr} + \frac{\nu}{r}\left[2\,\overline{v_r^2} - 2\,\overline{v_\theta^2}\right] = -\nu\frac{d\Phi}{dr}$$

Define the anisotropy function

$$\beta(r) = 1 - \overline{v_\theta^2}\,/\,\overline{v_r^2}$$

.

Clearly $\beta \leq 1$. We obtain one non-trivial Jeans equation

$$\frac{1}{\nu}\frac{d}{dr}\nu\,\overline{v_r^2} + 2\frac{\beta}{r}\,\overline{v_r^2} = -\frac{d\Phi}{dr}$$

Given $\beta(r)$, $\overline{v_r^2}$ and $\nu(r)$ we can derive the potential and mass distribution. Full knowledge of the full distribution function is not necessary to interpret observable parameters such as the velocity dispersion.

Total enclosed mass and rotation curve

For a circular orbit with velocity $v_c(r)$ we have:

$$\frac{d\Phi}{dr} = \frac{GM(<r)}{r^2} = \frac{v_c^2}{r}$$

So the Jeans equation can be written as

$$v_c^2 = \frac{GM(<r)}{r} = -\overline{v_r^2}\left(\frac{d\ln\nu}{d\ln r} + \frac{d\ln\overline{v_r^2}}{d\ln r} + 2\beta\right)$$

Measure: $\nu(r)$, $\overline{v_r^2}$ and β $\Rightarrow$ determine enclosed mass

11. Velocity Moments and the Jeans equations BT 4.2 p 195-198

We usually don't observe the motions of individual stars, but we can observe the average motions, and the spread in velocities (the velocity dispersion). Here we derive equations for the densities, average velocities, and dispersions.

We can derive these *without* taking into account the full distribution function.

Assume a population of objects with density ν and distribution function f in a potential Φ.

Notice that ν is not necessarily the same as ρ, which is the total matter density.

Integrate distribution function $f(\vec{x}, \vec{v})$ over velocities. This gives three velocity moments:

0. Spatial density of stars / 0th moment:

$$\nu(\vec{x}) = \int f(\vec{x}, \vec{v})\, \mathrm{d}^3\vec{v}$$

1. Mean stellar velocity / first moment:

$$\overline{v_i}\,(\vec{x}) \equiv \frac{1}{\nu} \int v_i f(\vec{x}, \vec{v})\, \mathrm{d}^3\vec{v}, \quad i = 1, 2, 3$$

2. Second moments:

$$\overline{v_i v_j}\,(\vec{x}) \equiv \frac{1}{\nu} \int v_i v_j f(\vec{x}, \vec{v})\, \mathrm{d}^3\vec{v}, \quad j = 1, 2, 3$$

Plus:

Velocity dispersion tensor:

$$\sigma_{ij}^2 \equiv \overline{(v_i - \overline{v_i}\,)(v_j - \overline{v_j}\,)} = \overline{v_i v_j} - \overline{v_i}\;\overline{v_j}$$

Much like fluid dynamics, the three moments and the velocity dispersion tensor are constrained by 3 equations: the Jeans equations. These three equations are:

Jeans equation 1 (the Continuity equation):

$$\boxed{\frac{\partial \nu}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \, \overline{v_i} = 0}$$

Jeans equation 2 (the Force equation):

$$\boxed{\frac{\partial (\nu \, \overline{v_j}\,)}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} (\nu \, \overline{v_i v_j}\,) + \nu \frac{\partial \Phi}{\partial x_j} = 0}$$

Jeans equation 3 (a common rewrite of Jeans-2):

$$\boxed{\nu \frac{\partial \, \overline{v_j}}{\partial t} + \sum_{i=1}^{3} \nu \, \overline{v_i} \, \frac{\partial \, \overline{v_j}}{\partial x_i} = -\nu \frac{\partial \Phi}{\partial x_j} - \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \sigma_{ij}^2}$$

Jeans equation 1
(the Continuity equation)

This equation is obtained by taking the 0th moment of the Collisionless Boltzmann Equation (CBE) in v.
Recall the CBE:

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0 \qquad \text{(CBE).}$$

The 0th moment in v of the CBE:

$$\int \text{CBE d}\vec{v}$$

or:

$$\int \frac{\partial f}{\partial t} \text{d}\vec{v} + \sum_{i=1}^{3} \int v_i \frac{\partial f}{\partial x_i} \text{d}\vec{v} - \sum_{i=1}^{3} \frac{\partial \Phi}{\partial x_i} \int \frac{\partial f}{\partial v_i} \text{d}\vec{v} = 0.$$

Using the divergence theorem, we can rewrite the last term as a surface integrale:

$$\int \frac{\partial f}{\partial t} \text{d}\vec{v} + \sum_{i=1}^{3} \int v_i \frac{\partial f}{\partial x_i} \text{d}\vec{v} - \sum_{i=1}^{3} \frac{\partial \Phi}{\partial x_i} \int [f]_{-\infty}^{\infty} d\vec{S} = 0.$$

Since $f(\vec{x}, \vec{v}) = 0$ for "infinite" velocities, the last term is zero.

The second term can be simplified by moving the derivative outside the integral:

$$\boxed{\frac{\partial \nu}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu\, \overline{v_i} = 0} \qquad \text{Jeans} - 1$$

or:

$$\frac{\partial \nu}{\partial t} + \nabla \cdot (\nu\, \overline{\vec{v}}\,) = 0$$

This is a *continuity equation* for the mean streaming motion $\overline{\vec{v}}$ of the stars in configuration space

Note the similarity with the the continuity equations for fluid mechanics:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0$$

Jeans equation 2 + 3
(the Force equation)

The first moment in v of the CBE:

$$\int \text{ CBE } v\, \mathrm{d}\vec{v}$$

or:

$$\frac{\partial}{\partial t} \int f v_j d\vec{v} + \sum_{i=1}^{3} \int v_i v_j \frac{\partial f}{\partial x_i} d\vec{v} - \sum_{i=1}^{3} \frac{\partial \Phi}{\partial x_i} \int v_j \frac{\partial f}{\partial v_i} d\vec{v} = 0$$

The last term can be simplified. Do the partial integration over dv_i and use the fact that f vanishes for large v:

$$\int v_j \frac{\partial f}{\partial v_i} d\vec{v} =$$

$$\int \int v_j (f(v_i = \infty) - f(v_i = -\infty)) d^2 v_{\neq i} - \int \frac{\partial v_j}{\partial v_i} f d\vec{v} =$$

$$0 - \int \delta_{ij} f d\vec{v} = -\delta_{ij} \nu$$

where $\delta_{ij} = 1$ for $i = j$ and 0 for $i \neq j$.

Hence:

$$\frac{\partial}{\partial t}\int f v_j d\vec{v} + \sum_{i=1}^{3}\int v_i v_j \frac{\partial f}{\partial x_i} d\vec{v} + \sum_{i=1}^{3}\frac{\partial \Phi}{\partial x_i}\delta_{ij}\nu = 0$$

or

$$\boxed{\frac{\partial(\nu\,\overline{v_j}\,)}{\partial t} + \sum_{i=1}^{3}\frac{\partial}{\partial x_i}(\nu\,\overline{v_i v_j}\,) + \nu\frac{\partial \Phi}{\partial x_j} = 0}$$

(Jeans – 2)

Multiply continuity equation (Jeans-1) by $\overline{v_j}$, and subtract from the last equation (Jeans-2):

$$\nu\frac{\partial\,\overline{v_j}}{\partial t} - \sum_{i=1}^{3}\overline{v_j}\,\frac{\partial \nu\,\overline{v_i}}{\partial x_i} + \sum_{i=1}^{3}\frac{\partial \nu\,\overline{v_i v_j}}{\partial x_i} + \nu\frac{\partial \Phi}{\partial x_j} = 0$$

Using

$$\frac{\partial \nu \sigma_{ij}^2}{\partial x_i} = \frac{\partial}{\partial x_i}\nu(\,\overline{v_i v_j} - \overline{v_i}\;\overline{v_j}\,) =$$

$$\frac{\partial(\nu\,\overline{v_i v_j}\,)}{\partial x_i} - \overline{v_j}\,\frac{\partial(\nu\,\overline{v_i}\,)}{\partial x_i} - \nu\,\overline{v_i}\,\frac{\partial\,\overline{v_j}}{\partial x_i}$$

We obtain the more frequently used variant of Jeans-2, Jeans equations 3:

$$\boxed{\nu\frac{\partial\,\overline{v_j}}{\partial t} + \sum_{i=1}^{3}\nu\,\overline{v_i}\,\frac{\partial\,\overline{v_j}}{\partial x_i} = -\nu\frac{\partial \Phi}{\partial x_j} - \sum_{i=1}^{3}\frac{\partial}{\partial x_i}\nu\sigma_{ij}^2}$$

(Jeans – 3)

Hence we obtain the analogue of the Euler equation:

$$\frac{\rho\partial\vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\rho\vec{\nabla}\Phi - \vec{\nabla}p = 0$$

- Almost the same as Euler equations for fluid, but instead of $\vec{\nabla}p$ we have the summation over the stress tensor $\partial\nu\sigma_{ij}^2\partial x_i$. For a stationary model the left terms disappear completely, and the velocity dispersion tensor counter-acts gravity, just like for a star made of gas. Note that the pressure in a galaxy is anisotropic ! But notice: no equation of state for our "gas" in a galaxy, in contrast to stars !

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The ellipsoid with semi-axes $\tilde{\sigma}_{11}$, $\tilde{\sigma}_{22}$, and $\tilde{\sigma}_{33}$, oriented along the local axes $\tilde{x}_1$, $\tilde{x}_2$, and $\tilde{x}_3$, is called the velocity ellipsoid. It is sometimes used to describe the local velocity distribution

Jeans equations for spherical models
BT 4.2d, page 203-209

Assume a coordinate system (r, θ, ϕ). We assume the system is invariant under rotations about the center. Hence we have

$$\overline{v_r} = \overline{v_\theta} = \overline{v_\phi} = 0$$

$$\overline{v_r v_\theta} = \overline{v_r v_\phi} = \overline{v_\theta v_\phi} = 0$$

$$\overline{v_\theta^2} = \overline{v_\phi^2}$$

so that velocity ellipsoid is everywhere aligned with (r, θ, ϕ) coordinates.

Now the Jeans equation(-2/3) in the stationary case reduces to:

$$\frac{d(\nu\,\overline{v_r^2}\,)}{dr} + \frac{\nu}{r}\left[2\,\overline{v_r^2} - 2\,\overline{v_\theta^2}\right] = -\nu\frac{d\Phi}{dr}$$

Define the anisotropy function:

$$\beta(r) = 1 - \overline{v_\theta^2}\,/\,\overline{v_r^2}\,.$$

Clearly $\beta \leq 1$. We obtain one non-trivial Jeans equation:

$$\frac{1}{\nu}\frac{d}{dr}\nu\,\overline{v_r^2} + 2\frac{\beta}{r}\overline{v_r^2} = -\frac{d\Phi}{dr}$$

Given $\beta(r)$, $\overline{v_r^2}$ and $\nu(r)$ we can derive the potential and mass distribution. Full knowledge of the full distribution function is not necessary to interpret observable parameters such as the velocity dispersion.

Total enclosed mass and rotation curve

For a circular orbit with velocity $v_c(r)$ we have:

$$\frac{d\Phi}{dr} = \frac{GM(<r)}{r^2} = \frac{v_c^2}{r}$$

So the Jeans equation can be written as

$$v_c^2 = \frac{GM(<r)}{r} = -\overline{v_r^2}\left(\frac{d\ln\nu}{d\ln r} + \frac{d\ln\overline{v_r^2}}{d\ln r} + 2\beta\right)$$

Measure: $\nu(r)$, $\overline{v_r^2}$ and β $\Rightarrow$ determine enclosed mass