# Relevant Material for Lecture 5

"Galaxies: Structure, Dynamics, and Evolution"

## Galactic Dynamics - Continued

3.6 Time scales (BT 4 to start 4.1)

dynamical timescale, particle interaction timescale

Is gravitational force dominated by short or long range encounters? (N.B. in a gas, only short range forces are relevant).

In a galaxy, the situation is different.

Consider force with which stars in cone attract star in apex of cone.

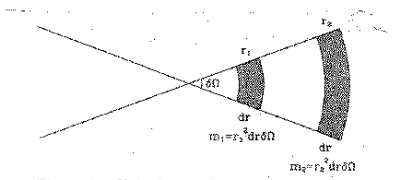



**Figure 4-1.** If the density of stars were everywhere the same, the stars in each of the shaded segments would make equal contributions to the net force on a star at the cone's apex. Thus the acceleration of a star at the apex is determined by the large-scale gradient in the density of stars within the galaxy.

Force $\sim 1/r^2$, with $r$ the distance from apex. If $\rho$ is almost constant, then the mass in a shell with width $dr$ increases as $r^2 dr$.

Hence differential force is constant at each $r$, and we have to integrate all the way out to obtain the total force.



Realistic densities decrease after some radius, so that the force will be determined by the density distribution on a galactic scale (characterized by the half mass radius).

3.7 Relaxation time

Short range encounters do not dominate $\rightarrow$
Approximate force field with a smooth density $\rho(x)$ instead of point masses.

- Contrary of situation in gas: only consider long range encounters (long range $\sim$ scale of the galaxy)

Assume all stars have mass $m$. Analyze perturbations due to the fact that density is not smooth, but consists of individual stars. Simplify, and look first at single star-star encounter.

What is effect of a single encounter with point mass on motion of star?

- Exact: BT §7.1: hyperbolic Keplerian encounter
- Estimate: straight line trajectory past stationary perturber

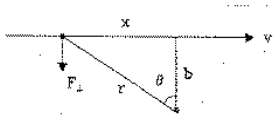



**Figure 4-2.** A field star approaches the test star at speed $v$ and impact parameter $b$. We estimate the resulting impulse to the test star by approximating the field star's trajectory as a straight line.

The perpendicular force $\vec{F}_\perp$ gives perturbation $\delta\vec{v}_\perp$:

$$\vec{F}_\perp = \frac{Gm^2\cos\theta}{r^2} = \frac{Gm^2\cos\theta}{(b^2+x^2)} = \frac{Gm^2 b}{(b^2+x^2)^{3/2}}$$

$$\sim \frac{Gm^2}{b^2[1+(vt/b)^2]^{3/2}}$$

Newton: $\frac{d}{dt}\delta\vec{v}_\perp = \frac{\vec{F}_\perp}{m} \quad \Rightarrow$

$$\delta\vec{v}_\perp = \int \mathrm{d}t \frac{\vec{F}_\perp}{m} = \int \frac{Gm}{b^2[1+(vt/b)^2]^{3/2}}\mathrm{d}t$$

$$= \frac{Gm}{bv}\int_{-\infty}^{\infty} \frac{\mathrm{d}s}{(1+s^2)^{3/2}} = \frac{Gm}{bv}\frac{s}{\sqrt{1+s^2}}\Big|_{-\infty}^{\infty}$$

$$= \frac{2Gm}{bv}$$

Note: approximation fails when
$\delta\vec{v}_\perp > v \quad \Rightarrow \quad b < Gm/v^2 = b_{\min}$

Galaxy has characteristic radius $R$.

Define crossing time $t_c$ as the time it takes a star to move through the galaxy $t_c = R/v$

Calculate number of perturbing encounters per crossing time $t_c$



In a crossing time, the star has 1 "encounter" with each other star in the galaxy

The impact parameter of each encounter can be derived by projecting each star onto a plane perpendicular to the unperturbed motion of the star

Hence "flatten" the galaxy in the plane perpendicular to the motion of the star, and assume that the stars are homogeneously distributed in that plane, out to a radius $R$, and no stars outside $R$. This is obviously a simplifying assumption, but it is reasonably accurate.

This can be used to derive the distribution of impact parameters:

$N$ stars in total in Galaxy, distributed over total surface $\pi R^2$

# per unit area: $\frac{N}{\pi R^2}$

In a crossing time, the star has $\delta n$ encounters with impact parameter between $b$ and $b+db$. $\delta n$ is given by the area of the annulus $2\pi b db$ times the density of stars on the surface, which is $N/(\pi R^2)$:

$\delta n = \frac{N}{\pi R^2} 2\pi b \mathrm{d}b = \frac{2N}{R^2} b\ db$

Result: $\langle \delta\vec{v}_\perp \rangle \equiv 0$

as the perturbations are randomly distributed, and will not change the average velocity

$$\langle \delta v_\perp^2 \rangle = \Big(\frac{2Gm}{bv}\Big)^2 \frac{2Nb}{R^2} \mathrm{d}b = 8N \Big(\frac{Gm}{Rv}\Big)^2 \frac{\mathrm{d}b}{b}$$

as each perturbation adds to $\langle \delta v_\perp^2 \rangle$ by an equal amount $(2Gm/bv)^2$.

The encounters do not produce an average perpendicular velocity, but they do produce an average (perpendicular velocity)$^2$. Hence, on average, the stars still follow their average path, but they tend to "diffuse" around it.

The total increase in rms perpendicular velocity can be calculated by integrating over all impact parameters from $b_{min}$ to infinity:

Total rms increase:

$$\langle \Delta v_\perp^2 \rangle = \int_{b_{\rm min}}^{R} \langle \delta v_\perp^2 \rangle \; db = \int_{b_{\rm min}}^{R} 8N \left(\frac{Gm}{Rv}\right)^2 \; db/b \; =$$

$$= 8N \Big(\frac{Gm}{Rv}\Big)^2 \ln \Lambda$$

with $\ln \Lambda$ = Coulomb logarithm $= \ln \dfrac{R}{b_{\rm min}}$

We can rewrite this equation. Use

$$b_{\rm min} = Gm/v^2$$

From virial theorem

$$v^2 = GM/R = GNm/R$$

Hence $b_{min} = Gm/(GNm/R) = R/N$

$$\ln \Lambda = \ln R/b_{min} = \ln \frac{R}{R/N} = \ln N$$



Furthermore from virial theorem:

$$\frac{GM^2}{R} = Mv^2 \rightarrow \frac{GM}{R} = v^2 \rightarrow \frac{GNm}{R} = v^2 \rightarrow N = \frac{v^2 R}{Gm}$$

so that: $\dfrac{\langle \Delta v_\perp^2 \rangle}{v^2} = \dfrac{8 \ln N}{N}$

This last number is the fractional change in energy per crossing time. Hence we need the inverse number of crossings $N/(8 \ln N)$ to get $\langle \Delta v_\perp^2 \rangle \sim v^2$

The timescale $t_{relax}$ is defined as the time it takes to deflect each star significantly by two body encounters, and it is therefore equal to

$$t_{relax} = \frac{N}{8 \ln N} t_c$$

Conclusions

- effect of point mass perturbations decreases as $N$ increases
- even for low N=50, $\langle \Delta v_\perp^2 \rangle / v^2 = 0.6$, hence deflections play a moderate role.
- for larger systems the effect of encounters become even less important

Notice: one derives the same equation when the exact formulas for the encounters are used. Put in another way, the encounters with $b < b_{min}$ do not dominate.

3.8 Relaxation time for large systems

$$t_{\rm relax} = \frac{0.1N}{\ln N} t_c$$

| System | $N$ | $t_c$ (yr) | $t_{\rm relax}$ (yr) |
|---|---|---|---|
| globular cluster | $10^5$ | $10^5$ | $2 \times 10^8$ |
| galaxy | $10^{11}$ | $10^8$ | $10^{17}$ |
| galaxy cluster | $10^3$ | $10^9$ | $3 \times 10^{10}$ |

Age of Universe $\sim$ Hubble time $\sim 1.5 \times 10^{10}$ yr

$\Rightarrow$ Galaxies are collisionless systems

- motion of a star accurately described by single particle orbit in smooth gravitational field of galaxy
- no need to solve $N$-body problem with $N = 10^{11}$ (!)

## Collisionless Boltzmann Equation

BT 4.1 p. 190-193

Consider a system with a large nunber of stars
At any $t$ define the distribution function $f(\vec{x},\vec{v},t)d\vec{x}d\vec{v}$ $= \#$ of stars in volume $d\vec{x}$ with velocities in range $d\vec{v}$ (centered on $\vec{x},\vec{v}$).

$f(\vec{x},\vec{v},t)$ is called the distribution function or the phase space density

at all $\vec{x},\vec{v}$: $f \geq 0$

We now have reduced $6N$ functions $\vec{x}_i,\vec{v}_i$ into one 7-DIMENSIONAL FUNCTION.

If we derive the time evolution of $f$, we can completely ignore individual particles !

(Notice that we can always rewrite $f(\vec{x},\vec{v},t)$ as a summation of $\delta$ functions. We would then get back our "original" particles. It shows that in some sense, one 7-dimentional function is "more complex" than 6N 1-dimensional functions. But if we take smooth distribution functions, they are much simpler than the 6N 1-dimensional functions.)

In order to derive the time evolution, first define a new coordinate $\vec{w}$:

$$\vec{w} = (\vec{x},\vec{v}) = (w_1, w_2, ...w_6)$$

where $w_1 = x_1$, $w_2 = x_2$, ... $w_4 = v_1$, etc. Hence the star has coordinate $\vec{w}$ in phase-space. The flow of the



star is given by

$$\dot{\vec{w}} = (\dot{\vec{x}},\dot{\vec{v}}) = (\vec{v}, -\vec{\nabla}\Phi)$$

The flow $\dot{\vec{w}}$ conserves stars.

Hence we have the continuity equation:

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \frac{\partial (f\dot{w}_\alpha)}{\partial w_\alpha} = 0$$

Why is this ? Integrate over any volume. The first term gives the increase in number of stars in the volume. The second term is equal to:

$$\int_V \vec{\nabla}\cdot(f\dot{\vec{w}}) = \int_S (f\dot{\vec{w}})\cdot d^2S$$

This is the surface integral over the flow **out** of the volume. Hence the equation guarantees that stars are conserved (the density can only increase if stars move into the volume).

A special property of $\dot{\vec{w}}$ is

$$\sum_{\alpha=1}^{6} \frac{\partial \dot{w}_\alpha}{\partial w_\alpha} = \sum_{\alpha=1}^{3} \left( \frac{\partial v_i}{\partial x_i} + \frac{\partial \dot{v}_i}{\partial v_i} \right)$$

Notice that by definition $\partial v_i/\partial x_i = 0$ because $x_i$ and $v_i$ are independent coordinates. The second term is equal to

$$\sum_{\alpha=1}^{3} -\frac{\partial}{\partial v_i}\left(\frac{\partial \Phi}{\partial x_i}\right)$$

This is also equal to zero because the potential does not depend on $v_i$.

Hence we can now write

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \frac{\partial f \dot{w}_\alpha}{w_\alpha} = 0$$

or

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \left[ \frac{\partial f}{\partial w_\alpha} . \dot{w}_\alpha + f \frac{\partial \dot{w}_\alpha}{\partial w_\alpha} \right] = 0$$

The last term on the right is zero, as we have seen above. Hence

$$\frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \dot{w}_\alpha \frac{\partial f}{\partial w_\alpha} = 0$$

Or we write this as

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i} \frac{\partial f}{\partial v_i} = 0$$

or

$$\frac{\partial f}{\partial t} + \vec{v} . \vec{\nabla} f - \vec{\nabla} \Phi . \frac{\partial f}{\partial \vec{v}} = 0$$

These equations are the Collisionless Boltzmann Equation (CBE).
The CBE is sufficient to calculate the evolution of any $f$ with time.
A different description of the same equation: consider the evolution of $f$ if one moves along with a particle (this is the Lagrangian derivative):



$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \frac{dw_\alpha}{dt} \frac{\partial f}{\partial w_\alpha} = \frac{\partial f}{\partial t} + \sum_{\alpha=1}^{6} \dot{w_\alpha} \frac{\partial f}{\partial w_\alpha} = 0$$

Hence the CBE can be written as

$$\frac{df}{dt} = 0$$

If you move along with the particles, their mass is conserved. If you move along with the particles, the density is conserved. Hence the flow in phase-space is incompressible (the density remains conserved along a flow-line).

## 4.2 Constants and Integrals of motion (BT 3.1 p 110-113)

First, we define the 6 dimensional "phase space" coordinates $(\vec{x}, \vec{v})$. They are conveniently used to describe the motions of stars. Now we introduce:

- *Constant of motion*: a function $C(\vec{x}, \vec{v}, t)$ which is constant along any orbit:

$$C(\vec{x}(t_1), \vec{v}(t_1), t_1) = C(\vec{x}(t_2), \vec{v}(t_2), t_2)$$

$C$ is a function of $\vec{x}$, $\vec{v}$, and time t.

- *Integral of motion*: a function $I(x, v)$ which is constant along any orbit:

$$I[\vec{x}(t_1), \vec{v}(t_1)] = I[\vec{x}(t_2), \vec{v}(t_2)]$$

$I$ is not a function of time ! Thus: integrals of motion are constants of motion,

but constants of motion are not always integrals of motion!

E.g.: for a circular orbit $\psi = \Omega\, t + \psi_o$, so that $C = t - \psi/\Omega$.

C is constant of motion, but not an integral as it depends on t.

### Constants of motion

6 for any arbitrary orbit:

Initial position $(\vec{x}_0, \vec{v}_0)$ at time $t = t_0$. Can always be calculated back from $\vec{x}, \vec{v}, t$.



### Integrals

Much harder to define. E.g.:

Energy (all static potentials): $E(\vec{x}, \vec{v}) = \frac{1}{2}v^2 + \Phi$

$L_z$ (axisymmetric potentials)

$\vec{L}$ (spherical potentials)

- Integrals constrain geometry of orbits.

examples:

- 1. Spherical potentials:

$E, L_x, L_y, L_z$ are integrals of motion, but also $E, |L|$ and the direction of $\vec{L}$ (given by the unit vector $\vec{n}$, which is defined by two independent numbers). $\vec{n}$ defines the plane in which $\vec{x}$ and $\vec{v}$ must lie. Define coordinate system with z axis along $\vec{n}$

$$\vec{x} = (x_1, x_2, 0)$$

$$\vec{v} = (v_1, v_2, 0)$$

$\rightarrow$ $\vec{x}$ and $\vec{v}$ constrained to 4D region of the 6D phase space. In this 4 dimensional space, $|L|$ and $E$ are conserved. This constrains the orbit to a 2 dimensional space. Hence the velocity is uniquely defined for a given $\vec{x}$

$$v_r = \pm\sqrt{2(E - \Phi) - L^2/r^2}$$

$$v_\psi = \pm L/r$$

- 2. Integrals in 2 dimensional flattened potentials

Examples:

Circular potential $V(x,y) = V(\vec{r})$

Two integrals: $E, L_z$.

Flattened potential $V(x,y) = \ln(x^2 + \frac{y^2}{a} + 1)$

Only "classic" integral of motion: $E$

Figures on the next page show the orbits that one gets by integrating the equations of motion.

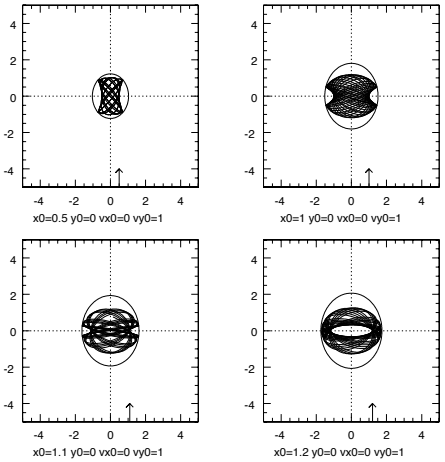
Clearly the orbits are regular and *do not fill equipotential surface*

Furthermore, they do not traverse each point in a random direction, but generally only in 2 directions

Conclusion: the orbits do not occupy a 3 dimensional space in the 4-dimensional phase-space, but they occopy only a 2-dimensional space !

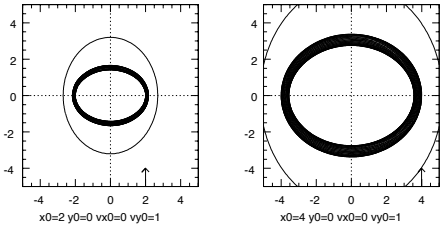
This indicates that there is an additional integral of motion: 'a non-classical integral'

The non-classical integral, plus the regular 'Energy', constrain the orbit to lie on a 2 dimensional surface in the 4 dimensional phase-space.




the first 3 are box orbits (no net angular momentum)

avoid outer x-axis




loop orbits (with net angular momentum)

avoid inner x-axis

can circulate in two directions.

## A homogeneous ellipsoid

The homogeneous ellipsoid helps us to understand how additional integrals of motions, and box orbits, exist. Consider a density distribution:

$\rho = \rho_0 H(1 - m^2)$, with

$m^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$

$H(x) = 1$ for $x \geq 0$, $H(x) = 0$ for $x < 0$

Potential inside the ellipsoid:

$$V = A_x x^2 + A_y y^2 + A_z z^2 + C_0$$

Forces are of the form $F_i = -A_i x_i$, i.e. 3 independent harmonic oscillators:

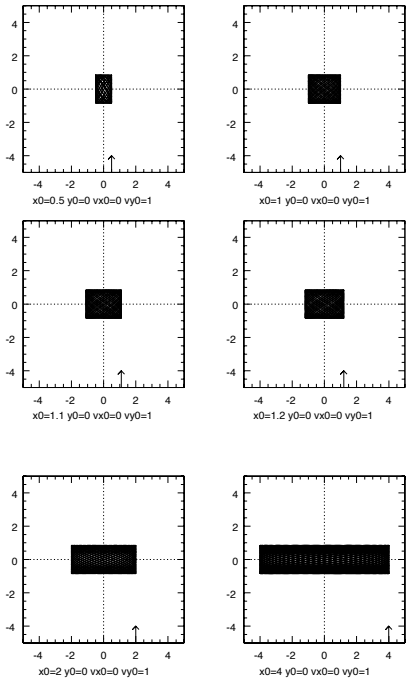
$$x_i = a_i \cos(\omega_i t + \psi_{0,i})$$

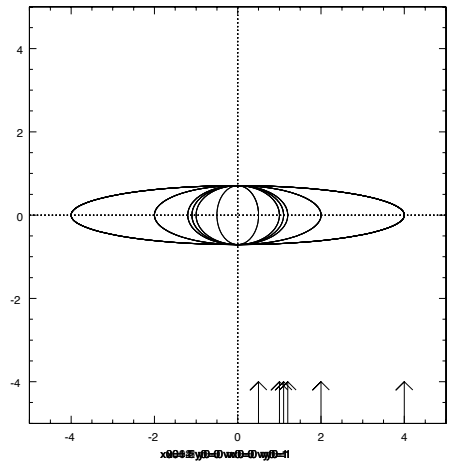
3 integrals of motion, $E_i$

Hence the orbits are as shown on the next page

## Orbits for general homogeneous ellipsoid

All orbits are box orbits






**Special case: a=b**

all orbits are loop orbits

• 3. Axisymmetric potentials (BT 3.2)

$\Phi = \Phi(R, |z|)$, with $R^2 = x^2 + y^2$.

For z=0: orbits as if potential were circular

For (R,z): 3D orbit can be described as 2D orbit in meridional plane:

$$\frac{d^2\vec{r}}{dt^2} = -\nabla\Phi(R, z)$$

with $\vec{e}_r, \vec{e}_\psi, \vec{e}_z$ unit vectors in $r$, $\psi$ and z direction:

$$\vec{r} = R\,\vec{e}_r + z\,\vec{e}_z$$

$$\nabla\Phi = \frac{\delta\Phi}{\delta R}\vec{e}_r + \frac{\delta\Phi}{\delta z}\vec{e}_z$$

The equations of motion reduce to

$$\frac{d^2R}{dt^2} - R\dot{\psi}^2 = \frac{-\delta\Phi}{\delta R}$$

$$\frac{d}{dt}R^2\dot{\psi} = -\frac{\delta\Phi}{\delta\psi} = 0$$

$$\frac{d^2z}{dt^2} = -\frac{\delta\Phi}{\delta z}$$

From the second equation we see that $L_z$ is constant. Hence $\dot{\psi} = L_z^2/R^2$. If we use this in the upper equation we obtain for the motion in two dimensions $R, z$:

$$\frac{d^2R}{dt^2} = -\frac{\delta\Phi_{eff}}{\delta R} \qquad \frac{d^2z}{dt^2} = -\frac{\delta\Phi_{eff}}{\delta z}$$

with $\Phi_{eff} = \Phi(R,z) + \frac{L_z^2}{2R^2}$. Hence 3D motion can be reduced to motion in (R,z) plane or *meridional* plane, under influence of the effective potential $\Phi_{eff}$. Total energy:

$$E = \Phi_{eff} + 1/2\dot{R}^2 + 1/2\dot{z}^2$$

.

**Allowed region in meridional plane**

For energy E motion only allowed in area where $\Phi_{eff} < E$. E.g.

If $\Phi = \frac{1}{2}v_0^2\ \ln(R^2 + \frac{z^2}{q^2})$, then

$$\Phi_{eff} = \frac{1}{2}v_0^2\ \ln(R^2 + \frac{z^2}{q^2}) + \frac{L_z^2}{2R^2}$$

, i.e. only motion if $E > \Phi_{eff}$.

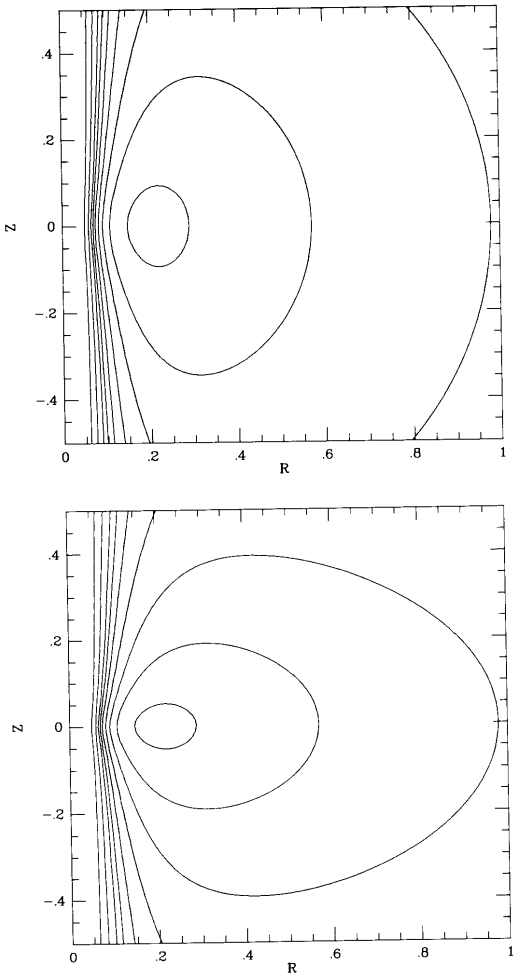



**Figure 3-2.** Level contours of the effective potential of equation (3-50) when $L_z = 0.2$ and: $q = 0.9$ (top); $q = 0.5$ (bottom). Contours are shown for $\Phi_{\rm eff} = -1, -0.5, 0, 0.5, 1, 1.5, 2, 3, 5$, assuming $v_0 = 1$.

Lines of constant $\Phi_{eff}$ are shown in Figure 3.2. Stars with energy E have zero velocity at curves of $\Phi_{eff} = E$.

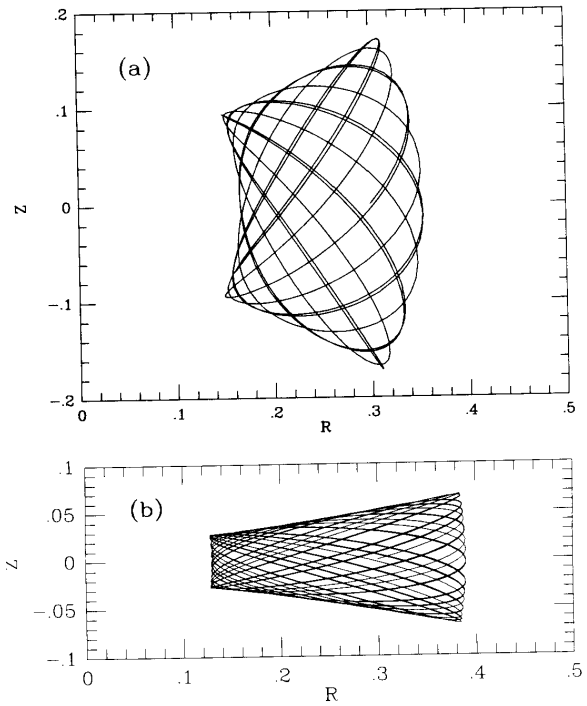



**Figure 3-3.** Two orbits in the potential of equation (3-50) with $q = 0.9$. Both orbits are at energy $E = -0.8$ and angular momentum $L_z = 0.2$, and we assume $v_0 = 1$.

Again: not all orbits fill the space $\Phi_{eff} < E$ fully!

Two integrals $(E, L_z)$ reduce the dimensionality of the orbit from 6 to 4 (e.g. R, z, $\psi$, $v_z$). Therefore another integral of motion must play a role → dimensionality reduced to 3 (e.g. $R, z, \psi$).

This integral is a non-classical integral of motion

## A general 3-dimensional potential

Stäckel potential( $\rho = 1/(1+m^2)^2$)

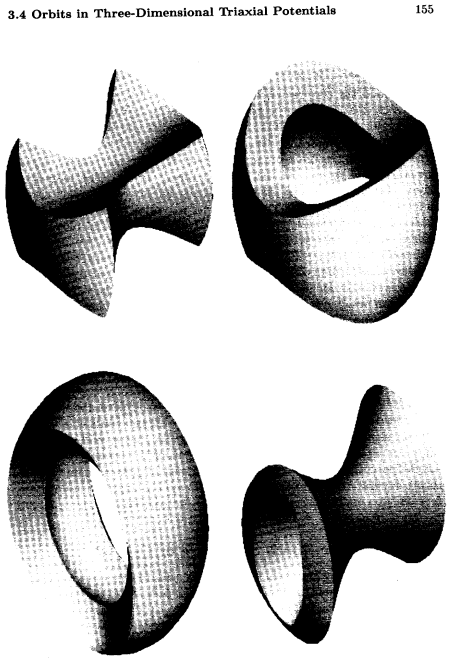



**Figure 3-20.** Orbits in a nonrotating triaxial potential. Clockwise from top left: (a) box orbit; (b) short-axis tube orbit; (c) inner long-axis tube orbit; (d) outer long-axis tube orbit. [Courtesy of T. Statler; see Statler (1986).]

## A Simple recipe to build galaxies

Schwarzschild's method:

- Define density $\rho$
- Calculate potential, forces
- Integrate orbits, find orbital densities $\rho_i$
- Calculate weights $w_i > 0$ such that

$$\rho = \sum \rho_i w_i$$

Examples: build a 2D galaxy in a logarithmic potential $\Phi = \ln(1 + x^2 + y^2/a)$.

- As we saw, box orbits void the outer x-axis
- As we saw, loop orbits void the inner x-axis

$\rightarrow$ both box and loop orbits are needed.

Suppose we have constructed a model.

- What kind of rotation can we expect ?

  box orbits: no net rotation

  loop orbits: can rotate either way: positive, negative, or "neutral".

  Hence: a maximum rotation is defined if all loop orbits rotate the same way. The rotation can vary between zero, and this maximum rotation

## 4 Orbits in stationary Potentials (BT 3 to page 107)

Now we have seen how to calculate forces and potentials from the smoothed density $\rho$. We can now analyse how stars move in this potential. Because two body interactions can be ignored, we can analyse each star by itself. We therefore speak of "orbits"

### 4.1 Orbits in spherical potentials

Potential function of $r = |\vec{r}|$: $\Phi = \Phi(r)$
equation of motion for star with unit mass

$$\frac{d^2\vec{r}}{dt^2} = F(r)\vec{e}_r$$

recall that $\vec{r} \times \vec{r} = 0$ for any $\vec{r}$

$$\frac{d}{dt}\left(r \times \frac{\mathrm{d}\vec{r}}{dt}\right) = \frac{\mathrm{d}\vec{r}}{dt} \times \frac{\mathrm{d}\vec{r}}{dt} + \vec{r} \times \frac{d^2\vec{r}}{dt^2} = F(r)\vec{r} \times \vec{e}_r = 0$$

Hence $\vec{L} = \vec{r} \times \dot{\vec{r}}$ is constant with time. $\vec{L}$ = angular momentum/unit mass . $\vec{L}$ is always perpendicular to the plane in which $\vec{r}$ and $\vec{v}$ lie. Since it is constant with time, these vectors always lie in the same plane. Hence the orbit is constrained to this plane.

Use polar coordinates $(r, \psi)$ in orbital plane:

rewrite equations of motion in polar coordinates

$$\ddot{r} - r\dot{\psi}^2 = F(r)$$
$$2\dot{r}\dot{\psi} + r\ddot{\psi} = F_\psi$$

Because of the circular symmetry, we have $F_\psi = 0$.
Hence:

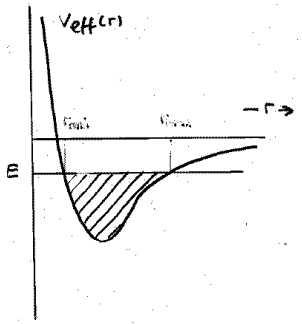
$$2\dot{r}\dot{\psi} + r\ddot{\psi} = \frac{1}{r}\frac{dr^2\dot{\psi}}{dt} = 0 \Rightarrow r^2\dot{\psi} = rv_\perp = L = \mathrm{c}^{st}$$
$$\ddot{r} - r\dot{\psi}^2 = \ddot{r} - \frac{L^2}{r^3} = -\frac{d\Phi}{dr}$$

where $\Phi$ is the potential.

Multiply the last equation by $\dot{r}$, and integrate w.r.t. $t$:

$$\tfrac{1}{2}\dot{r}^2 = E - \Phi - \frac{L^2}{2r^2} = E - \Phi_{\mathrm{eff}}(r)$$

with $E$ the energy. This equation governs radial motion in *effective potential* $\Phi_{\mathrm{eff}}(r)$




Motion possible only when $\dot{r}^2 \geq 0$

$$r_{\min} \leq r \leq r_{\max}$$

pericenter apocenter

Typical orbit in a spherical potential is a planar rosette

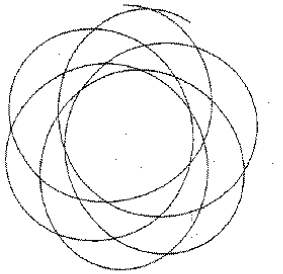


**Figure 3-1.** A typical orbit in a spherical potential forms a rosette.

Angle $\Delta\psi$ between successive apocenter passages depends on mass distribution:

$$\pi < \Delta\psi < 2\pi$$

homogeneous sphere point mass

Special cases

$r_{\min} = r_{\max}$ circular orbit $\frac{v_\perp^2}{r} = \frac{d\Phi}{dr} = \frac{GM(r)}{r^2}$

$L = 0 \quad \Rightarrow \quad$ radial orbit $\quad \frac{1}{2}\dot{r}^2 = E - \Phi(R)$

Homogeneous sphere

$$\Phi(r) = \tfrac{1}{2}\Omega^2 r^2 + Constant$$

In radial coordinates

$$\ddot{\vec{r}} = -\Omega^2 \vec{r}$$

or in cartesian coordinates $x, y$

$$\ddot{x} = -\Omega^2 x \qquad \ddot{y} = -\Omega^2 y$$

Hence solutions are

$$x = X\cos(\Omega t + c_x) \qquad y = Y\cos(\Omega t + c_y)$$

where $X, Y, c_x$ and $c_y$ are arbitracy constants. Hence, even though energy and angular momentum restrict orbit to a "rosetta", these orbits are even more special: they do not fill the area between the minimum and maximum radius, but are always closed !

The same holds for Kepler potential. But beware, for the homogeneous sphere the particle does two radial excursions per cycle around the center, for the Kepler potential, it does one radial excursion per angular cycle.

We now wish to "classify" orbits and their density distribution in a systematic way. For that we use Integrals of motion.

## A general 3-dimensional potential

Stäckel potential( $\rho = 1/(1+m^2)^2$)

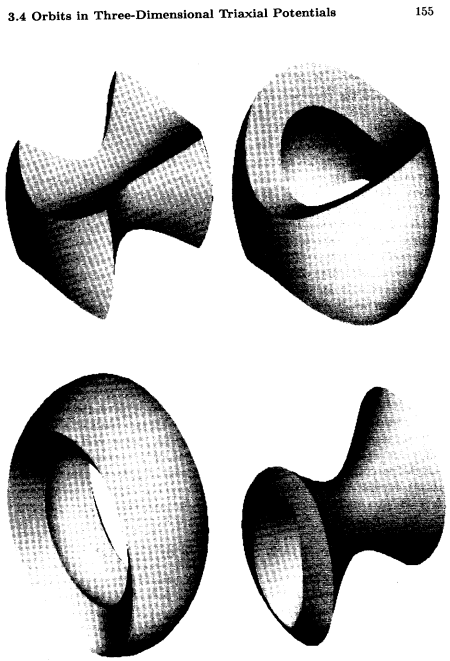



**Figure 3-20.** Orbits in a nonrotating triaxial potential. Clockwise from top left: (a) box orbit; (b) short-axis tube orbit; (c) inner long-axis tube orbit; (d) outer long-axis tube orbit. [Courtesy of T. Statler; see Statler (1986).]



## A Simple recipe to build galaxies

Schwarzschild's method:

- Define density $\rho$
- Calculate potential, forces
- Integrate orbits, find orbital densities $\rho_i$
- Calculate weights $w_i > 0$ such that

$$\rho = \sum \rho_i w_i$$

Examples: build a 2D galaxy in a logarithmic potential $\Phi = \ln(1 + x^2 + y^2/a)$.

- As we saw, box orbits void the outer x-axis
- As we saw, loop orbits void the inner x-axis

$\rightarrow$ both box and loop orbits are needed.

Suppose we have constructed a model.

- What kind of rotation can we expect ?

  box orbits: no net rotation

  loop orbits: can rotate either way: positive, negative, or "neutral".

  Hence: a maximum rotation is defined if all loop orbits rotate the same way. The rotation can vary between zero, and this maximum rotation

- Is the solution unique ?

  the density is two-dimensional function

  box orbits are defined by 2 integrals of motion, say the coordinates of the corner

  loop orbits have two integrals of motion

  Hence, we have two construct a 2 dimensional function from the superposition of two 2-2dimentional functions

$$\rho(\vec{x},\vec{y}) = w_{box}(I_1,I_2)\rho_{box}(I_1,I_2)+$$

$$w_{loop}(I_1,I_2)\rho_{loop}(I_1,I_2)$$

The unknown functions are $w_{box}(I_1,I_2)$ and $w_{loop}(I_1,I_2)$. The system is underdetermined. Hence, many solutions are possible.