Key Point #9 from Bachelor Course: Interactions with Other Stars Not Especially Important

Galactic Dynamics - Continued

3.6 Time scales (BT 4 to start 4.1)

dynamical timescale, particle interaction timescale

Is gravitational force dominated by short or long range encounters? (N.B. in a gas, only short range forces are relevant).

In a galaxy, the situation is different.

Consider force with which stars in cone attract star in apex of cone.

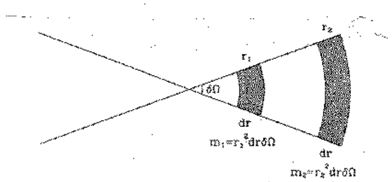


Figure 4-1. If the density of stars were everywhere the same, the stars in each of the shaded segments would make equal contributions to the net force on a star at the cone's apex. Thus the acceleration of a star at the apex is determined by the large-scale gradient in the density of stars within the galaxy.

Force $\sim 1/r^2$, with r the distance from apex. If ρ is almost constant, then the mass in a shell with width dr increases as $r^2 dr$.

Hence differential force is constant at each r, and we have to integrate all the way out to obtain the total force.

Realistic densities decrease after some radius, so that the force will be determined by the density distribution on a galactic scale (characterized by the half mass radius).

3.7 Relaxation time

Short range encounters do not dominate $\rightarrow$
Approximate force field with a smooth density $\rho(x)$ instead of point masses.

- Contrary of situation in gas: only consider long range encounters (long range $\sim$ scale of the galaxy)

Assume all stars have mass m. Analyze perturbations due to the fact that density is not smooth, but consists of individual stars. Simplify, and look first at single star-star encounter.

What is effect of a single encounter with point mass on motion of star?

- Exact: BT §7.1: hyperbolic Keplerian encounter
- Estimate: straight line trajectory past stationary perturber

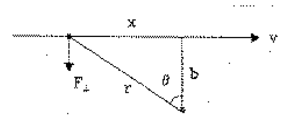


Figure 4-2. A field star approaches the test star at speed v and impact parameter b. We estimate the resulting impulse to the test star by approximating the field star's trajectory as a straight line.

The perpendicular force $\vec{F}_\perp$ gives perturbation $\delta\vec{v}_\perp$:

$$\vec{F}_\perp = \frac{Gm^2 \cos\theta}{r^2} = \frac{Gm^2\cos\theta}{(b^2+x^2)} = \frac{Gm^2 b}{(b^2+x^2)^{3/2}}$$

$$\sim \frac{Gm^2}{b^2[1+(vt/b)^2]^{3/2}}$$

Newton: $\frac{d}{dt}\delta\vec{v}_\perp = \frac{\vec{F}_\perp}{m} \quad \Rightarrow$

$$\delta\vec{v}_\perp = \int \mathrm{d}t \frac{\vec{F}_\perp}{m} = \int \frac{Gm}{b^2[1+(vt/b)^2]^{3/2}} \mathrm{d}t$$

$$= \frac{Gm}{bv}\int_{-\infty}^{\infty} \frac{\mathrm{d}s}{(1+s^2)^{3/2}} = \frac{Gm}{bv}\left.\frac{s}{\sqrt{1+s^2}}\right|_{-\infty}^{\infty}$$

$$= \frac{2Gm}{bv}$$

Note: approximation fails when

$\delta\vec{v}_\perp > v \quad \Rightarrow \quad b < Gm/v^2 = b_{\min}$

Galaxy has characteristic radius R.

Define crossing time t_c as the time it takes a star to move through the galaxy $t_c = R/v$

Calculate number of perturbing encounters per crossing time t_c

In a crossing time, the star has 1 "encounter" with each other star in the galaxy

The impact parameter of each encounter can be derived by projecting each star onto a plane perpendicular to the unperturbed motion of the star

Hence "flatten" the galaxy in the plane perpendicular to the motion of the star, and assume that the stars are homogeneously distributed in that plane, out to a radius R, and no stars outside R. This is obviously a simplifying assumption, but it is reasonably accurate.

This can be used to derive the distribution of impact parameters:

N stars in total in Galaxy, distributed over total surface πR^2

per unit area: $\quad \frac{N}{\pi R^2}$

In a crossing time, the star has δn encounters with impact parameter between b and $b + db$. δn is given by the area of the annulus $2\pi b db$ times the density of stars on the surface, which is $N/(\pi R^2)$:

$\delta n = \frac{N}{\pi R^2} 2\pi b \mathrm{d}b = \frac{2N}{R^2} b \; db$

Result: $\quad \langle \delta\vec{v}_\perp \rangle \equiv 0$

as the perturbations are randomly distributed, and will not change the average velocity

$$\langle \delta v_\perp^2 \rangle = \left(\frac{2Gm}{bv}\right)^2 \frac{2Nb}{R^2} \mathrm{d}b = 8N \left(\frac{Gm}{Rv}\right)^2 \frac{\mathrm{d}b}{b}$$

as each perturbation adds to $\langle \delta v_\perp^2 \rangle$ by an equal amount $(2Gm/bv)^2$.

The encounters do not produce an average perpendicular velocity, but they do produce an average (perpendicular velocity)^2. Hence, on average, the stars still follow their average path, but they tend to "diffuse" around it.

The total increase in rms perpendicular velocity can be calculated by integrating over all impact parameters from b_{min} to infinity:

Total rms increase:

$$\langle \Delta v_\perp^2 \rangle = \int_{b_{\min}}^{R} \langle \delta v_\perp^2 \rangle \; db = \int_{b_{\min}}^{R} 8N \left(\frac{Gm}{Rv}\right)^2 \; db/b =$$

$$= 8N \left(\frac{Gm}{Rv}\right)^2 \ln \Lambda$$

with $\ln \Lambda =$ Coulomb logarithm $= \ln \dfrac{R}{b_{\min}}$

We can rewrite this equation. Use

$b_{\min} = Gm/v^2$

From virial theorem

$v^2 = GM/R = GNm/R$

Hence $b_{min} = Gm/(GNm/R) = R/N$

$$\ln \Lambda = \ln R/b_{min} = \ln \frac{R}{R/N} = \ln N$$

Furthermore from virial theorem:

$$\frac{GM^2}{R} = Mv^2 \rightarrow \frac{GM}{R} = v^2 \rightarrow \frac{GNm}{R} = v^2 \rightarrow N = \frac{v^2 R}{Gm}$$

so that: $\dfrac{\langle \Delta v_\perp^2 \rangle}{v^2} = \dfrac{8 \ln N}{N}$

This last number is the fractional change in energy per crossing time. Hence we need the inverse number of crossings $N/(8 \ln N)$ to get $\langle \Delta v_\perp^2 \rangle \sim v^2$

The timescale t_{relax} is defined as the time it takes to deflect each star significantly by two body encounters, and it is therefore equal to

$$t_{relax} = \frac{N}{8 \ln N} t_c$$

Conclusions

- effect of point mass perturbations decreases as N increases
- even for low N=50, $\langle \Delta v_\perp^2 \rangle / v^2 = 0.6$, hence deflections play a moderate role.

- for larger systems the effect of encounters become even less important

Notice: one derives the same equation when the exact formulas for the encounters are used. Put in another way, the encounters with $b < b_{min}$ do not dominate.

3.8 Relaxation time for large systems

$$t_{\rm relax} = \frac{0.1N}{\ln N} t_c$$

System	N	t_c (yr)	$t_{\rm relax}$ (yr)
globular cluster	10^5	10^5	2×10^8
galaxy	10^{11}	10^8	10^{17}
galaxy cluster	10^3	10^9	3×10^{10}

Age of Universe $\sim$ Hubble time $\sim 1.5 \times 10^{10}$ yr

$\Rightarrow$ Galaxies are collisionless systems

- motion of a star accurately described by single particle orbit in smooth gravitational field of galaxy
- no need to solve N-body problem with $N = 10^{11}$ (!)

Key Point #10 from Bachelor Course: Galaxy Formation is Driven by the Collapse of the Dark Matter Halo (when any specific region of the universe exceeds the critical density)

Galaxy Formation

Summary

- We can't say from equilibrium physics what the structure is of a galaxy
- We don't know what a galaxy consists of
  Dark matter (90% ?)
  some stars, gas (10%?)

  we don't even know where a galaxy stops !

Why are galaxies like we see them ?
How is the structure of galaxies determined ?

the bad news

equilibrium physics does NOT fix galaxy structure
(No HR diagram for galaxies as for stars)

the good news

equilibrium physics does NOT fix galaxy structure
galaxy structure is determined by GALAXY FORMATION, i.e., the process by which galaxies formed

The big question in galaxy research is that of **GALAXY FORMATION**

HOW DO GALAXIES FORM ?

We know that

- Universe was very smooth at z=1000 from the Cosmic Background Radiation (fluctuations $\propto 10^{-5}$)
- Universe is not so smooth now: galaxies, clusters, large scale structure

Where does this come from ?

SIMPLEST HYPOTHESIS:

gravitational collapse of very small density enhancements
- We start with a homogeneous universe, with a very small section at slightly higher density
- we notice that the relative density contrast $\delta = \delta\rho/\rho$ grows with time.

this is easy to derive using simple equations, and we will show this below

The expanding universe

it is observed that the universe expands

- nearby: $v = H_0 D$
  - v=velocity , D = distance, H_0= 100 h km/s/Mpc

  h=0.73 $\pm$ 0.03

What are the equations of motion ?

- The complete answer follows from General Relativity
- the correct answer can also be derived from basic, Newtonian physics

Consider a homogeneous sphere, with density ρ, and uniformly expanding
Consider the force on a shell of the sphere, at radius r, and velocity $\dot{r}$:

$$\ddot{r} = -\frac{GM(<r)}{r^2}$$

As the sphere expands, the mass is conserved. Multiply both sides with $\dot{r}$

$$\dot{r}\ddot{r} = -\frac{GM\dot{r}}{r^2}$$

Integrate once:

$$\tfrac{1}{2}(\dot{r})^2 = \frac{GM}{r} + c$$

The terms here can be easily identified: on the left is the kinetic energy, on the right is the gravitational energy, and the integration constant. The total energy is given by

$$E = \tfrac{1}{2}(\dot{r})^2 - \frac{GM}{r} = c$$

Now write $M = \frac{4}{3}\pi r^3 \rho$

$$\tfrac{1}{2}(\dot{r})^2 = \frac{G\frac{4}{3}\pi r^3 \rho}{r} + c$$

$$= G\tfrac{4}{3}\pi \rho r^2 + c$$

$$\left(\frac{\dot{r}}{r}\right)^2 = \tfrac{8}{3}\pi G\rho + \frac{c}{r^2}$$

This is the final equation of motion. Remember, however, that ρ is not a constant, it varies like $\propto r^{-3}$

The left term is special, since it is equal to H, the Hubble "constant". The consequence is, that the hubble constant is not constant !

Universe models

Notice that E is the energy of the sphere, normalized to $r = \infty$. The value of E will determine the evolution of the sphere. The equations above imply that

$$E = c$$

We have different types of models:

- $E = c = 0$. The total energy is zero. The gravitational and kinetic energy compensate each other. The expansion halts at $t = \infty$. Require that $r = t^\alpha$, we find for α: $\alpha = 2/3$. Hence, the solution is $r = t^{2/3}$
- $E = c < 0$ Gravitational energy dominates. The universe will halt, and collapse again !
- $E = c > 0$ Kinetic energy dominates. The universe will keep expanding. Gravitational energy will become less and less important, and at some phase expansion will be at a constant rate.

We can rewrite the Energy criterium as a density criterium. If $c = E = 0$, we define a critical density ρ_c from the last equation:

$$H^2 = \tfrac{8}{3}\pi G\rho_c$$

which can be rewritten as

$$\rho_c = \frac{H^2}{\frac{8}{3}\pi G} = \frac{3H^2}{8\pi G}$$

This density has the property that it would halt the expansion at infinity.
The true density of the universe is usually expressed as Ω

$$\Omega = \rho/\rho_c = \frac{8\pi G\rho}{3H^2}$$

If $\Omega = 1$, then the expansion stops at infinity. If $\Omega < 1$, the expansion continues forever. If $\Omega > 1$, the expansion halts and reverses.

Looking back in time

The remarkable thing is, that as the universe expands, the photon "expands" as well.

first consider objects close to each other

$$v = H_0 D$$

where D is the distance between the 2 objects.
Hence the light will be shifted in wavelength by (simple Doppler)

$$\frac{\delta\lambda}{\lambda} = \frac{v}{c} = \frac{H_0 D}{c}$$

But now compare this to the expansion of the universe in the time it took for the photon to travel from the

moment of emission t_{emit} to the moment of detection t_{obs}

$$\delta D = v_{expand} \times (t_{emit} - t_{obs})$$
$$= H_0 \times D \times (t_{emit} - t_{obs})$$

It follows what

$$\frac{\delta D}{D} = H_0 \times (t_{emit} - t_{obs})$$

But since $(t_{emit} - t_{obs}) = \frac{D}{c}$

$$\frac{\delta D}{D} = \frac{H_0 \times D}{c}$$

and we obtain

$$\frac{\delta\lambda}{\lambda} = \frac{\delta D}{D}$$

Hence the wavelength of the photons expand like the distance between objects ! In short

$$\lambda \propto r$$

where r is the radius which we introduced earlier. This leads us to introduce the "redshift" z:

$$1 + z = \frac{\lambda(observed)}{\lambda(emitted)} = \frac{r(t(observed))}{r(t(emitted))}$$

The redshift is easily measured for galaxies from emission lines, absorption lines, etc. For nearby galaxies

$$v = cz$$

We know galaxies with redshifts of 5 and higher - i.e., the universe was 5 times smaller, and 125 times denser, when that light was emitted than it is now...

We generally use redshift to indicate the relative size of the universe, because it is so easy to use.

How can we derive time of emission from redshift ?

for $\Omega = 1$ universe

$$r \propto t^{2/3}$$

Redshift is defined by

$$(1+z) = \frac{\lambda_{obs}}{\lambda_{emit}} = \frac{r_{obs}}{r_{emit}} = \left(\frac{t_{obs}}{t_{emit}}\right)^{2/3}$$

Hence

$$t_{emit} = \frac{t_{obs}}{(1+z)^{1.5}}$$

Take MicroWaveBackground: $z = 1000$, $t_{obs} = 13\ 10^9$ year, hence $t_{emit} = 4\ 10^5$ year.

How does Ω evolve with redshift ?

Take the equation

$$\left(\frac{\dot{r}}{r}\right)^2 = \tfrac{8}{3}\pi G\rho + \frac{c}{r^2}$$

- if $\Omega = 1$, the total energy is zero, and remains zero - hence Ω remains 1
- if $\Omega < 1$, the energy was defined as:

$$E = c = \tfrac{1}{2}(\dot{r})^2 - \frac{GM}{r}$$

and $\dot{r}$ increases at increasing redshift (i.e., decreases with increasing time). Since

$$\Omega = \frac{8\pi G\rho}{3H^2}$$

we can rewrite the energy equation

$$\frac{E}{r^2} = \tfrac{1}{2}\left(\frac{\dot{r}}{r}\right)^2 - \frac{G\frac{4}{3}\pi\rho r^3}{r^3}$$

$$= \tfrac{1}{2}(H^2 - \tfrac{8}{3}\pi G\rho)$$

Now devide both sides by $4/3\pi G\rho$:

$$\frac{3E}{4\pi G\rho r^2} = \frac{3H^2}{8\pi G\rho} - 1$$

or

$$\frac{3E}{4\pi G}\frac{1}{\rho r^2} = \frac{1}{\Omega} - 1$$

Now we know that $\rho \propto r^{-3}$, hence $\rho r^2 \propto r^{-1} \propto 1 + z$ In short, the left term is proportional to $1/(1+z)$, and gets smaller and smaller with increasing redshift z. Hence the term on the right also gets smaller as $1/(1+z)$. Now write $\Omega = 1 + \delta\Omega$. Hence

$$\frac{1}{\Omega} - 1 = -\delta\Omega$$

And since $\frac{1}{\Omega} - 1$ evolves like $(1/(1+z))$

$$\delta\Omega \propto 1/(1+z)$$

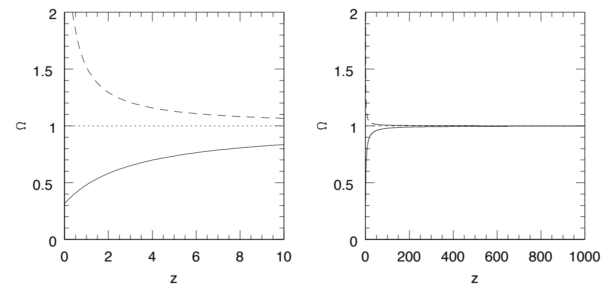
Hence, with increasing redshift, Ω gets closer and closer to 1
As a consequence, Ω was very, very close to 1 at high redshifts, independent of the current value !

How do we now form a galaxy ?

Answer: look back in the distant past (very high redshift). Ω was very close to 1 at that time. Assume that in some volume, the density ρ was enhanced by a minute fraction $\delta\rho$. Since Ω was almost 1 , the smallest $\delta\rho$ is high enough to push the *local* Ω above 1. This local volume has Ω higher than 1, and the total energy is lower than 0. In short, it will not keep expanding like the rest would (if, e.g., $\Omega \leq 1$). Hence it will collapse at some time, and will form a galaxy. Or a cluster...

Evolution of a overdense region

Ω will be close to 1 in the distant past of our universe, independent of the current value. This is put graphically below



Now, take the universe at a nominal redshift of say, $z = 10^6$. We know that Ω is very close to 1. Assume that for some reason or another, density fluctuations are present in this universe:

$$\delta = \rho/\bar{\rho} - 1$$

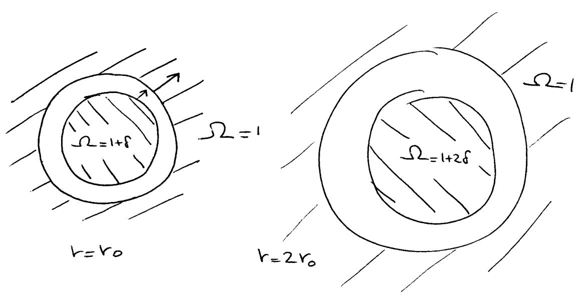
where $\bar{\rho}$ is the mean density of the universe at that epoch, which we take to be the critical density.

We now wish to understand how a overdensity $\delta > 0$ evolves with time.

It turns out that the overdensity grows with expansion like

$$\delta \propto r \propto t^{2/3}$$

We can prove this by considering a homogeneous sphere at $z = 10^6$ with overdensity δ. The sphere is embedded in a $\Omega = 1$ universe.



The particles in the sphere do not feel anything from the outside universe. Hence the sphere will evolve like it is a "separate universe", with $\Omega = (1 + \delta)$. Since a universe with $\Omega > 1$ will collapse at some time, the sphere will collapse at some time $t_{collapse}$. Before that, the density of the sphere will evolve like

$$\frac{1}{\Omega(t)} - 1 = \frac{\frac{1}{\Omega_0} - 1}{1 + z(t)} = \frac{\frac{1}{\Omega_0} - 1}{1/r(t)} = (\frac{1}{\Omega_0} - 1)r(t)$$

where r is the "radius" of the universe (expansion parameter). Take at $z = 10^6 : \Omega_0 = 1 + \delta_0$, then we find

$$\frac{1}{\Omega_0} - 1 = \frac{1}{1 + \delta_0} - 1 = 1 - \delta_0 - 1 = -\delta_0$$

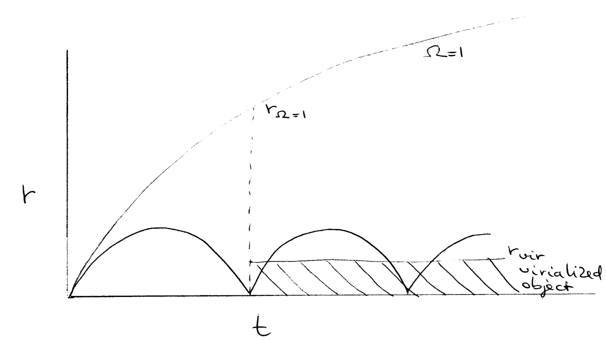
Hence, the equation results into

$$\delta(t) = \delta_0 \times \frac{r(t)}{r_0} = \delta_0 * \left(\frac{t}{t_0}\right)^{2/3}$$

Hence, the density contrast of the sphere increases linearly with expansion radius. As a result, if a fluctuation was present of 10^{-6} at $z = 10^6$, it would have grown to a fluctuation of $\delta = 1$ at $z = 0$ under its own gravity.

This mechanism is the basic mechanism to form galaxies

In detail, what happens is the following:



The sphere will collapse, and start oscillating (if we ignore the material just outside of the sphere). In reality, the sphere will have internal density fluctuations, and it will settle to an equilibrium structure, with a radius of about half the "maximum expansion " radius.

This radius is often called the "virialization radius" , r_{vir}. The sphere will obtain this radius at the first collapse time, which is also called the "virialization time" (or "formation time")
By comparing the maximum expansion radius of the sphere to the "normal expansion" radius of the universe (with $\Omega = 1$) at the virialization time, we can derive:

$$r_{max} = \frac{1}{\frac{1}{4}(12\pi)^{2/3}} r_{\Omega=1}(t_{collapse}) \quad (\approx 0.36 r_{\Omega=1}(t_{collapse}))$$

This is derived using analytical solutions for the expansion of the universe. Since $r_{vir} = 1/2 r_{max}$

$$r_{vir} = \frac{1}{\frac{1}{2}(12\pi)^{2/3}} r_{\Omega=1}$$

The relative density of the sphere, compared to the rest of the universe, is simply given by the ratio $(r_{\Omega=1}/r_{vir})^3$, since the mass of the sphere is conserved, but the density is increased compared to the $\Omega = 1$ universe since the mass is put in a smaller density structure.
Hence

$$\frac{\rho_{vir}}{\rho(universe)(z = z_{vir})} = (r_{\Omega=1}/r_{vir})^3$$

$$= (1/2(12\pi)^{2/3})^3 = 18\pi^2 = 178$$

This makes a very specific prediction for the density of objects (galaxies, clusters, etc : If a galaxy forms at a redshift z_{form}, it will have a density which is 178 times higher the density of the universe at z_{form}.

After the galaxy has formed, it will remain the same, whereas the universe will keep expanding. Hence, the density contrast will increase with time

$$\frac{\rho_{vir}}{\rho(universe)} = 178 * (r/r_{form})^3$$

$$= 178 * ((1 + z_{form})/(1 + z))^3 = 178 * (t/t_{form})^2$$

This can be used as a simple recipe: we now measure that galaxies have an overdensity of about 10^5 inside the optical radius. This part would be formed at a redshift of

$$(1 + z_{form}) = (10^5/178)^{1/3} = 8$$

The galaxy is much bigger, however, than the optical radius. The halo has a density profile which goes like $\rho \propto r^{-2}$. The average density goes down like r^{-2}, and the density contrast will be a lot smaller if we take the halo into account. If we assume that the halo extends to 100 kpc (10 times further), the density will be lower by a factor of 100, and the formation redshift will be

$$(1 + z_{form}) = (10^3/178)^{1/3} = 1.8$$

Hence, the fact galaxies have halos has a very important consequence for galaxy formation: it makes them bigger, have lower mean density, and thereby form much later !
At maximum expansion, the region has an overdensity of about 5, this increases very rapidly to 178 in the next half of the total collapse time.

the smallest $\delta\rho$ is high enough to push the *local* Ω above 1. This local volume has Ω higher than 1, and the total energy is lower than 0. In short, it will not keep expanding like the rest would (if, e.g., $\Omega \leq 1$). Hence it will collapse at some time, and will form a galaxy. Or a cluster...

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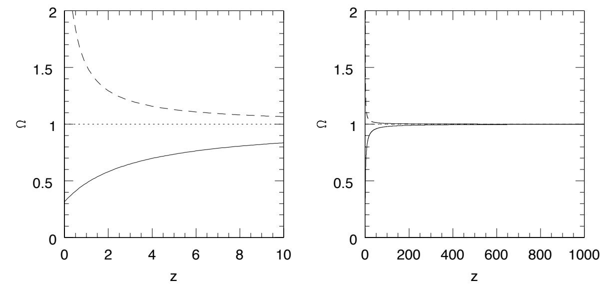

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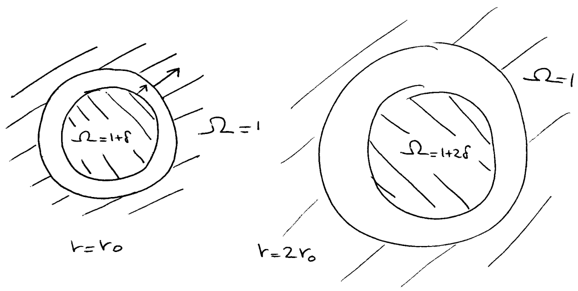

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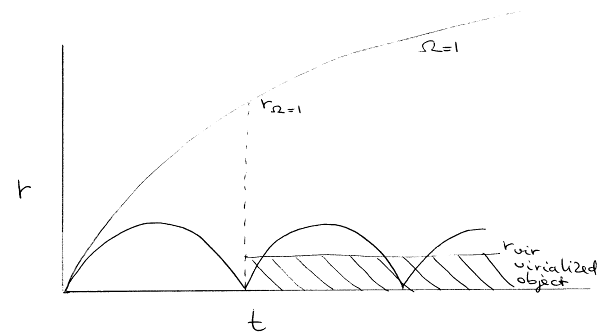

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$$(1 + z_{form}) = (10^5/178)^{1/3} = 8$$

The galaxy is much bigger, however, than the optical radius. The halo has a density profile which goes like $\rho \propto r^{-2}$. The average density goes down like r^{-2}, and the density contrast will be a lot smaller if we take the halo into account. If we assume that the halo extends to 100 kpc (10 times further), the density will be lower by a factor of 100, and the formation redshift will be

$$(1 + z_{form}) = (10^3/178)^{1/3} = 1.8$$

Hence, the fact galaxies have halos has a very important consequence for galaxy formation: it makes them bigger, have lower mean density, and thereby form much later !
At maximum expansion, the region has an overdensity of about 5, this increases very rapidly to 178 in the next half of the total collapse time.