Galaxies: Structure, Dynamics, and Evolution
Problem Set 1
Instructor: Dr. Bouwens

Here is problem set #1. The entire problem set will be due before class on Monday, February 23 (email them to Wout and hand them before class). Be sure to pay extra attention to problem 3, as your solution to that problem will be checked carefully and used in determining your homework grade.

1. Derive the potential from the density for a point-source mass M, uniform density ρ sphere, and a singular isothermal sphere ρ_0/r^2 (where ρ_0 is the density at radius 1 and r is the radius) using the following equation presented in class:

$$\Phi = -4\pi G\left[\frac{1}{r}\int_0^r \rho(r')r'^2 dr' + \int_r^\infty \rho(r')r'dr'\right] \quad (1)$$

Show your work. As the potential for a singular isothermal sphere blows up at radius 0, please derive an expression for the potential such that the potential equals zero at r_0.

$$\Phi(r) = -4\pi G\left[\frac{1}{r}\int_0^r \rho(r')r'^2\,dr' + \int_r^\infty \rho(r')r'\,dr'\right].$$

Case 1. Point source of mass M

For a point mass,

$$\rho(r) = M\delta^{(3)}(\mathbf{r}).$$

For $r > 0$, the enclosed mass is

$$4\pi\int_0^r \rho(r')r'^2\,dr' = M,$$

so

$$\int_0^r \rho(r')r'^2\,dr' = \frac{M}{4\pi}.$$

The second integral vanishes outside the point source. Therefore

$$\Phi(r) = -4\pi G\left[\frac{1}{r}\frac{M}{4\pi}\right] = -\frac{GM}{r}.$$

$$\boxed{\Phi(r) = -\frac{GM}{r}}$$

Case 2. Uniform-density sphere

Let the sphere have radius r and constant density ρ for $r' < r$, with $\rho = 0$ for $r' > r$. Its total mass is

$$M = \frac{4\pi}{3}\rho r^3.$$

The second integral is zero, so

$$\Phi(r) = -4\pi G\frac{1}{r}\frac{\rho^3}{3} = -\frac{GM}{r}.$$

$$\boxed{\Phi(r) = -\frac{GM}{r}}$$

Case 3. Singular isothermal sphere

For a singular isothermal sphere,

$$\rho(r) = \frac{\rho_0}{r^2}.$$

Then

$$\int_0^r \rho(r')r'^2\,dr' = \int_0^r \rho_0\,dr' = \rho_0 r.$$

The first term becomes

$$\frac{1}{r}\int_0^r \rho(r')r'^2\,dr' = \rho_0.$$

The second term is

$$\int_r^\infty \rho(r')r'\,dr' = \int_r^\infty \frac{\rho_0}{r'}\,dr',$$

which diverges logarithmically. Therefore the absolute potential is not well-defined if we set $\Phi(\infty) = 0$. Instead, define the potential relative to some reference radius r_0, with

$$\Phi(r_0) = 0.$$

Introduce a large cutoff radius $R_{\max}$. Then

$$\Phi(r) = -4\pi G\rho_0\left[1 + \int_r^{R_{\max}} \frac{dr'}{r'}\right] = -4\pi G\rho_0\left[1 + \ln\left(\frac{R_{\max}}{r}\right)\right].$$

Similarly,

$$\Phi(r_0) = -4\pi G\rho_0\left[1 + \ln\left(\frac{R_{\max}}{r_0}\right)\right].$$

Subtracting,

$$\Phi(r) - \Phi(r_0) = -4\pi G\rho_0\left[\ln\left(\frac{R_{\max}}{r}\right) - \ln\left(\frac{R_{\max}}{r_0}\right)\right].$$

Thus

$$\Phi(r) - \Phi(r_0) = -4\pi G\rho_0 \ln\left(\frac{r_0}{r}\right) = 4\pi G\rho_0 \ln\left(\frac{r}{r_0}\right).$$

Since $\Phi(r_0) = 0$,

$$\boxed{\Phi(r) = 4\pi G\rho_0 \ln\left(\frac{r}{r_0}\right)}$$

This potential diverges to $-\infty$ as $r \to 0$, as expected for a singular isothermal sphere.

2. The model given by $\rho = 1/(1+r^2)^{2.5}$ is a Plummer model. Derive the potential of this model. What is the total mass?

For the Plummer model,

$$\rho(r) = \rho_0 \left(1+r^2\right)^{-5/2}.$$

The enclosed mass is

$$M(r) = 4\pi \int_0^r \rho(s)\, s^2\, ds$$

so

$$M(r) = 4\pi\rho_0 \int_0^r s^2(1+s^2)^{-5/2}\, ds.$$

Notice that

$$\frac{d}{ds}\left[\frac{s^3}{(1+s^2)^{3/2}}\right] = 3s^2(1+s^2)^{-5/2},$$

hence

$$\int s^2(1+s^2)^{-5/2}\, ds = \frac{1}{3}\frac{s^3}{(1+s^2)^{3/2}}.$$

Therefore,

$$M(r) = \frac{4\pi\rho_0}{3}\frac{r^3}{(1+r^2)^{3/2}}.$$

Taking $r \to \infty$,

$$M_{\rm tot} = \lim_{r\to\infty} M(r) = \frac{4\pi\rho_0}{3}.$$

So the total mass is

$$\boxed{M_{\rm tot} = \frac{4\pi\rho_0}{3}}.$$

For a spherically symmetric system,

$$\frac{d\Phi}{dr} = \frac{GM(r)}{r^2}.$$

Substituting $M(r)$,

$$\frac{d\Phi}{dr} = \frac{GM_{\rm tot}\, r}{(1+r^2)^{3/2}}.$$

Using the boundary condition

$$\Phi(\infty) = 0,$$

we integrate inward:

$$\Phi(r) = -\int_r^\infty \frac{GM_{\rm tot}\, s}{(1+s^2)^{3/2}}\, ds.$$

Since

$$\frac{d}{ds}\left[\frac{1}{\sqrt{1+s^2}}\right] = -\frac{s}{(1+s^2)^{3/2}},$$

the integral gives

$$\Phi(r) = -\frac{GM_{\rm tot}}{\sqrt{1+r^2}}.$$

Thus the Plummer potential is

$$\boxed{\Phi(r) = -\frac{GM_{\rm tot}}{\sqrt{1+r^2}}}.$$

With a scale radius a, the standard form is

$$\rho(r) = \frac{3M}{4\pi a^3}\left(1+\frac{r^2}{a^2}\right)^{-5/2},$$

and the corresponding potential is

$$\boxed{\Phi(r) = -\frac{GM}{\sqrt{r^2+a^2}}}.$$

3. Assume that the age of the universe is 13 Gyr and $\Omega = 1$ and $\sim$100% of the mass-energy density of the universe is in the form of matter.
(a) Using the equation

$$\left(\frac{\dot{r}}{r}\right)^2 = \frac{8}{3}\pi G\rho + \text{const}/r^2 \tag{2}$$

where r is the scale factor of the universe and $\rho = \rho_0/r^3$, show that r increases with time as $t^{2/3}$. What does const equal for a universe where $\Omega = 1$?
(b) What is the Hubble constant $H_0 = (\dot{r}/r)_0$ that would yield a universe with an age of 13 Gyr?
(c) Calculate the age of the universe at redshifts z of 1, 5, and 10. Note that for redshifts z of 1, 5, and 10, the scale factor r for the universe was $(1+z)$ smaller than it is today (i.e., $r = r_0/(1+z)$ where r_0 is the scale factor today).
(d) How long has the light travelled which was emitted at $z = 1$?

$$\left(\frac{\dot{r}}{r}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\text{const}}{\text{r}^2}, \qquad \rho = \frac{\rho_0}{r^3}.$$

(a) Evolution of the scale factor

For a universe with $\Omega = 1$, the universe is spatially flat, so

$$\rho = \rho_{crit} = \frac{3(\frac{\dot{r}}{r})^2}{8\pi G}$$

and thus

$$\text{const} = 0.$$

So

$$\left(\frac{\dot{r}}{r}\right)^2 = \frac{8\pi G}{3}\frac{\rho_0}{r^3}.$$

Therefore

$$\frac{\dot{r}^2}{r^2} = \frac{8\pi G\rho_0}{3r^3},$$

so

$$\dot{r}^2 = \frac{8\pi G\rho_0}{3}\frac{1}{r}.$$

Taking the square root,

$$\dot{r} = \left(\frac{8\pi G\rho_0}{3}\right)^{1/2} r^{-1/2}.$$

Therefore

$$r^{1/2}dr = \left(\frac{8\pi G\rho_0}{3}\right)^{1/2} dt.$$

Integrating,

$$\frac{2}{3}r^{3/2} = \left(\frac{8\pi G\rho_0}{3}\right)^{1/2} t.$$

Thus

$$r^{3/2} \propto t,$$

so

$$\boxed{r \propto t^{2/3}}.$$

(b) Hubble constant for a 13 Gyr universe

For a matter-dominated, flat universe,

$$r \propto t^{2/3}.$$

Therefore

$$H = \frac{\dot{r}}{r} = \frac{2}{3t}.$$

At the present time,

$$H_0 = \frac{2}{3t_0}.$$

With

$$t_0 = 13 \text{ Gyr},$$

we get

$$H_0 = \frac{2}{3(13 \text{ Gyr})} = \frac{2}{39} \text{ Gyr}^{-1}.$$

Thus

$$H_0 = 0.0513 \text{ Gyr}^{-1}.$$

Using

$$1 \text{ Gyr}^{-1} \approx 978 \text{ km s}^{-1} \text{ Mpc}^{-1},$$

we find

$$H_0 \approx 0.0513(978) \approx 50.1 \text{ km s}^{-1} \text{ Mpc}^{-1}.$$

Therefore

$$\boxed{H_0 \approx 50 \text{ km s}^{-1} \text{ Mpc}^{-1}}.$$

(c) Age of the universe at redshift z

Since

$$r \propto t^{2/3},$$

we have

$$t \propto r^{3/2}.$$

At redshift z,

$$r = \frac{r_0}{1+z}.$$

Therefore

$$\frac{t(z)}{t_0} = \left(\frac{r}{r_0}\right)^{3/2} = \left(\frac{1}{1+z}\right)^{3/2}.$$

Thus

$$t(z) = \frac{t_0}{(1+z)^{3/2}}.$$

With $t_0 = 13$ Gyr,

$$\boxed{t(z) = \frac{13\ \text{Gyr}}{(1+z)^{3/2}}}.$$

For $z = 1$,

$$t(1) = \frac{13}{2^{3/2}} = 4.60\ \text{Gyr}.$$

For $z = 5$,

$$t(5) = \frac{13}{6^{3/2}} = 0.884\ \text{Gyr}.$$

For $z = 10$,

$$t(10) = \frac{13}{11^{3/2}} = 0.356\ \text{Gyr}.$$

Therefore

$$\boxed{t(z=1) = 4.60\ \text{Gyr}}$$

$$\boxed{t(z=5) = 0.884\ \text{Gyr}}$$

$$\boxed{t(z=10) = 0.356\ \text{Gyr}}.$$

(d) Light travel time from $z = 1$

The light travel time is the difference between the present age of the universe and the age when the light was emitted:

$$t_{\text{travel}} = t_0 - t(z=1).$$

Thus

$$t_{\text{travel}} = 13\ \text{Gyr} - 4.60\ \text{Gyr} = 8.40\ \text{Gyr}.$$

Therefore

$$\boxed{t_{\text{travel}} \approx 8.4\ \text{Gyr}}.$$

4. (a) Consider that there was some overdense region in the universe which had a density ρ which was $2\rho_{crit}$ (the critical density) which otherwise had spherical symmetry. What was the density of that sphere relative to ρ_{crit} when that sphere was 10 times smaller?
(b) Imagine that the universe as a whole had an average density ρ equal to $2\rho_{crit}$ at the present time. How overdense was the universe when the universe was 10 times smaller?

Solution

In class we derived the relation

$$\frac{1}{\Omega} - 1 = \frac{\text{const}}{1+z}.$$

We are told that at the present time,

$$\Omega_0 = 2.$$

At the present epoch,

$$z = 0.$$

Therefore,

$$\frac{1}{\Omega_0} - 1 = \text{const}.$$

Substituting $\Omega_0 = 2$,

$$\text{const} = \frac{1}{2} - 1 = -\frac{1}{2}.$$

Thus the evolution equation becomes

$$\frac{1}{\Omega(z)} - 1 = -\frac{1}{2}\frac{1}{1+z}.$$

When the universe (or spherical region) was 10 times smaller,

$$1 + z = 10.$$

Therefore,

$$\frac{1}{\Omega} - 1 = -\frac{1}{2} \cdot 110 = -\frac{1}{20}.$$

So

$$\frac{1}{\Omega} = 1 - \frac{1}{20} = \frac{19}{20}.$$

Hence

$$\Omega = \frac{20}{19} = 1.0526.$$

Therefore the density relative to the critical density was

$$\boxed{\rho = 1.053\,\rho_{\text{crit}}}.$$

The solution is the same both for an isolated region of the universe and for the universe as a whole:

$$\boxed{\text{(a) } \rho = 1.053\,\rho_{\text{crit}}}$$

$$\boxed{\text{(b) } \rho = 1.053\,\rho_{\text{crit}}}$$

5. In lecture, we examined an arbitrary dynamical system and determined how that dynamical system can be scaled in position, mass, and velocity and still maintain the same qualitative form.
(a) Show explicitly that the virial theorem produces the same result for the scaling relations.
(b) Derive Kepler's Third Law using the scaling relations found in class.
(c) Do the same sort of scaling relations exist for stars? Is it possible to scale the position, velocity, and mass for particles in a star in the same way – and have a system with the same qualitative form? Which equilibrium is retained and which is lost?

(a)

$$\text{virial theorem: } v'^2 = \frac{Gm'}{x'} \quad \rightarrow \quad (a_v v)^2 = \frac{G a_m m}{a_x x} \quad \rightarrow \quad a_x^2 a_t^2 = \frac{a_m}{a_x} \quad \rightarrow \quad a_x^3 a_t^2 = a_m \quad (1)$$

(b)

$$\text{Period } P' = \frac{2\pi x'}{v'} = \frac{2\pi a_x x}{a_v v} = P\frac{a_x}{a_x a_t} = \frac{P}{a_t}$$

$$a_x^3 a_t^2 = a_m \quad \rightarrow \quad \frac{a_x^3 x^3}{x^3}\frac{a_t^2 P^2}{P^2} = \frac{a_m m}{m} \quad \rightarrow \quad \frac{x'^3}{x^3}\frac{P^2}{P'^2} = \frac{m'}{m}$$

$$\frac{P^2}{x^3} \propto \frac{1}{m} \quad \rightarrow \quad \text{Kepler's third law} \qquad (1)$$

(c)
Hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2} \quad \rightarrow \quad a_P = \frac{a_m a_\rho}{a_r} = \frac{a_m^2}{a_r^4}$$

Assume the star is an ideal gas cloud without radiation pressure:

$$P = \frac{R}{\mu}\rho T \quad \rightarrow \quad a_P = a_\rho a_T = \frac{a_m}{a_r^3}a_v^2 = \frac{a_m^2}{a_r^4}$$

A star therefore retains hydrostatic equilibrium with the scaling relation $a_P = a_m^2 / a_r^4$. There may be no scaling relation for hydrostatic equilibrium to hold if radiation pressure is taken into account.

Radiative transfer:

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{F}{4\pi r^2}$$

Thermal equilibrium:

$$\frac{dF}{dr} = 4\pi r^2 \rho q$$

Both the opacity κ and the energy release rate per unit mass q have complicated and only approximate dependence on ρ and T. Therefore, we would not expect deriving exact identical scaling relations from the above equations. (1)

6. Prove that $M \propto T^{3/2}/n^{1/2}$. Use the fact that $\sigma^2 \propto T$ and $n \propto M/R^3$. Comment on the importance of this scaling relative to the T vs. n diagram used to understand for which mass sources $T_{cool} < T_{dyn}$ (i.e., where galaxy formation is efficient).

Virial Theorem: 2 K.E. = P.E. $\rightarrow$ $\sigma^2 \propto \frac{M}{R}$

Given $\sigma^2 \propto$ T and n $\propto \frac{M}{R^3}$,

$\mathrm{T} \propto \mathrm{M} \cdot (\frac{n}{M})^{1/3} \propto \mathrm{n}^{1/3}\, \mathrm{M}^{2/3}$ $\rightarrow$ $\mathrm{M} \propto \mathrm{T}^{3/2}\, \mathrm{n}^{-1/2}$ (1)

On the n-T diagram, this relation draws constant mass lines, which shows gas clouds with masses more massive than $10^{12-13}\ M_\odot$ cannot be cooled efficiently, thus explaining the large quantities of hot gas still seen today in galaxies and galaxy clusters. (1)

7. If you assume that the rotational speed of the Milky Way is constant as a function of radius and has a value of 220 km/sec, what is the epicyclic frequency at the sun (distance from the center = 8 kpc) ? How does this compare to the orbital frequency Ω?

Constant rotational speed: $\mathrm{v}_c = \Omega\, \mathrm{R}$

$$\kappa^2 = R\frac{d\Omega^2}{dR} + 4\Omega^2 = R\frac{d}{dR}\left(\frac{v_c^2}{R^2}\right) + 4\Omega^2 = \frac{2v_c^2}{R^2}$$

$$\kappa = \sqrt{2}\cdot\frac{220\ \text{km/sec}}{8\ \text{kpc}} = 1.2\times10^{-15}s^{-1} \tag{1}$$

$$\kappa = \sqrt{2}\Omega \tag{1}$$

Galaxies: Structure, Dynamics, and Evolution
Problem Set 2
Instructor: Dr. Bouwens

Here is Problem Set 2. The entire problem set will be due before class on Monday, March 16 (email them to Wout). Be sure to pay extra attention to problem 4, as your solution to that problem will be checked carefully and used in determining your homework grade.

1. How does the epicyclic (radial) frequency κ for both an isothermal potential (v_c = constant) and a potential with the form $\phi(r) = r^{-3/4}$ compare with the azimuthal frequency Ω? How many radial/epicyclic oscillations will a star undergo for each orbit around a galaxy?

Isothermal:

$$\kappa^2 = R\frac{d\Omega^2}{dR} + 4\Omega^2 = R\frac{d}{dR}\left(\frac{v_c^2}{R^2}\right) + 4\Omega^2 = \frac{2v_c^2}{R^2}$$

$$\kappa = \sqrt{2}\Omega \qquad (1)$$

$\phi(r) \propto r^{-3/4}$:

$F(r) = -\nabla\phi \propto r^{-7/4} \quad \rightarrow \quad R\,\Omega^2 = \text{constant} \times R^{-7/4} \quad \rightarrow \quad \Omega^2 = \text{constant} \times R^{-11/4}$

$$\kappa^2 = R\frac{d\Omega^2}{dR} + 4\Omega^2 = R \cdot \text{constant} \cdot \frac{-11}{4} R^{-15/4} + 4\Omega^2 = 1.25\Omega^2$$

$\kappa = 1.12\ \Omega$, ~1.12 radial oscillations per orbit. (1)

2. Calculate the epicyclic frequency for both an isothermal potential (v_c = constant) and a potential with the form $\phi = -1/(1+r^2)$. Draw the behaviour of $\Omega - n\kappa/m$ as in figure 6.11.

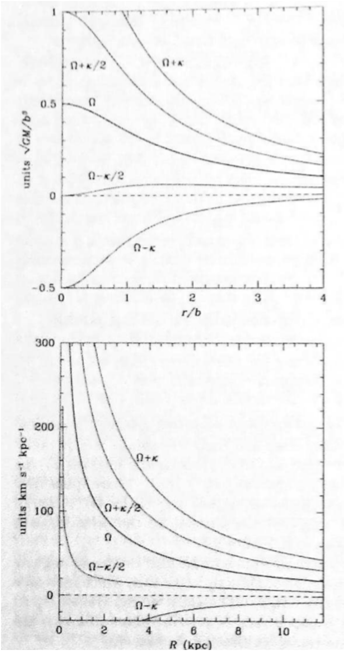
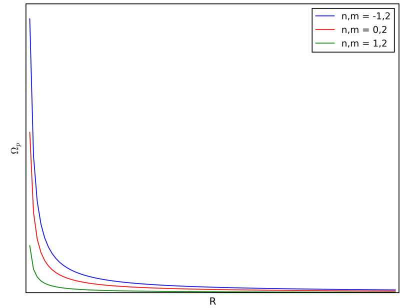


Figure 6.11 Behavior of $\Omega - n\kappa/m$ in: (top) the isochrone potential (eq. 2.47) (bottom) Model I for our Galaxy, described in §2.7.

Isothermal potential:
Same as question 3, $\kappa = \sqrt{2}\Omega$.

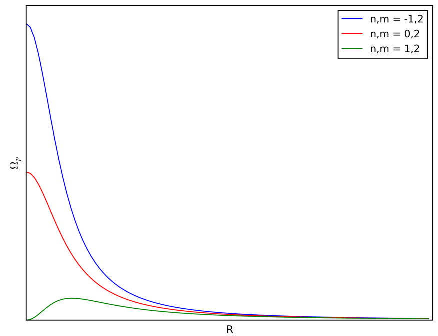
$$\Omega_p(R) = \Omega(R) - \frac{n\kappa(R)}{m}$$
$$= \left(1 - \sqrt{2}\frac{n}{m}\right)\frac{v_c}{R}$$



(1)

$\phi(\mathrm{R}) = \frac{-1}{1+R^2}$:
$|\,\mathrm{F(R)}\,| = |\,\frac{d}{dR}\,(1 + \mathrm{R}^2)^{-1}\,| = \frac{2R}{(1+R^2)^2} = \mathrm{R}\,\Omega^2 \quad \rightarrow \quad \Omega^2 = \frac{2}{(1+R^2)^2}$

$$\kappa^2 = R\frac{d\Omega^2}{dR} + 4\Omega^2 = \frac{8}{(1+R^2)^3}$$
$$\Omega_p = \frac{\sqrt{2}}{1+R^2} - \frac{n}{m}\frac{\sqrt{8}}{(1+R^2)^{3/2}} \quad 3$$



(1)

3. In class, we calculated the relaxation time for a star in some dynamical system (e.g., a galaxy) by considering the effect of its interactions with other stars in a galaxy. In calculating the relaxation time, we only considered impact parameters b from b_{min} to the size of the galaxy R (where b_{min} was the minimum impact parameter where our simple formula for the velocity kick was approximately valid). What about the effect of impact parameters $b = 0$ to b_{min}? How does this impact the relaxation time?
(a) Calculate the probability that a star will pass by another star with impact parameter b less than or equal to b_{min}? (Ignore the curvature of the orbit.) Adopt the standard variables N, v, m, and G used in the derivation during lecture.

(b) If any star passes by another star with impact parameter $b < b_{min}$, its velocity will be so perturbed that it "will lose all memory of its initial orbit." Let us say it relaxes with just one encounter. Calculate the relaxation time assuming that $b < b_{min}$ encounters are the only meaningful relaxation process. Calculate the relaxation time for the same choice of N, m, G, and v considered in class for a galaxy (i.e., $N = 10^{10}$, $v = 100$ km s^{-1}, $r = 10$ kpc, $t_{cross} = 10^8$ yr). How does this compare with the relaxation time derived in class considering only $b > b_{min}$ encounters?

(a) Rephrased question (not graded): Calculate the *rate* that a star will pass by another star with impact parameter b less than or equal to b_{min} ? (Ignore the curvature of the orbit.) Adopt the standard variables N, v, m, and G used in the derivation during lecture.

Rate = cylindrical volume (of trajectory) × number density / time of crossing the radius

$$= \frac{\pi b_{min}^2 R \times \frac{N}{4\pi R^3/3}}{R/v}$$

$$= \frac{3 v b_{min}^2 N}{4R^3}$$

Convert R to v, m, and G using virial theorem: $v^2 = GM/R$
Also, $b_{min} = GM/v^2$ (see source material 2).

$$\text{Rate} = \frac{3 v b_{min}^2 N}{4R^3}$$

(b) Plug in the provided values, $t_{relax} = \text{rate}^{-1} \simeq 1 \times 10^{18}$ years (2)

4. To help yourself visualize the Inner and Outer Lindblad Resonances work, as well as the corotational radius, I will ask you to sketch out a number of snapshots of the movement of a spiral arm around a galaxy. Assume that a galaxy has two spiral arms and that it is an isothermal sphere.
(a) Suppose that circular velocity of the galaxy is 200 km/s and that the inner Lindblad resonance for a galaxy is at a radius of 2.5 kpc. What is the pattern speed (or frequency) Ω_p?
(b) What is the period of epicyclic motion at the Inner Lindblad resonance?
(c) What is the corotation radius and radius of the Outer Lindblad resonance for the pattern speed you computed in part (a)?
(d) In multiples of the epicyclic period of stars at the inner Lindblad resonance (consider multiples out to 10), please sketch out the motion of stars in a spiral galaxy (similar to what I do in lecture, but now moving indicative stars at the Inner and Outer Lindblad resonances and corotation radius self consistently). Please indicate the position of the spiral arms, a star at the radius of the Inner Lindblad resonance, a star at the corotation radius, and a star at the radius of the outer Lindblad resonance. Assume that the position angle of the spiral arms and the stars at all three radii are all 0 at time t = 0.

The pattern speed of a spiral arm is given by $\Omega_\mathrm{p} = \Omega + n\kappa/m$. For a galaxy with two spiral arms ($m = 2$), the inner Lindblad resonance condition ($n = -1$) simplifies the expression to $\Omega_\mathrm{p} = \Omega - \kappa/2$, where Ω is the orbital frequency and κ the epicyclic frequency. In problem set 2, we showed that for a constant circular velocity $v_\mathrm{c}(r) =$ constant, the orbital and epicyclic frequencies are related by $\kappa = \sqrt{2}\Omega$. Thus, we have

$$\Omega_\mathrm{p}(r) = \Omega(r)\left(1 - \frac{\sqrt{2}}{2}\right) = \frac{\Omega(r)}{2}\left(2 - \sqrt{2}\right). \tag{1}$$

For a constant circular velocity, we have $\Omega(r) = v_\mathrm{c}/r$, and thus

$$\Omega_\mathrm{p}(r) = \frac{v_\mathrm{c}}{2r}\left(2 - \sqrt{2}\right). \tag{2}$$

For the given value of the circular velocity, we may write

$$\begin{aligned} \Omega_\mathrm{p}(r) &= 1.71 \times 10^{-15} \left(\frac{1\,\mathrm{kpc}}{r}\right) \mathrm{s}^{-1} \\ &= 53.9 \left(\frac{1\,\mathrm{kpc}}{r}\right) \mathrm{Gyr}^{-1}. \end{aligned} \tag{3}$$

At 2.5 kpc, the pattern speed is thus 21.6 Gyr^{-1}.

As mentioned in part (a), the relation between the epicyclic and orbital frequencies for a constant circular velocity is $\kappa(r) = \sqrt{2}\Omega(r)$. We thus have

$$\begin{aligned} \kappa(r) = \sqrt{2}\Omega(r) &= \frac{\sqrt{2}v_\mathrm{c}}{r} \\ &= 8.25 \times 10^{-15} \left(\frac{1\,\mathrm{kpc}}{r}\right) \mathrm{s}^{-1} \\ &= 260 \left(\frac{1\,\mathrm{kpc}}{r}\right) \mathrm{Gyr}^{-1}. \end{aligned} \tag{4}$$

The period is then

$$P_\mathrm{epi} = \frac{2\pi}{260}\left(\frac{r}{1\,\mathrm{kpc}}\right) \mathrm{Gyr} = 24.2 \left(\frac{r}{1\,\mathrm{kpc}}\right) \mathrm{Myr}. \tag{5}$$

At 2.5 kpc, the epicyclic period is about 60 Myr.

The corotation condition ($n = 0$) means $\Omega_{\rm p}(r) = \Omega(r) = v_{\rm c}/r$. Thus,

$$r_{\rm corot} = \frac{v_{\rm c}}{\Omega_{\rm p}}. \tag{6}$$

Using the pattern speed $\Omega_{\rm p} = 21.6\,{\rm Gyr}^{-1}$ as calculated in part (a), we have $r_{\rm corot} = 8.54\,{\rm kpc}$. The condition for the outer Lindblad resonance is $\Omega_{\rm p} = \Omega + \kappa/2$. Following a similar procedure as in part (a), we get

$$\Omega_{\rm p} = \frac{v_{\rm c}}{2r}\left(2 + \sqrt{2}\right). \tag{7}$$

Solving for the radius,

$$r_{\rm OL} = \frac{v_{\rm c}}{2\Omega_{\rm p}}(2 + \sqrt{2}). \tag{8}$$

For our numbers, we get $r_{\rm OL} = 14.58\,{\rm kpc}$.

The angular distance ϕ traversed by a star in time steps t_n of the epicyclic period at the inner Lindblad resonance $P_{\rm epi}$ is

$$\phi(t_n) = 360^\circ n \left[\frac{P_{\rm epi}}{P(r)}\right] = 360^\circ n \left[P_{\rm epi}\frac{v_{\rm c}}{2\pi r}\right], \tag{9}$$

where n is an integer and $P(r)$ is the period of the star at orbital distance r. Figures 1 and 2 trace the movements of three stars (red dots) at the inner Lindblad resonance (inner circle), corotation radius (middle circle), and outer Lindblad resonance (outer circle) in steps of $P_{\rm epi} = 60\,{\rm Myr}$. At $t_0 = 0\,{\rm Myr}$, the stars begin aligned. For reference, the orbital periods for the inner Lindblad resonance, corotation radius, and outer Lindblad resonance are 85 Myr, 292 Myr, and 500 Myr, respectively. The blue lines denote the spiral arms, which move at the same rotational speed as the star at the corotation radius.

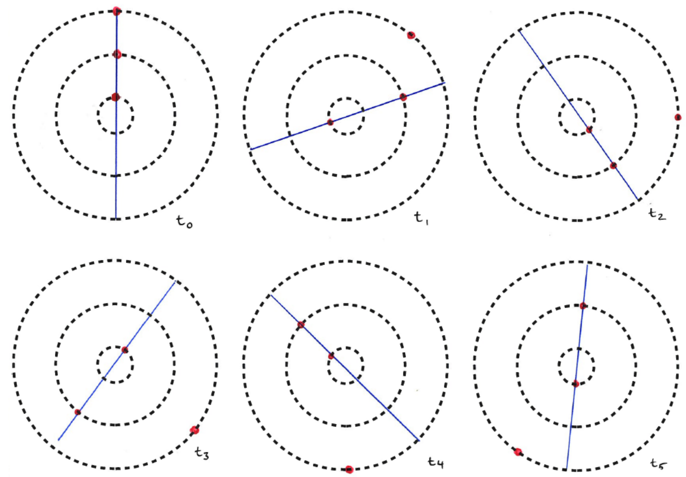


Figure 1: Time-lapse of stars (red dots) orbiting at the inner Lindblad resonance (inner circle), corotation radius (middle circle), and outer Lindblad resonance (outer circle). Each time step t_n is the nth multiple of the epicyclic period of the inner Lindblad resonance, $P_{\rm epi} = 60\,{\rm Myr}$. The spiral arms, which follow the same rotational speed as the star at the corotation radius, are denoted by blue lines.

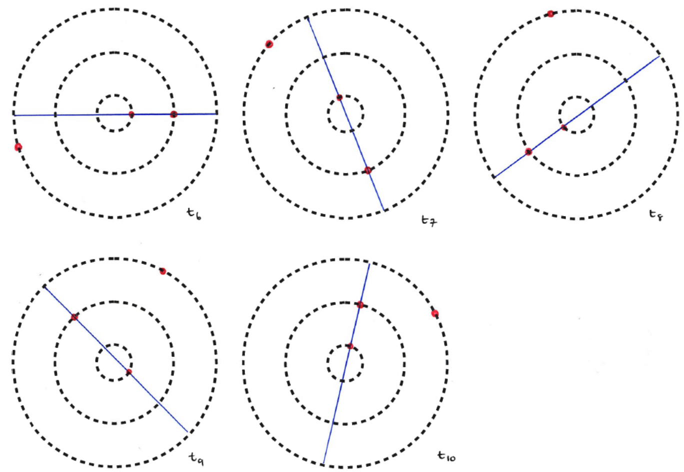


Figure 2: Same as Figure 1, but for additional time steps.

Figure 3 shows the first three time steps of Figure 1 in the rotating frame of the star at the corotation radius. Obviously, the star at the corotation radius as well as the spiral arms will appear motionless. Further, this frame clearly shows the star at the inner Lindblad resonance hopping between the two spiral arms with each epicyclic period. The star at the outer Lindblad resonance does not appear invariant. A similar situation would occur if the chosen frame was that of the star at the inner Lindblad resonance. In this case, the inner Lindblad resonance star would appear motionless, while the star at the corotation radius would hop from one side of the galaxy to the other. The star at the outer Lindblad resonance would still not appear invariant at all.

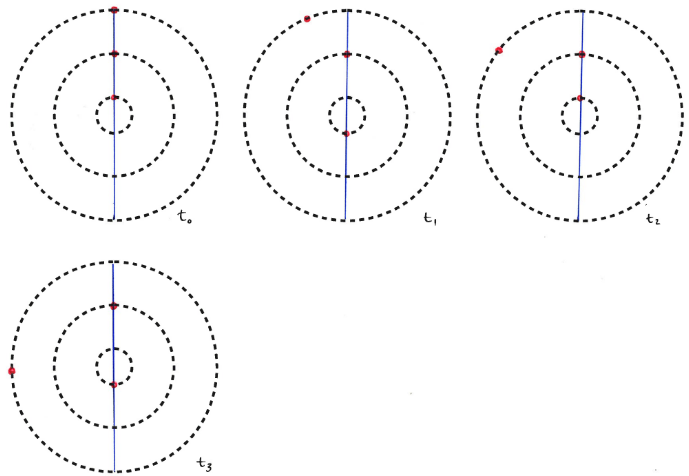


Figure 3: First three time steps from Figure 1 in the rotating frame of the star at the corotation radius.

5. How many integrals of motion are there for a particle with the following force law $F(r) = -\hat{r}/r^2$ where r is the radius?

There are five integrals of motion associated with the given force law. As the corresponding potential is time-independent—that is, static—the energy is conserved. Further, the associated potential is spherically symmetric, and thus all three components of the angular momentum are conserved. We have E, L_x, L_y, and L_z.
Moreover since the orbit is closed, its shape and orientation is conserved. The conserved quantity here is called the Laplace-Runge-Lenz Vector.
In total we have 5 integrals of motion.

6. Show that the distribution function $f(\epsilon, L)$

$$f(\epsilon, L) = \begin{cases} F\delta(L^2)(\epsilon - \epsilon_0)^{-1/2} & \text{for } \epsilon > \epsilon_0, \\ 0 & \text{otherwise.} \end{cases}$$

where F and ϵ_0 are constants and δ is the familiar delta function. Show that this distribution function self-consistently generates a model with density

$$\rho(r) = \begin{cases} Cr^{-2} & \text{for } r < r_0 \\ 0 & \text{otherwise.} \end{cases}$$

where C is a constant and the relative potential at r_0 satisfies $\Psi(r_0) = \epsilon_0$. This is the only analytic stellar system known to us in which all stars are on perfectly radial orbits.

We want to find the density for the distribution function:

$$f(\epsilon, L) = \begin{cases} F\delta(L^2)(\epsilon - \epsilon_0)^{-1/2} & \text{if } \epsilon > \epsilon_0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We therefore need to equate the integral:

$$\rho(r) = \int f(\epsilon, L) d^3 v$$

We rewrite this integral so that we integrate over radial, tangential and angular velocity. This is just like rewriting an integral over Cartesian coordinates to spherical coordinates:

$$\iiint f(x, y, z) dx dy dz = \int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} f(r, \theta, \phi) r^2 \sin\theta dr d\theta d\phi$$

First we give the integral over velocity in spherical coordinates:

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{\infty} f(v, \theta, \phi) v^2 \sin\theta dv d\theta d\phi$$

We now want to rewrite this integral in terms of the radial velocity, tangential velocity and the angle phi. The first two are given by $v_r = v\cos\theta$ and $v_r = v\sin\theta$. The Jacobian of the new integral is given by $|\frac{\partial(v,\theta,\phi)}{\partial(v_r,v_t,\phi)}| = 1/v$. The integral now becomes:

$$\int_0^{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(v_r, v_t, \phi) v \sin\theta dv_r dv_t d\phi$$

With the definition of v_t this becomes:

$$\int_0^{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v_r, v_t, \phi) v_t dv_r dv_t d\phi$$

We just integrate over the angular velocity because both the energy and angular momentum do not depend on this. This integration just gives a factor 2π. The angular momentum depends on the tangential velocity ($L = rv_t$) and the energy depends on both the radial and tangential velocity ($\epsilon = \Psi - v_r^2/2 - v_t^2/2$). We now can write:

$$\rho(r) = 2\pi \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\epsilon, L) v_t dv_r dv_t$$

Because we only have radial orbits we can split the integral, the energy is then only depended on the radial velocity.

$$\rho(r) = 2\pi \int_{-\infty}^{\infty} f(\epsilon) dv_r \int_{-\infty}^{\infty} f(L) v_t dv_t$$

Now filling in the distribution function:

$$\rho(r) = 2\pi F \int_0^{v_{max}} (-v_r^2/2 + \Psi + \epsilon_0)^{-1/2} dv_r \int_{-\infty}^{\infty} \delta(L^2) v_t dv_t$$

The definition of angular momentum is $L^2 = r^2 v_t^2$. We can write $x = r^2 v_t^2$ and $dx = 2r^2 v_t dv_t$ plugging this in we get:

$$\begin{aligned}\rho(r) &= 2\pi F \int_0^{v_{max}} (-v_r^2/2 + \Psi + \epsilon_0)^{-1/2} dv_r \int_{-\infty}^{\infty} \delta(x) dx/r^2 \\ &= \frac{\pi F}{r^2} \int_0^{v_{max}} (-v_r^2/2 + \Psi + \epsilon_0)^{-1/2} dv_r\end{aligned}$$

We know v_{max} is reached when the energy is the lowest thus $\epsilon = \epsilon_0$ from $\epsilon = \Psi - v_r^2/2$ we now know $v_{max} = \sqrt{2(\Psi - \epsilon_0)}$. We thus can write:

$$\begin{aligned}\rho(r) &= \frac{\pi F}{r^2} \int_0^{\sqrt{2(\Psi-\epsilon_0)}} (-v_r^2/2 + \Psi + \epsilon_0)^{-1/2} dv_r \\ &= \frac{\pi F}{r^2\sqrt{2}} \int_0^{\sqrt{2(\Psi-\epsilon_0)}} \frac{1}{\sqrt{-v_r^2 + 2(\Psi + \epsilon_0)}} dv_r \\ &= \frac{\pi F}{r^2\sqrt{2}} \int_0^{\sqrt{2(\Psi-\epsilon_0)}} \frac{1}{\sqrt{-v_r^2 + \sqrt{2(\Psi + \epsilon_0)}^2}} dv_r\end{aligned}$$

This is a standard integral of the form $\int_0^a \frac{1}{\sqrt{-x^2+a^2}} dx = \pi/2$ and we find thus:

$$\rho(r) = \frac{\pi^2 F}{2r^2\sqrt{2}}$$

We find thus the following density:

$$\rho(r) = \begin{cases} \frac{C}{r^2} & \text{if } r < r_0 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Galaxies: Structure, Dynamics, and Evolution
Problem Set 3
Instructor: Dr. Bouwens

Here is Problem Set 3. The entire problem set will be due before class on Monday, April 6 (email them to Wout and include GSD in the subject line). Be sure to pay extra attention to problem 1, as your solution to that problem will be checked carefully and used in determining your homework grade.

1. Consider the case of the homogeneous ellipsoid potential discussed on page 3 of this handout. For this potential, the motion of a star in the x, y, and z directions can be described as harmonic oscillators. Suppose that we represent the motion of each particle as $x = a_x cos(\omega_x t + \phi_x)$, $y = a_y cos(\omega_y t + \phi_y)$, $z = a_y cos(\omega_z t + \phi_z)$ where ω_x does not equal ω_y does not equal ω_z and ω_x/ω_y, ω_y/ω_z, and ω_x/ω_z are not rational numbers. Particles in such a potential follow box orbits.
(a) Argue that a particle travels arbitrarily close to every spatial position in the entire volume $(-a_x, a_x) \times (-a_y, a_y) \times (-a_z, a_z)$. If you cannot prove it explicitly, demonstrate the plausibility of this statement by showing 2-D projections of the orbital tracks for two separate choices of $(\omega_x, \omega_y, \omega_z)$. Plot out the orbital tracks for varying integration times (short time, intermediate time, long time intervals).
(b) Write down a formula for the angular momentum of a particle in this potential. Is the angular momentum conserved? Why or why not? What is the average value of the angular momentum averaged over time?
(c) If a were equal to b (and hence ω_x were equal to ω_y), would the angular momentum be conserved? Why or why not? Would particles still travel on box orbits, or would the orbits be loop orbits?

a) Solution 1.

Consider first the problem in 2 dimensions. The motion of a particle is now given by

$$\begin{aligned} x &= a_x \cos(\omega_x t + \phi_x) \\ y &= a_y \cos(\omega_y t + \phi_y) \end{aligned} \tag{1}$$

We can define the quantities P_x and P_y as the periods of the harmonic oscillators in the $x-$ resp. $y-$direction. We write $P_x = 2\pi/\omega_x$ and $P_y = 2\pi/\omega_y$. This 2D-orbit will be periodic with period P if we can find a least common multiple (LCM) of these two periods. In other words, the orbit is periodic if we can find two numbers $A, B \in \mathbb{N}$ such that $A \times P_x = B \times P_y$. This implies

$$\frac{P_x}{P_y} = \frac{B}{A} = \frac{\omega_y}{\omega_x} \tag{2}$$

If $\omega_y/\omega_x \notin \mathbb{Q}$ by definition no such integers A, B exist. Thus the orbit will be aperiodic.

This immediately implies that the full square $(-a_x, a_x), (-a_y, a_y)$ will be filled given enough time. To see this, assume we have only an harmonic oscillator in the $x-$direction. In one full period, it will obviously traverse all points in $(-a_x, a_x)$. When we add another independent harmonic oscillator in the $y-$direction, still all values in the $x-$direction will be reached, although now for different values on the $y-$axis. Of course, the harmonic oscillator in the $y-$direction will also reach all points between $-a_y$ and a_y. And, since the system of oscillators is not periodic, it will never repeat itself, hence within a finite time it will reach a point it has not reached before. Thus given infinite time, every point in the $(x, y)-$plane will be traversed.

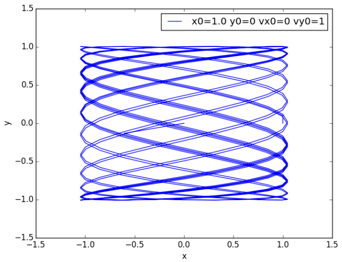
Adding a third dimension does not change what was outlined above. As long as none of the ratios of angular frequencies is a rational number, the orbit that is drawn out by the particle will have no period. Since the system is a combination of independent harmonic oscillators, the full volume $(-a_x, a_x), (-a_y, a_y), (-a_z, a_z)$ is traversed for $t \to \infty$.

a) Solution 2

We demonstrate the plausibility of the statement that a particle travels arbitrarily close to every spatial position in the entire volume by showing 2-D projections of the orbital tracks for two separate choices of $(\omega_x, \omega_y, \omega_z)$. We also vary the integration time. So, we get a plot for short, intermediate and long time intervals. We have chosen $a_x, a_y, a_z = 1$.

We did the plotting as follows: we start the star at a given location and velocity which is indicated in the plots. Then we calculate the new location and new velocity at the next time $t+dt$ by shifting the position with $dt \times v$ with velocity from the 'old' position and the velocity with $dt \times F$ with the force at the 'old' position. The force for this potential $(\Phi = x^2 + y^2 + z^2)$ is given by $F_x = -2x$, $F_y = -2y$ and $F_z = -2z$.

For the first three plots we choose: $(\omega_x, \omega_y, \omega_z) = (\pi, 1, \sqrt{2})$. This gives irrational numbers for $\omega_x/\omega_y = \pi/1$, $\omega_y/\omega_z = 1/\sqrt{2}$ and $\omega_x/\omega_z = \pi/\sqrt{2}$.



From the plots we can see that the particle travels arbitrarily close to every spatial position in the volume.

Figure 1: Orbital track plotted for short integration time

For the second three plots we choose: $(\omega_x, \omega_y, \omega_z) = (\sqrt{8}, \sqrt{33}, \sqrt{1.6})$. This gives irrational numbers for $\omega_x/\omega_y = \sqrt{8}/\sqrt{33}$, $\omega_y/\omega_z = \sqrt{33}/\sqrt{1.6}$ and $\omega_x/\omega_z = \sqrt{8}/\sqrt{1.6}$.

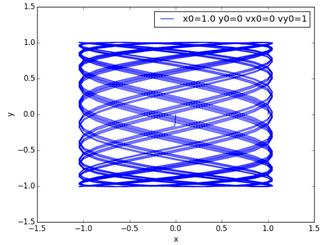


Figure 2: Orbital track plotted for intermediate integration time

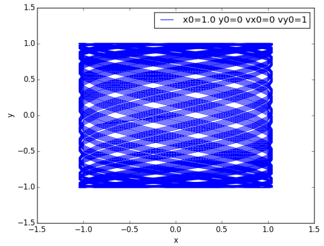


Figure 3: Orbital track plotted for long integration time

b The formula for angular momentum is given by $L = m\vec{r} \times \vec{v}$, writing out gives:

$$
\begin{aligned}
L &= m \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} \\
&= m \begin{pmatrix} a_x \cos(\omega_x t + \psi_x) \\ a_y \cos(\omega_y t + \psi_y) \\ a_z \cos(\omega_z t + \psi_z) \end{pmatrix} \times \begin{pmatrix} -a_x \omega_x \sin(\omega_x t + \psi_x) \\ -a_y \omega_y \sin(\omega_y t + \psi_y) \\ -a_z \omega_z \sin(\omega_z t + \psi_z) \end{pmatrix} \\
&= m \begin{pmatrix} -a_z \omega_z \sin(\omega_z t + \psi_z) a_y \cos(\omega_y t + \psi_y) + a_y \omega_y \sin(\omega_y t + \psi_y) a_z \cos(\omega_z t + \psi_z) \\ a_z \omega_z \sin(\omega_z t + \psi_z) a_x \cos(\omega_x t + \psi_x) - a_x \omega_x \sin(\omega_x t + \psi_x) a_z \cos(\omega_z t + \psi_z) \\ -a_y \omega_y \sin(\omega_y t + \psi_y) a_x \cos(\omega_x t + \psi_x) + a_x \omega_x \sin(\omega_x t + \psi_x) a_y \cos(\omega_y t + \psi_y) \end{pmatrix}
\end{aligned}
$$

If angular momentum is conserved we need $\frac{dL}{dt}$, we write this down for the first element

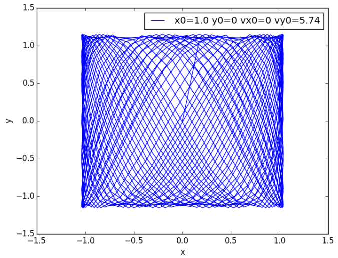


Figure 4: Orbital track plotted for short integration time

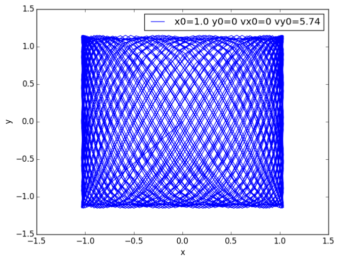


Figure 5: Orbital track plotted for intermediate integration time

(The other elements can be found similarly):

$$
\begin{aligned}
\frac{dL}{dt} &= m\frac{d}{dt}(-a_z\omega_z\sin(\omega_z t+\psi_z)a_y\cos(\omega_y t+\psi_y)+a_y\omega_y\sin(\omega_y t+\psi_y)a_z\cos(\omega_z t+\psi_z))\\
&= m(-a_z\omega_z^2\cos(\omega_z t+\psi_z)a_y\cos(\omega_y t+\psi_y)+a_z\omega_z\sin(\omega_z t+\psi_z)a_y\omega_y\sin(\omega_y t+\psi_y)\\
&+a_y\omega_y^2\cos(\omega_y t+\psi_y)a_z\cos(\omega_z t+\psi_z)-a_y\omega_y\sin(\omega_y t+\psi_y)a_z\omega_z\sin(\omega_z t+\psi_z))\\
&= -a_z\omega_z^2\cos(\omega_z t+\psi_z)a_y\cos(\omega_y t+\psi_y)+a_y\omega_y^2\cos(\omega_y t+\psi_y)a_z\cos(\omega_z t+\psi_z)\\
&= (-a_z\omega_z^2 a_y+a_y\omega_y^2 a_z)\cos(\omega_y t+\psi_y)\cos(\omega_z t+\psi_z)
\end{aligned}
$$

As ω_z and ω_y are not equal this can only be zero when $\cos(\omega_y t+\psi_y)\cos(\omega_z t+\psi_z)=0$ so when $\frac{y}{a_y}\frac{z}{a_z}=0$. The same will be found for the other two elements in the same manner. We thus see from this that the angular momentum changes along the orbit and is thus not conserved.

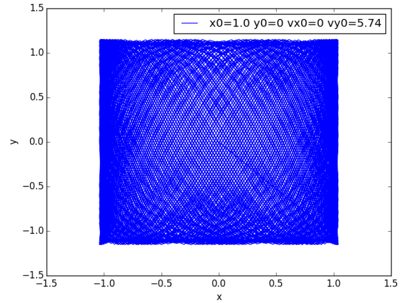


Figure 6: Orbital track plotted for long integration time

The average value of the angular momentum averaged over time will be zero. Every position (x, y, z) in is covered in plus and min in this time (as we also saw from the plots) and these will cancel each other as the total angular momentum is calculated.

c If $\omega_x=\omega_y$ then the angular momentum becomes:

$$
L = m\begin{pmatrix} -a_z\omega_z\sin(\omega_z t+\psi_z)a_y\cos(\omega_x t+\psi_y)+a_y\omega_x\sin(\omega_x t+\psi_y)a_z\cos(\omega_z t+\psi_z)\\ a_z\omega_z\sin(\omega_z t+\psi_z)a_x\cos(\omega_x t+\psi_x)-a_x\omega_x\sin(\omega_x t+\psi_x)a_z\cos(\omega_z t+\psi_z)\\ -a_y\omega_x\sin(\omega_x t+\psi_y)a_x\cos(\omega_x t+\psi_x)+a_x\omega_x\sin(\omega_x t+\psi_x)a_y\cos(\omega_x t+\psi_y)\end{pmatrix}
$$

For the last term we get:

$$
\begin{aligned}
L &= m(-a_y\omega_x\sin(\omega_x t+\psi_y)a_x\cos(\omega_x t+\psi_x)+a_x\omega_x\sin(\omega_x t+\psi_x)a_y\cos(\omega_x t+\psi_y))\\
&= m(a_y\omega_x a_x(-\sin(\omega_x t+\psi_y)\cos(\omega_x t+\psi_x)+\sin(\omega_x t+\psi_x)\cos(\omega_x t+\psi_y)))\\
&= m(a_y\omega_x a_x(-\sin(\psi_y-\psi_x)))\\
&= constant
\end{aligned}
$$

We see that angular momentum is conserved in the z-direction. Also in this case there will be a net angular momentum. Because of the constant angular momentum in the z-direction, the star will not cover every position in the volume and the angular momentum can therefore not cancel as it did in the previous exercise. Box orbits are characterized by the fact that they go through the center and have no net angular momentum. This orbit becomes thus a loop orbit.

2. Finding a solution to the collisionless Boltzmann equation using the Jeans theorem.
(a) Derive ρ and Ψ for a spherically-symmetric system with some distribution function f of the form $f(\epsilon) = \begin{cases} \epsilon^{n-3/2}, & \text{if } \epsilon > 0 \\ 0, & \text{if } \epsilon < 0 \end{cases}$. where $n = 1$, $\epsilon = -E + \Phi_0$ and E is the energy of a particle orbitting around the system. Adopt the standard definition that $\Psi = \epsilon + (1/2)v^2$. Show that the total mass of the model is $(1/2)\Psi_0 G^{-3/2}\sqrt{\pi/c_1}$ where c_1 is defined by equation (4-107b) from BT. Hint this is problem 4-16 from Binney & Tremaine (BT) and is discussed in some depth on BT 223-225.

The density follows from the distribution function as

$$\begin{aligned} \rho(r) &= 4\pi \int_0^\Psi f(\epsilon)\sqrt{2(\Psi-\epsilon)}d\epsilon = 4\sqrt{2}\pi \int_0^\Psi \epsilon^{-1/2}\sqrt{\Psi-\epsilon}d\epsilon \\ &= 4\sqrt{2}\pi \left[\sqrt{\epsilon}\sqrt{\Psi-\epsilon} + \Psi \arctan\left(\frac{\sqrt{\epsilon}}{\sqrt{\Psi-\epsilon}}\right)\right]_0^\Psi \\ &= 4\sqrt{2}\pi\Psi\left[\arctan(\infty) - \arctan(0)\right] = 4\sqrt{2}\pi\Psi\left(\frac{\pi}{2} - 0\right) \\ &= 2\sqrt{2}\pi^2\Psi \equiv A\Psi. \end{aligned} \tag{22}$$

As in Problem 3, we may write the Poisson equation in terms of Ψ and replace for ρ:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi}{\partial r}\right) = -4\pi G\rho = -4\pi G A\Psi. \tag{23}$$

Defining $\psi = \Psi/\Psi_0$ and $s = r/b$ where $b = (4\pi GA)^{-1/2}$, we may rewrite the Poisson equation as

$$\begin{aligned} \frac{\Psi_0}{b^2 s^2}\frac{\partial}{b\partial s}\left(b^2 s^2\frac{\partial\psi}{b\partial s}\right) &= -b^{-2}\Psi_0\psi \\ \frac{1}{s^2}\frac{\partial}{\partial s}\left(s^2\frac{\partial\psi}{\partial s}\right) &= -\psi, \end{aligned} \tag{24}$$

which is the Lane-Emden equation for $n = 1$. We demonstrate that the solution is

$\psi = \sin(s)/s$:

$$\begin{aligned} \frac{1}{s^2}\frac{\partial}{\partial s}\left(s^2\frac{\partial\psi}{\partial s}\right) &= \frac{1}{s^2}\frac{\partial}{\partial s}\left(s^2\frac{\partial}{\partial s}\left[\frac{\sin(s)}{s}\right]\right) = \frac{1}{s^2}\frac{\partial}{\partial s}\left(s^2\left[\frac{\cos(s)}{s} - \frac{\sin(s)}{s^2}\right]\right) \\ &= \frac{1}{s^2}\frac{\partial}{\partial s}\left[s\cos(s) - \sin(s)\right] = \frac{1}{s^2}\left[\cos(s) - s\sin(s) - \cos(s)\right] \\ &= -\frac{\sin(s)}{s} \\ \frac{1}{s^2}\frac{\partial}{\partial s}\left(s^2\frac{\partial\psi}{\partial s}\right) &= -\psi. \end{aligned} \tag{25}$$

We can then write the relative potential

$$\begin{aligned} \Psi(r) &= \Psi_0\psi = \Psi_0\frac{\sin(s)}{s} = \Psi_0\left(\frac{b}{r}\right)\sin\left(\frac{r}{b}\right) \\ &= \frac{\Psi_0}{\sqrt{4\pi GA}}\frac{1}{r}\sin\left(\sqrt{4\pi GA}r\right) \\ &= \frac{\Psi_0}{\sqrt{8\sqrt{2}\pi^3 G}}\frac{1}{r}\sin\left(\sqrt{8\sqrt{2}\pi^3 G}r\right), \end{aligned} \tag{26}$$

and the density

$$\rho(r) = A\Psi = \frac{\Psi_0}{2^{1/4}}\sqrt{\frac{\pi}{G}}\frac{1}{r}\sin\left(\sqrt{8\sqrt{2}\pi^3 G}r\right). \tag{27}$$

The total mass is then

$$
\begin{aligned}
M &= \frac{1}{G}\left(r^2\frac{\partial\Phi}{\partial r}\right)_{r_{\rm cross}} = -\frac{1}{G}\left(b^2s^2\frac{\partial\Psi}{b\partial s}\right)_{s_{\rm cross}} \\
&= -\frac{b}{G}\left(s^2\frac{\partial\Psi}{\partial s}\right)_{s_{\rm cross}},
\end{aligned} \tag{28}
$$

where $s_{\rm cross} = \pi$ refers to the value of s that corresponds to the first zero-crossing for Ψ. Replacing for Ψ,

$$
\begin{aligned}
M &= -\frac{b}{G}\left(s^2\frac{\partial\Psi}{\partial s}\right)_{s=\pi} = -\frac{b}{G}\left(s^2\frac{\partial}{\partial s}\left[\Psi_0\frac{\sin(s)}{s}\right]\right)_{s=\pi} \\
&= -\frac{b\Psi_0}{G}\left(s^2\left[\frac{\cos(s)}{s} - \frac{\sin(s)}{s^2}\right]\right)_{s=\pi} \\
&= -\frac{\Psi_0 b}{G}\left[s\cos(s) - \sin(s)\right]_{s=\pi} = -\frac{\Psi_0 b}{G}(-\pi - 0) \\
&= \frac{\Psi_0\pi b}{G} = \frac{\Psi_0\pi}{G}(4\pi GA)^{-1/2} = \frac{\Psi_0\pi}{G}(8\sqrt{2}\pi^3G)^{-1/2} \\
M &= 2^{-7/4}\pi^{-1/2}G^{-3/2}\Psi_0.
\end{aligned} \tag{29}
$$

Equation (4.85b) in Binney & Tremaine for $n = 1$ gives

$$
\begin{aligned}
c_n &= \frac{(2\pi)^{3/2}(n-\frac{3}{2})!}{n!} \\
\Rightarrow c_1 &= (2\pi)^{3/2}\left(-\frac{1}{2}\right)! = 2^{3/2}\pi^2.
\end{aligned} \tag{30}
$$

Thus, the given expression for the mass becomes

$$
\begin{aligned}
M &= \frac{1}{2}\Psi_0G^{-3/2}\sqrt{\frac{\pi}{c_1}} \\
&= \frac{1}{2}\Psi_0G^{-3/2}\sqrt{\pi}\left(\frac{1}{2^{3/4}\pi}\right) \\
M &= 2^{-7/4}\pi^{-1/2}G^{-3/2}\Psi_0,
\end{aligned} \tag{31}
$$

which matches our previous result.

(b) Derive ρ and Ψ for some spherically symmetric system with the distribution function f with the form $f(\epsilon) = \begin{cases} \epsilon^{n-3/2}, & \text{if } \epsilon > 0 \\ 0, & \text{if } \epsilon < 0 \end{cases}$. where $n = 5$. This is the distribution function for a Plummer model. Find the expression for ρ and Ψ. Derive also the formula for the total mass of the system.

b) $n = 5$

$$\psi = (1+s^2)^{-1/2}$$

$$\frac{1}{s^2}\frac{d}{ds}\left(s^2 \frac{d}{ds}(1+s^2)^{-1/2}\right) = \frac{1}{s^2}\frac{d}{ds}\left(s^2 \cdot \frac{-1}{2}(2s)(1+s^2)^{-3/2}\right)$$

$$= \frac{1}{s^2}\frac{d}{ds}\left(-s^3(1+s^2)^{-3/2}\right) = \frac{-1}{s^2}\left(3s^2(1+s^2)^{-3/2} + s^3(2s)\left(\frac{-3}{2}\right)(1+s^2)^{-5/2}\right)$$

$$= -1\left(3s^2 - 3(1+s^2)\right)(1+s^2)^{-5/2}$$

$$= -3(1+s^2)^{-5/2} = -3\psi^5 \checkmark$$

$$M = -\frac{b}{G}\left(s^2 \Psi_0 \frac{d\psi}{ds}\right)_{s\to\infty}$$

$$= -\frac{b\Psi_0}{G}\left(s^2 \frac{-1}{2}(2s)(1+s^2)^{-3/2}\right)$$

$$= -\frac{b\Psi_0}{G} \frac{s^3}{(1+s^2)^{-3/2}}\Bigg|_{s\to\infty} = \lim_{u\to 0} \frac{b\Psi_0}{G}\frac{1}{u^3(1+u^{-2})^{-3/2}}$$

$$= \lim_{u\to 0} -\frac{b\Psi_0}{G}\frac{1}{(u^2+1)^{-3/2}} = \frac{b\Psi_0}{G}$$

$u = 1/s$

$$= \frac{\Psi_0}{G}\left(\frac{3}{4\pi G C \Psi_0^{7/2}}\right)^{1/2} = \frac{1}{2}\sqrt{\frac{3}{\pi}}\, G^{-3/2} \frac{1}{\sqrt{\Psi_0}},$$

7/4

3. Derive the third Jeans equation by subtracting the second Jeans equation from the first Jeans equation multiplied by $\bar{v_j}$. See the supplementary reading posted on the course website for hints on how to do this.

We know the first Jeans equation:

$$\frac{\partial \nu}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \bar{v}_i$$

and the second Jeans equation:

$$\frac{\partial \nu \bar{v}_i}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \overline{v_i v_j} + \nu \frac{\partial \Phi}{\partial x_j}$$

Now to get the third Jeans equation we have to multiply the first Jeans equation with $\bar{v}_j$ and subtract the second Jeans equation from this. We now can write:

$$\bar{v}_j \frac{\partial \nu}{\partial t} + \bar{v}_j \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \bar{v}_i = \frac{\partial \nu \bar{v}_i}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \overline{v_i v_j} + \nu \frac{\partial \Phi}{\partial x_j}$$

We now use the definition of the velocity dispersion tensor ($\sigma_{ij}^2 = \overline{v_i v_j} - \bar{v}_i \bar{v}_j$) to write:

$$\bar{v}_j \frac{\partial \nu}{\partial t} + \bar{v}_j \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \bar{v}_i = \frac{\partial \nu \bar{v}_i}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu (\sigma_{ij}^2 + \bar{v}_i \bar{v}_j) + \nu \frac{\partial \Phi}{\partial x_j}$$

$$\bar{v}_j \frac{\partial \nu}{\partial t} + \bar{v}_j \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \bar{v}_i = \frac{\partial \nu \bar{v}_i}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \sigma_{ij}^2 + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \bar{v}_i \bar{v}_j + \nu \frac{\partial \Phi}{\partial x_j}$$

We now rewrite $\bar{v}_j \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \bar{v}_i$ and $\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \bar{v}_i \bar{v}_j$ as follows:

$$\bar{v}_j \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \bar{v}_i = \sum_{i=1}^{3} \bar{v}_j \nu \frac{\partial}{\partial x_i} \bar{v}_i + \sum_{i=1}^{3} \bar{v}_j \bar{v}_i \frac{\partial}{\partial x_i} \nu$$

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \bar{v}_i \bar{v}_j = \sum_{i=1}^{3} \nu \bar{v}_i \frac{\partial}{\partial x_i} \bar{v}_j + \sum_{i=1}^{3} \bar{v}_i \bar{v}_j \frac{\partial}{\partial x_i} \nu + \sum_{i=1}^{3} \nu \bar{v}_j \frac{\partial}{\partial x_i} \bar{v}_i$$

Plugging these in the previous equation:

$$\bar{v}_j \frac{\partial \nu}{\partial t} + \sum_{i=1}^{3} \bar{v}_j \nu \frac{\partial}{\partial x_i} \bar{v}_i + \sum_{i=1}^{3} \bar{v}_j \bar{v}_i \frac{\partial}{\partial x_i} \nu$$

$$= \frac{\partial \nu \bar{v}_i}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \sigma_{ij}^2 + \sum_{i=1}^{3} \nu \bar{v}_i \frac{\partial}{\partial x_i} \bar{v}_j + \sum_{i=1}^{3} \bar{v}_i \bar{v}_j \frac{\partial}{\partial x_i} \nu + \sum_{i=1}^{3} \nu \bar{v}_j \frac{\partial}{\partial x_i} \bar{v}_i + \nu \frac{\partial \Phi}{\partial x_j}$$

$$\bar{v}_j \frac{\partial \nu}{\partial t} = \frac{\partial \nu \bar{v}_i}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \sigma_{ij}^2 + \sum_{i=1}^{3} \nu \bar{v}_i \frac{\partial}{\partial x_i} \bar{v}_j + \nu \frac{\partial \Phi}{\partial x_j}$$

We know:

$$\frac{\partial \nu \bar{v}_i}{\partial t} = \bar{v}_j \frac{\partial \nu}{\partial t} + \nu \frac{\partial \bar{v}_j}{\partial t}$$
$$\frac{\partial \nu \bar{v}_i}{\partial t} - \bar{v}_j \frac{\partial \nu}{\partial t} = \nu \frac{\partial \bar{v}_j}{\partial t}$$

We can now write by plugging this in:

$$0 = \nu \frac{\partial \bar{v}_j}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \sigma_{ij}^2 + \sum_{i=1}^{3} \nu \bar{v}_i \frac{\partial}{\partial x_i} \bar{v}_j + \nu \frac{\partial \Phi}{\partial x_j}$$
$$-\nu \frac{\partial \bar{v}_j}{\partial t} - \sum_{i=1}^{3} \nu \bar{v}_i \frac{\partial}{\partial x_i} \bar{v}_j = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \sigma_{ij}^2 + \nu \frac{\partial \Phi}{\partial x_j}$$
$$\nu \frac{\partial \bar{v}_j}{\partial t} + \sum_{i=1}^{3} \nu \bar{v}_i \frac{\partial}{\partial x_i} \bar{v}_j = -\sum_{i=1}^{3} \frac{\partial}{\partial x_i} \nu \sigma_{ij}^2 - \nu \frac{\partial \Phi}{\partial x_j}$$

This is the third Jeans equation.

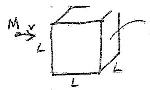
4. How many integrals of motion does a particle have in a Kepler potential? What are they?

5 Integrals of Motion: Energy, Angular Momentum (L_x, L_y, L_z), and the Laplace-Runge-Lenz vector

5. Dynamical Friction. Dynamical friction is an important mechanism which causes colliding galaxies to rapidly merge. How important would this mechanism be, if we consider the collision of a galaxy with an isolated star (wandering alone through the universe)? Make use of the following Chandrasekhar dynamical friction formula presented in class:

$$\frac{dv_M}{dt} = -\frac{4\pi \ln(\Lambda) G^2 (M+m)\rho_m}{v_M^2}$$

Assume that a galaxy is a 3 kpc x 3 kpc x 3 kpc cube with mass $3 \times 10^{10}\ M_\odot$ and is entirely composed of stars with 1 $M_\odot$. Assume that a star with one solar mass $M = 1\ M_\odot$ approaches the galaxy at velocity $v_M = 200$ km/s and at an angle perpendicular to the surface of the cube. How much will dynamical friction change the velocity of the star if it falls in from infinity and continues to infinity? Feel free to assume that the dynamical friction is constant throughout the entire passage of the star through the galaxy (i.e., that the slowing velocity of the star has no effect on the amplitude of the dynamical friction). How would the impact of dynamical friction change if the star (unrealistically) had a mass of $10^9\ M_\odot$? We ignored the effect of dynamical friction in calculating the relaxation time for a star in lecture 5. Is this assumption justified?



$\rho_m = \frac{Nm}{L^3}$

$L = 3$ kpc $\quad m = 1 M_\odot \quad N = 3 \times 10^{10}$

$V = 200$ kms^{-1}

$$\frac{dv}{dt} = -4\pi \ln(\Lambda) G^2 (M+m) \rho_m V^{-2} , \qquad \Lambda = \frac{b_{max} V^2}{G(M+m)}$$

$$b_{max} \sim \frac{L}{2}$$

$$\Delta t \approx \frac{L}{V}$$

$$\Delta V = \frac{dv}{dt} \Delta t = -4\pi G^2 (M+m) \ln\left(\frac{L V^2}{2G(M+m)}\right) \frac{Nm}{L^3} \frac{L}{V} V^{-2}$$

a) $M = 1 M_\odot$

$$\Delta V = -4\pi (6.67 \times 10^{-11})^2 (2 (1.99 \times 10^{30})) \ln\left(\frac{3(3.09 \times 10^{19})(200 \times 10^3)^2}{2(6.67 \times 10^{-11})(2(1.99 \times 10^{30}))}\right) \times \frac{3 \times 10^{10} (1.99 \times 10^{30})}{3(3.09 \times 10^{19})^2} \frac{1}{(200 \times 10^3)^3}$$

$= -4.38 \times 10^{-2}$ ms^{-1}

$\sim$ [illegible] -10^{-2} m s^{-1} (SLOW)

b) $M = 10^9$

$\Delta V = -2.55 \times 10^5$ m s^{-1}

$\sim$ [illegible] 10^5 m s^{-1} (Rapid)

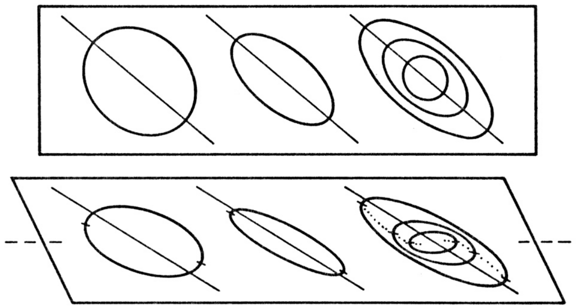
$\frac{\Delta V}{V} \approx -1$ 1 interaction with "stop" other galaxy

Ignoring dynamical for 1 star is justified
ΔV very small for small M

Galaxies: Structure, Dynamics, and Evolution
Problem Set 4
Instructor: Dr. Bouwens

Here is Problem Set 4. The entire problem set will be due before class on Monday, April 27 (email them to Wout and include GSD in the subject line). Be sure to pay extra attention to problem 6, as your solution to that problem will be checked carefully and used in determining your homework grade.

1. Determine the impact of projection effects on the apparent isophotal twist (for elliptical galaxies). Consider two ellipses with their major axis oriented 45 degrees away from some line (that line would be horizontal on the following diagram):



Suppose that the axial ratio is 1.15 for the one ellipse (similar to the leftmost ellipse shown in the above figure) and 2.8 for the other ellipse (similar to the center ellipse shown in the above figure). Suppose that we are viewing the ellipses face on and then we rotate the ellipses by 60 degrees about an axis (parallel to the aforementioned line) so that the ellipses are viewed almost edge on. What ellipticity would we measure for each of our two ellipses? What would be the apparent position angle of the major axis of each ellipse relative to aforementioned line?

The x and y coordinates on an ellipse are given by:

$$x = a\cos t$$
$$y = b\sin t$$

where a is the major axis and b the minor axis. We know want to describe this ellipse rotated by 45 degrees and projected by 60 degrees. First the rotation can be described by multiplying the rotation matrix with the position vector. We then get the following for the x and y coordinates:

$$x = a\cos t\cos\theta - b\sin t\sin\theta$$
$$y = a\cos t\sin\theta + b\sin t\cos\theta$$

The projection can be done by multiplying the y coordinate with $\cos i$. We now can write:

$$x = a\cos t\cos\theta - b\sin t\sin\theta$$
$$y = a\cos t\sin\theta\cos i + b\sin t\cos\theta\cos i$$

Now filling in the values for $\theta = \pi/4$ and $i = \pi/3$:

$$x = a\cos t\frac{1}{\sqrt{2}} - b\sin t\frac{1}{\sqrt{2}}$$
$$y = a\cos t\frac{1}{\sqrt{2}}\frac{1}{2} + b\sin t\frac{1}{\sqrt{2}}\frac{1}{2}$$

The minor axis and major axis are given by the maximum and minimum of r which is given by $r = \sqrt{x^2+y^2}$. So we calculate $\frac{dr^2}{dt}$ and equal this to zero to find where the maximum and minimum value are:

$$r^2 = (\frac{1}{2}a^2\cos^2 t - ab\cos t\sin t + \frac{1}{2}b^2\sin^2 t) + (\frac{1}{8}a^2\cos^2 t + \frac{1}{4}ab\cos t\sin t + \frac{1}{8}b^2\sin^2 t)$$
$$= (\frac{5}{8}a^2\cos^2 t - \frac{3}{4}ab\cos t\sin t + \frac{5}{8}b^2\sin^2 t)$$

$$\frac{dr^2}{dt} = (\frac{5}{8}a^2\cos^2 t - \frac{3}{4}ab\cos t\sin t + \frac{5}{8}b^2\sin^2 t)$$
$$= -a^2\frac{5}{4}\sin t\cos t + \frac{3}{4}ab\sin^2 t - \frac{3}{4}ab\cos^2 t + b^2\frac{5}{4}\sin t\cos t$$

Setting this equal to zero gives:

$$0 = -a^2\frac{5}{4}\sin t\cos t + \frac{3}{4}ab\sin^2 t - \frac{3}{4}ab\cos^2 t + b^2\frac{5}{4}\sin t\cos t$$

The axial ratio a/b is given for both ellipsis. For the first ellipse $a/b = 1.15$ we get two values for t which will give then the position of the minor and major axis:

$$0 = -(1.15b)^2\frac{5}{4}\sin t\cos t + \frac{3}{4}1.15b^2\sin^2 t - \frac{3}{4}1.15b^2\cos^2 t + b^2\frac{5}{4}\sin t\cos t$$
$$0 = -(1.15)^2\frac{5}{4}\sin t\cos t + \frac{3}{4}1.15\sin^2 t - \frac{3}{4}1.15\cos^2 t + \frac{5}{4}\sin t\cos t$$
$$\frac{3}{4}1.15\cos 2t = -(1.15)^2\frac{5}{4}\sin t\cos t + \frac{5}{4}\sin t\cos t$$
$$t = -2.2$$
$$t = -0.67$$

Plugging t in the definition of the position vector gives then the position of the major and minor axis, the axis is given by $r = \sqrt{x^2 + y^2}$:

$$x = 1.15\cos t\frac{1}{\sqrt{2}} - \sin t\frac{1}{\sqrt{2}}$$
$$y = 1.15\cos t\frac{1}{\sqrt{8}} + \sin t\frac{1}{\sqrt{8}}$$

For the first ellipse we calculate $b = 0.53$ and $a = 1.08$. The axial ration is now calculated to be $a/b = 2.04$ and the ellipticity is given by $e = 1 - b/a = 0.51$

Using the same strategy for the second ellipse with $a/b = 2.8$, we find $t = -1.8$ and $t = -0.23$

$$x = 2.8\cos t\frac{1}{\sqrt{2}} - \sin t\frac{1}{\sqrt{2}}$$
$$y = 2.8\cos t\frac{1}{\sqrt{8}} + \sin t\frac{1}{\sqrt{8}}$$

And we calculate $a = 2.27$, $b = 0.62$, $a/b = 3.7$, $e = 1 - b/a = 0.73$. The position angles can be found by using $arctan(y/x) = \theta$ where y and x are the coordinates for the position of the major axis. We find $\theta = 5.3$ degrees for the first ellipse and $\theta = 22.92$ degrees for the second.

2. (a) Derive the enclosed mass M($< r$) for the NFW profile $\rho(r) = \rho_s/[(r/r_s)(1+r/r_s)^2]$. Use $r/(1+r)^2 = 1/(r+1) - 1/(1+r)^2$
(b) Use this to show $\rho_s = \frac{200}{3}\rho_{cr}(z)\frac{c^3}{\ln(1+c)-c/(1+c)}$ given our parameterization $\rho(r) = \frac{\rho_s}{(r/r_s)(1+r/r_s)^2}$
(c) Derive the circular velocity as a function of radius for an NFW profile.

a) We begin with the NFW-profile for the density distribution:

$$\rho(r) = \frac{\rho_s}{\frac{r}{r_s}\left(1+\frac{r}{r_s}\right)^2} \tag{21}$$

We calculate the mass enclosed within a radius r as $M(r) = \int_0^r \rho(r)dV$. Due to spherical symmetry of the NFW-profile, we may write $dV = 4\pi r^2 dr$. The enclosed mass is thus:

$$M_{\text{encl}}(r) = 4\pi\rho_s \int_0^r \frac{r^2 dr}{\frac{r}{r_s}\left(1+\frac{r}{r_s}\right)^2} = 4\pi\rho_s \int_0^r \frac{r r_s dr}{\left(1+\frac{r}{r_s}\right)^2} \tag{22}$$

We introduce the dimensionless integration variable $z = r/r_s$, such that $dr = \frac{1}{r_s}dz$. The enclosed mass is now written as

$$\begin{aligned} M_{\text{encl}}(r) &= 4\pi\rho_s \int_0^{z} \frac{r_s^3 z\,dz}{(1+z)^2} = 4\pi\rho_s r_s^3 \int_0^{z} \left[\frac{1}{1+z} - \frac{1}{(1+z)^2}\right] dz \\ &= 4\pi\rho_s r_s^3 \left\{ [\log(1+z)]_0^{z} + \left[\frac{1}{1+z}\right]_0^{z} \right\} \\ &= 4\pi\rho_s r_s^3 \left\{ \log(1+z) + \frac{1}{1+z} - 1 \right\} \\ &= 4\pi\rho_s r_s^3 \left\{ \log\left(1+\frac{r}{r_s}\right) - \frac{\frac{r}{r_s}}{1+\frac{r}{r_s}} \right\}. \end{aligned} \tag{23}$$

b) We know that, for a collapsed halo, $\rho_{\rm halo} \simeq 200\rho_{\rm crit}$. Thus the mass of a halo, which we define to have a size of r_{200} is given by

$$M_{\rm halo} = \frac{4\pi}{3}\rho_{\rm halo} r_{200}^3 = \frac{800\pi\rho_{\rm crit}(z)}{3} r_{200}^3. \tag{24}$$

We now require that $M_{\rm halo} = M_{\rm encl}(r_{200})$. From this follows that:

$$\begin{aligned} 4\pi\rho_s r_s^3 \left\{ \log\left(1 + \frac{r_{200}}{r_s}\right) - \frac{r_{200}/r_s}{1 + r_{200}/r_s} \right\} &= \frac{800\pi\rho_{\rm crit}(z)}{3} r_{200}^3 \\ \Rightarrow \rho_s(z) = \frac{200}{3}\rho_{\rm crit}(z) \left(\frac{r_{200}}{r_s}\right)^3 &\left\{ \log\left(1 + \frac{r_{200}}{r_s}\right) - \frac{r_{200}/r_s}{1 + r_{200}/r_s} \right\}^{-1}. \end{aligned} \tag{25}$$

We define the concentration parameter to be $c \equiv r_{200}/r_s$ and obtain:

$$\rho_s(z) = \frac{200}{3}\rho_{\rm crit}(z) \frac{c^3}{\log(1+c) - \frac{c}{1+c}}. \tag{26}$$

c) The circular velocity can be found via $v_c^2 = r\frac{d\Phi}{dr} = GM(r)/r$. We immediately obtain from equation 23:

$$v_c^2(r, z) = 4\pi G\rho_s(z) r_s^3 \left\{ \frac{\log\left(1 + \frac{r}{r_s}\right)}{r} - \frac{1/r_s}{1 + r/r_s} \right\}. \tag{27}$$

3. Consider the collapse of a uniform cloud of stars initially at rest. Assume the cloud has a total mass of $5 \times 10^{10}\ M_\odot$, is entirely composed of stars with $1\ M_\odot$, and has approximate dimensions of 2 kpc × 2 kpc × 2 kpc. Assume that the collapse finishes in one free fall time, $1/\sqrt{G\rho}$. What is the time scale for violent relaxation? [Approximate order-of-magnitude estimates are fine for this first step.] If the system were instead in equilibrium (i.e., not undergoing collapse), what relaxation time scale would we estimate for stars in this system using the equations we derived in Lecture #5? How do these time scales compare?

We assume a uniform, cubical cloud, such that its density is given by $\rho = M_{\text{cloud}}/L^3$ with $L = 2$ kpc. The free-fall time, which was shown to be a good approximation of the violent relaxation time by Lynden-Bell, is then given by

$$t_{\text{ff}} = \frac{1}{\sqrt{G\rho}} = \sqrt{\frac{L^3}{GM_{\text{cloud}}}} \simeq 6 \times 10^6 \text{ yr} \approx t_{\text{vr}}. \tag{3}$$

Thus we obtain a violent relaxation timescale of $t_{\text{vr}} \approx 6 \times 10^6$ yr. We can compare this to the collisional relaxation timescale. We found in lecture 2 that this timescale is given by:

$$t_{\text{relax}} \simeq t_{\text{cross}} \frac{N}{8 \log N} \quad \text{where } t_{\text{cross}} \simeq \frac{L}{v}. \tag{4}$$

Here N is the number of stars in the cloud, given by $N = M_{\text{cloud}}/M_*$. The crossing time depends on the average velocity of particles in the cloud v, which we find via the virial theorem. To good approximation we can write $v^2 = GM_{\text{cloud}}/L$, where we are off by a factor of unity due to the potential energy depending on the precise density distribution and shape of the cloud. We now write the relaxation time as

$$t_{\text{relax}} \simeq \frac{L}{\sqrt{\frac{GM_{\text{cloud}}}{L}}} \frac{N}{8 \log N} = \sqrt{\frac{L^3}{GM_{\text{cloud}}}} \frac{N}{8 \log N} = t_{\text{ff}} \frac{N}{8 \log N}. \tag{5}$$

We obtain that $t_{\text{relax}} \simeq 1.5 \times 10^{15}$ yr. We see immediately that the two timescales differ by more than 8 orders of magnitude.

4. Determine what the b_n normalization factor in the Sersic law must be such that the integral of the surface brightness profile $10^{b_n[(R/R_e)^{1/n}-1]}$ over all radii is equal to one. What is this normalization factor in the case $n = 1$ and $n = 4$?

The Sérsic profile, which describes the surface brightness of a galaxy, is given by

$$I_n(R) = I_e \times 10^{b_n\left[\left(\frac{R}{R_e}\right)^{1/n}-1\right]}. \tag{6}$$

We can always write $10^x = \mathrm{e}^y$ for $y = x\log(10)$. Note that log denotes the natural logarithm. We write $\beta_n \equiv -b_n \log(10)$ such that the Sérsic profile is rewritten to

$$I_n(R) = I_e \mathrm{e}^{-\beta_n\left[\left(\frac{R}{R_e}\right)^{1/n}-1\right]} = I_e \mathrm{e}^{\beta_n} \mathrm{e}^{-\beta_n\left(\frac{R}{R_e}\right)^{1/n}}. \tag{7}$$

We now require that the integral of this profile over all radii equals unity. We write this as follows:

$$1 = 2\pi \int I_n(R) R dR = 2\pi I_e \mathrm{e}^{\beta_n} \int_0^\infty R \mathrm{e}^{-\beta_n\left(\frac{R}{R_e}\right)^{1/n}} dR. \tag{8}$$

We now substitute $x = \beta_n (R/R_e)^{1/n}$ such that $dx/dR = \beta_n/(nR_e)\,(R/R_e)^{1/n-1}$. This implies that

$$R = R_e \left(\frac{x}{\beta_n}\right)^n, \qquad dR = \frac{nR_e}{\beta_n}\left(\frac{R}{R_e}\right)^{1-1/n} dx = \frac{nR_e}{\beta_n}\left(\frac{x}{\beta_n}\right)^{n-1} dx. \tag{9}$$

We plug this into equation 8 to obtain:

$$\begin{aligned} 1 &= 2\pi I_e \mathrm{e}^{\beta_n} \int \mathrm{e}^{-x} R_e \left(\frac{x}{\beta_n}\right)^n \frac{nR_e}{\beta_n}\left(\frac{x}{\beta_n}\right)^{n-1} dx \\ &= 2\pi I_e \mathrm{e}^{\beta_n} \frac{nR_e^2}{\beta_n^{2(n-1)}} \int_0^\infty e^{-x} x^{2n-1} dx \\ &= 2\pi I_e n R_e^2 \frac{e^{\beta_n}\Gamma(2n)}{\beta_n^{2(n-1)}}. \end{aligned} \tag{10}$$

The final expression determines the normalization of β_n, and thus also that of b_n. Note that $\Gamma(2n)$ is the Gamma-function, which equals $(2n-1)!$ when n is an integer (as is the case for us). We plug $\beta_n = -b_n \log(10)$ back into this equation to obtain

$$1 = 2\pi I_e R_e^2 \frac{\mathrm{e}^{-b_n \log 10} n\Gamma(2n)}{(b_n \log 10)^{2(n-1)}} \tag{11}$$

The minus sign in the denominator has disappeared because it is taken to the $2(n-1)^{\text{th}}$ power, which is always an even number for n an integer. We cast the previous equation in a slightly nicer form:

$$10^{b_n} b_n^{2(n-1)} = \frac{2\pi n\Gamma(2n) I_e R_e^2}{(\log 10)^{2(n-1)}}. \tag{12}$$

For $n = 1$ this becomes

$$10^{b_1} = 2\pi I_e R_e^2 \quad \Rightarrow \quad b_1 = \log_{10}\left(2\pi I_e R_e^2\right). \tag{13}$$

For $n = 4$ we need to solve the equation numerically:

$$10^{b_4} b_4^6 = \frac{8\pi \times 7! \times I_e R_e^2}{(\log 10)^6} \simeq 850 I_e R_e^2 \tag{14}$$

Actually, we can find another expression for b_n, by making use of the fact that, by definition, the following is true:

$$\frac{1}{2} = 2\pi \int_0^{R_e} I_n(R) R dR \tag{15}$$

.

The radius R_e is namely defined to contain half of all the light (hence its name being the half-light radius). We can use our results from the above to find the following relation:

$$2 = \frac{\int_0^\infty 2\pi I_n(R) R dR}{\int_0^{R_e} 2\pi I_n(R) R dR} = \frac{\Gamma(2n)}{\Gamma(2n) - \Gamma(2n, \beta_n)}. \tag{16}$$

Here I made use of the following integral definition of the incomplete Gamma-function:

$$\Gamma(2n) - \Gamma(2n, \beta_n) = \int_0^{\beta_n} e^{-x} x^{2n-1} dx. \tag{17}$$

This is the same integral as in equation 10, except with a different upper limit $x(R_e) = \beta_n$, where x is the same parameter as the one introduced via substitution in equation 9. We rewrite equation 17 to

$$\Gamma(2n, \beta_n) = \frac{1}{2}\Gamma(2n). \tag{18}$$

Now, our values for β_n are no longer dependent on R_e and I_e, which they were in equation 12. We solve the above equation numerically for $n = 1$ and $n = 4$.

$$\begin{aligned} \Gamma(2, \beta_1) &= \frac{1}{2}\Gamma(2) = \frac{1}{2} \quad && \Rightarrow \beta_1 \simeq 1.68 \\ \Gamma(8, \beta_4) &= \frac{1}{2}\Gamma(8) = 2520 \quad && \Rightarrow \beta_4 \simeq 7.67 \end{aligned} \tag{19}$$

By using our definition of $\beta_n = -b_n \log 10$ we find that $b_1 = -0.73$ and $b_4 = -3.33$.

5. Look at the angular correlation functions for luminous galaxies $-22 < M_{UV,AB} < -21$ and lower luminosity galaxies $-19 < M_{UV,AB} < -18$ (shown in the last lecture). What is the ratio of bias factors for these galaxies at a scale of 1.5 h^{-1} Mpc? [Make your best guess for the bias factors based on the figure shown in lecture.]

For this we consult slides 75 and 76 of lecture 9, showing the projected correlation function as function of distance, for different galaxy luminosities and colours. At a projected separation of $1.5h^{-1}$ Mpc we find for the galaxies with absolute magnitude $-22 < M < -21$ that $\xi(r) \simeq 70h^{-1}$ Mpc. For the fainter galaxies ($-19 < M < -18$), we find instead $\xi(r) \simeq 30h^{-1}$ Mpc.
We know that $\xi(r)_{\rm galaxies} = b^2\xi(r)_{\rm DM}$, where b is the bias factor. We find that

$$\frac{\xi(r)_{\rm bright}}{\xi(r)_{\rm faint}} = \left(\frac{b_{\rm bright}}{b_{\rm faint}}\right)^2 \frac{\xi(r)_{\rm DM}}{\xi(r)_{\rm DM}} = \left(\frac{b_{\rm bright}}{b_{\rm faint}}\right)^2. \tag{20}$$

From the numbers we found above, we then see that $b_{\rm bright}/b_{\rm faint} \simeq 1.5$. We apply the same to the red and blue galaxies. We find, again at a projected distance of $1.5h^{-1}$ Mpc, that $\xi(r)_{\rm red} \simeq 80h^{-1}$ Mpc and $\xi(r)_{\rm blue} \simeq 30h^{-1}$ Mpc. We find that $b_{\rm red}/b_{\rm blue} \simeq 1.6$.

6. Derive the Fundamental Plane that one would find if the mass-to-light ratio is a function of mass only $M/L = M^{0.25}$ and more generally $M/L = M^\gamma$. (The Fundamental Plane is the relation of the form $R_e \propto \sigma^\alpha \mu_e^\beta$ where R_e is the half-light radius.) Assume that the galaxies are homologous, i.e., they have similar density profiles, but scaled up or down with respect to each other. Note that the assumption of homology results in the following relation: $\sigma^2 \propto M/R_e$.

We assume that elliptical galaxies have a mass-to-light ratio that depends only on their mass, i.e. $M/L \propto M^\gamma$ for some power-law index γ. We can write the luminosity of an elliptical galaxy as $L \propto R_e^2\mu_e$, where μ_e is the surface brightness profile.

We also assume the galaxies are virialized systems, which implies that the virial theorem holds: $2K + U = 0$, where K is the total kinetic energy per unit mass, and U the potential energy, also per unit mass. In general, the potential energy can be written as $U = \alpha\frac{GM}{R}$ where α is a constant which depends on the density profile. We assume elliptical galaxies are homologous, thus this constant will be the same for galaxies of different masses and sizes. In that case, it follows from the virial theorem that $\sigma^2 \propto M/R \propto M/R_e$. The generality of the second proportionality is also a consequence of homology. We take this all together and write

$$\frac{M}{L} \propto \frac{M}{\mu_e R_e^2} \propto M^\gamma \quad \Rightarrow \quad M^{\gamma-1} \propto \mu_e^{-1}R_e^{-2} \propto \left(\sigma^2 R_e\right)^{\gamma-1}. \tag{1}$$

From the final proportionality we may write:

$$R_e^{1+\gamma} \propto \sigma^{2(1-\gamma)}\mu_e^{-1} \quad \Rightarrow \quad R_e \propto \sigma^{\frac{2(1-\gamma)}{1+\gamma}}\mu_e^{-\frac{1}{1+\gamma}}. \tag{2}$$

This is the fundamental plane relation in its general form. If we take $\gamma = 0.25$ we find that $R_e \propto \sigma^{1.2}\mu_e^{-0.8}$, which is close to the relation that is observed between these three parameters (in the lecture we saw $R_e \propto \sigma^{1.4}\mu_e^{-0.9}$).

7. The number density of galaxies is about 0.01 h^3 Mpc^{-3}. The correlation length r_0 is $5h^{-1}$ Mpc.
a) Why does the density and the correlation length depend on h ($= H_0/$ (100 km/s/Mpc))
b) The correlation function gives the relative excess of galaxies at a given radius. Calculate the integrated correlation function, i.e., the excess from within a radius smaller than r.
c) Now combine this with the average number density to estimate the radius r within which each galaxy has on average 1 neighbor.
d) What would this radius be if the galaxies are not correlated?

(a) Distances can be converted from measured angles by assuming a set of cosmological parameters including the Hubble constant, H_0. The dependence of measured quantities on H_0, thus h, can usually be factored out nicely. Singling out h from the measured value makes it easier for users to substitute in the latest estimate of H_0. (1)
See: `https://arxiv.org/pdf/1308.4150.pdf`

(b) Integrated correlation function:

$$4\pi \int_0^r \xi(r')r'^2 dr' = 4\pi \int_0^r \left(\frac{r}{r_0}\right)^{-\gamma} r'^2 dr' = \frac{4\pi r^{3-\gamma}}{(3-\gamma)r_0^{-\gamma}} \quad (1)$$

(c) Number of neighbors within r = average number density × integrated 1+ξ(r), since ξ(r) only gives the *fractional* excess. (1)

Solve for r in:

$$0.008h^3 Mpc^3 \times 4\pi \int_0^r (1 + \left(\frac{r}{5.2h^{-1}Mpc}\right)^{-\gamma}) r'^2 dr' = 1$$

$$r = 0.66h^{-1} Mpc \quad (1)$$

(d) No correlation means there is no excess at any radii, i.e. ξ(r) = 0. (1)
Again, solve for r:

$$0.008h^3 Mpc^3 \times 4\pi \int_0^r r'^2 dr' = 1$$

$$r = 3.1h^{-1} Mpc \quad (1)$$

Galaxies: Structure, Dynamics, and Evolution
Problem Set 5
Instructor: Dr. Bouwens

Here is Problem Set 5. The entire problem set will be due before class on Monday, May 4 (email them to Wout and include GSD in the subject line). Be sure to pay extra attention to problem 1, as your solution to that problem will be checked carefully and used in determining your homework grade.

1. We can see from the figure from Springel et al. that about 30-40% of the mass of a halo is in subhalos. This appears quite different from the situation in clusters, where the light is dominated by the ensemble of regular cluster galaxies, and NOT by the brightest cluster galaxy. Can you think of an explanation ?

We consult slide 20 of lecture 9, which zooms in on a DM halo containing a main, central halo with numerous subhaloes. It is evident that the central halo outshines all the subhaloes, even though the latter may contain 30-40% of the mass of the total, encompassing halo. If we instead look at galaxy clusters, there is also a large, bright galaxy in the centre. However, this galaxy does not outshine the rest of the cluster in the way a central galaxy like the Milky Way outshines all its satellites/dwarf galaxies.

A likely explanation for this is that subhaloes around a galaxy are inefficient regions of star formation, such that they do not emit much light. A possible reason for this might be the result of the small binding energy such subhaloes have, due to their low masses. Energetic events, such as supernovae, are able to blow out much of the gas in small haloes, which impedes their star formation. Furthermore, the central galaxy might also strip the dwarf galaxies of baryons, further disabling their star formation. Individual galaxies in a galaxy cluster are more likely to keep their baryons due to their higher mass, hence they will be able to form stars much more efficiently.

Furthermore, in a galaxy cluster, all large galaxies likely have sufficient binding energy to maintain a substantial part of their baryonic content, despite feedback processes as supernovae and AGN feedback. They therefore remain important sources of star formation, and will have a non-negligible impact on the total cluster brightness.

2. One result which has been found in the astronomical literature (Adelberger 2005) is that the observed clustering of quasars does not depend on the luminosity of the quasar. What does this suggest about the relationship between the quasar luminosity and the mass of the underlying halo in which it lives. Can you think of any physical reason why this might be the case?

As we have seen in the lecture, as well as in the previous exercise, more luminous galaxies generally live in higher mass haloes. Because quasars do not cluster depending on their luminosity, this implies that more luminous quasars do not necessarily live in higher mass haloes / more luminous galaxies.

There does therefore not appear to be a quasar-luminosity vs. halo-mass relation, i.e. the luminosity of a quasar does not depend on the mass of the underlying halo. This would imply that quasars can have a broad range of luminosities for a narrow range of masses for the underlying halo.

A possible physical reason for this is energy input into the galaxy by the central quasar: luminous quasars accrete a lot of baryonic material, causing emission of hard radiation and possible outflows. These mechanisms will impede accretion, and baryons will be blown away from the central quasar. In turn, it will therefore become less active due to the lack of fuel.

There might thus be an equilibrium situation in the amount of material a quasar accretes, which dictates its luminosity. The total luminosity of a quasar might therefore not be dependent on the underlying halo mass, as even though more massive haloes contain more baryons and thus more potential fuel as well as a deeper gravitational potential well, the quasar will 'regulate' its luminosity via feedback processes.

Another possibility is that quasars simply only exist in a very small range of underlying halo masses. In this case there would be no observed clustering, as these haloes all cluster in the same way, even though quasar luminosities might differ a lot from halo to halo.

3. Approximate the rotation curve of UGC 4325 by a straight line, through (0 arcsec, 0 km/s) and (60 arcsec, 110 km/s). What is the best fitting NFW model ? This would be the model for which $\chi^2 = \int (V_{obs} - V_{model})^2 dr$ is minimized.

TBD

4. Images of the bulge show that it has a very regular appearance. However, we have seen that the halo is quite irregular, with stellar streams, etc. Why might the bulge be so regular, whereas the halo is irregular? Be quantitative.

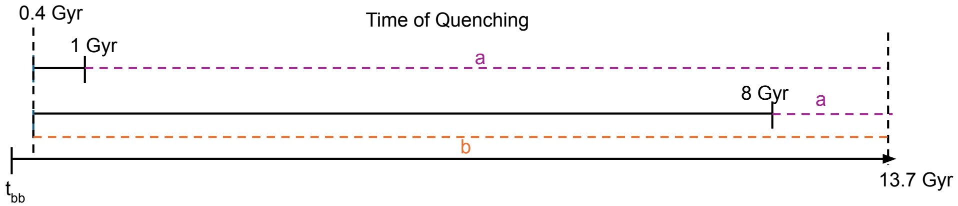
There are many different factors which contribute to making the structure in the bulge much smoother than in the halo.

Violent Relaxation: As the bulge is the end destination for baryons merging into a galaxy, stars in the bulge will experience a rapidly changing potential which would tend to thermalize the velocity profile (maximum entropy).

Dynamical friction: The density of stars in the bulge is much, much higher than in the outer parts of the galaxy. As a result, any spatially significant substructure (with velocity and spatial coherence) would sink to the center of the bulge, where it would meet other equally dense or significant substructure.

5. Assume that red galaxies form in random bursts from $t = 0.4$ Gyr to $t = T_b$ where T_b is 1 Gyr and 8 Gyr and t is the time after the Big Bang. Calculate the scatter in the color $U-V$ magnitude that one would derive for a population of such galaxies, assuming $U - V = 0.65 \log_{10} \text{time} + b$ where b is some constant. The current age of the universe is 13.7 Gyr.

Problem Set 5 Question 5



We are looking at two cases for the quenching times of red galaxies. For T_b = 1: galaxies formed early and were quenched over a short period of time (0.4-1Gyr). For T_b = 8: galaxies formed early and were quenched over a longer period of time (0.4-8Gyr). The quenching times are related to the spread in colors. Therefore, we want to look at the scatter in the color. We would expect a smaller scatter for 0.4-1Gyr, as most of the galaxies are red. We would expect a larger scatter for 0.4-8Gyr as there would be some galaxies that would be slightly bluer.

Since we can think of time here as stellar age and the universe is 13.7 Gyr old, we can say time is $t_{age} = 13.7 - t$. So the intervals in which the stars form and quench are $t_{age} = [12.7, 13.3]$ for $T_b = 1$ and $t_{age} = [5.7, 13.3]$ for $T_b = 8$. The scatter is the standard deviation, which we can get by finding the variance $\sigma^2 = < f^2 > - < f >^2$ and taking the square root. We are given that $f = 0.65 \log_{10} t_{age} + b$. We can rewrite $\log_{10} t_{age}$ as $\frac{\ln(t_{age})}{\ln(10)}$ and then drop the constant b (it doesn't affect the variance). We can rewrite $Var(0.65 \frac{\ln(t_{age})}{\ln(10)}) = (\frac{0.65}{\ln(10)})^2 Var(\ln(t_{age}))$. Since we are looking at a random uniform distribution over the interval we can take the expectation values as the average. So $< f > = \frac{1}{b-a}\int_a^b \ln(t_{age})dt$.

Use integration by parts to solve this with $u = \ln(t_{age})$, $du = \frac{1}{t_{age}}dt, v = t, and\ dv = dt$.

$$< \boldsymbol{f} > = \frac{1}{b-a}\int_a^b \ln(t_{age})dt = \frac{1}{b-a}\left[t_{age} ln(t_{age})|_a^b - \int_a^b t_{age} \times \frac{1}{t_{age}} dt\right] = \boxed{\frac{\mathbf{1}}{\boldsymbol{b-a}}[\boldsymbol{t_{age} ln(t_{age}) - t_{age}}]|_a^b}$$

We can also find $< f^2 >$ through integration by parts, with $u = \ln(t_{age})^2$, $du = \frac{2\ln(t_{age})dt}{t_{age}}, v = t, and\ dv = dt$:

$$< f^2 > = \frac{1}{b-a}\int_a^b \ln(t_{age})^2 dt = \frac{1}{b-a}\left[t_{age} ln(t_{age})^2|_a^b - 2\int_a^b \boldsymbol{\ln(t_{age})dt}\right]$$

We have already solved $\int_a^b \boldsymbol{\ln(t_{age})dt}$ previously, so plugging it in gives:

$$< \boldsymbol{f^2} > = \frac{1}{b-a}\int_a^b \ln(t_{age})^2 dt = \boxed{\frac{\mathbf{1}}{\boldsymbol{b-a}}[\boldsymbol{t_{age} ln(t_{age})^2 - 2t_{age} ln(t_{age}) + 2t_{age}}]|_a^b}$$

For case $T_b = 1$: a = 12.7 and b=13.3

$< f > \approx 2.5648605$, $< f >^2 \approx 6.5785098$, $< f^2 > \approx 6.578687$, $\boxed{\boldsymbol{\sigma^2 \approx 0.00001412} \textbf{ and } \boldsymbol{\sigma \approx 0.0038}}$

For case $T_b = 8$: a=5.7 and b=13.3

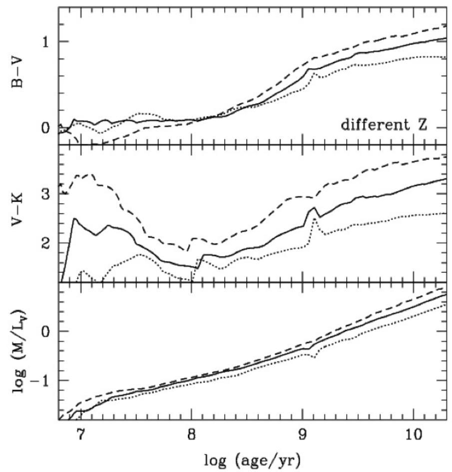
$< f > \approx 2.223$, $< f >^2 \approx 4.9427$ and $< f^2 > \approx 5.0005$, $\boxed{\boldsymbol{\sigma^2 \approx 0.0456} \textbf{ and } \boldsymbol{\sigma \approx 0.068}}$

So, just like we expected, galaxies that were quenched over a longer time indeed show a larger scatter than galaxies that were quenched over a shorter time.

Galaxies: Structure, Dynamics, and Evolution
Problem Set 6
Instructor: Dr. Bouwens

Here is Problem Set 6. The entire problem set will be due before class on Monday, May 11 (email them to Wout and include GSD in the subject line). Be sure to pay extra attention to problem 1, as your solution to that problem will be checked carefully and used in determining your homework grade.

1. Evolution of the mass-to-light ratio. (a) The mass-to-light ratio is roughly a power-law with time. Measure by hand the coefficient α for the mass-to-light ratio $M/L = t^\alpha$ for the V band from the following figure shown in lecture (choose the middle line):



(b) The $B-V$ and $V-K$ color of a SSP (simple stellar population) depend more or less linearly on $\log t$ for ages about 10^8 years. Determine this dependence from the figure shown above (take the middle line again). Use the result to derive the coefficient α for the mass-to-light ratio dependence on time for the B band and the K band.

(c) Use the following figure (also shown in lecture) to derive α for the U-band (the U band curve is the steepest one):

(a)

$$\text{slope } = \alpha_V = \frac{\Delta log(M/L)}{\Delta log(t)} \sim \frac{0.4-(-1.6)}{10-7} = 0.67 \qquad (1)$$

(b)

$$\frac{\Delta B-V}{\Delta \log(t)} \sim \frac{0.5-0.1}{9-8} = 0.4 \quad \text{and} \quad \frac{\Delta V-K}{\Delta \log(t)} \sim \frac{2.4-1.6}{9-8} = 0.8 \qquad (1)$$

$$B-V = 0.4\log_{10} t + \text{constant} \quad \text{and} \quad V-K = 0.8\log_{10} t + \text{constant}$$

$$-2.5\log_{10}\frac{F_B}{F_V} = -2.5\log_{10}\frac{M/L_V}{M/L_B} \propto 0.4\log_{10} t \quad \text{and} \quad -2.5\log_{10}\frac{M/L_K}{M/L_V} \propto 0.8\log_{10} t$$

$$\frac{M}{L_B} \propto \frac{M}{L_V t^{-0.16}} \propto t^{0.83} \quad \text{and} \quad \frac{M}{L_K} \propto t^{0.35} \qquad (1)$$

(c)

$$\alpha_U \sim \frac{-1.1-(-2.7)}{9-8} = 1.60 \qquad (1)$$

2. (a) Assume that the time dependence of the mass-to-light ratio derived in problem #1 for all t below 10^{10} years. The equations above were derived for single burst stellar populations. Now assume a population with constant star formation. Calculate the evolution of the M/L ratio with time T for the U, B, V and K band. Do this by calculating the light from a populations formed at a time interval t, $t + dt$, and then integrating from $t = 0$ to $t = T$, where T varies from 1 to 10 Gyr. The only thing we care about is the dependence of the M/L ratio with time, not the absolute value of the M/L ratio.

(b) Use the results obtained in (a) to derive the dependence of the $U - B$, $B - V$, and $V - K$ colors with time. Compare these numbers to the time dependence of the same colors for an SSP.

(a) A population of constant star formation is equivalent to an infinite number of single bursts over infinitesimal time intervals. (1)

$$\text{For U band, } \frac{M}{L_U}(T) = \int_0^T \frac{M}{L_U}(t)dt = \int_0^T t^{1.60} dt \propto T^{2.60}$$

$$\text{Similarly, } \frac{M}{L_B}(T) \propto T^{1.83}, \frac{M}{L_V}(T) \propto T^{1.67}, \text{ and } \frac{M}{L_K}(T) \propto T^{1.35} \quad (1)$$

(b)

$$(U-B)(T) = -2.5\log_{10}\frac{F_U}{F_B} = -2.5\log_{10}\frac{M/L_B}{M/L_U} = -2.5\log_{10}\frac{T^{1.83}}{T^{2.6}} = 1.9\log_{10}T$$

$$(B-V)(T) = -2.5\log_{10}\frac{F_B}{F_V} = -2.5\log_{10}\frac{M/L_V}{M/L_B} = -2.5\log_{10}\frac{T^{1.67}}{T^{1.83}} = 0.4\log_{10}T$$

$$(V-K)(T) = -2.5\log_{10}\frac{F_V}{F_K} = -2.5\log_{10}\frac{M/L_K}{M/L_V} = -2.5\log_{10}\frac{T^{1.35}}{T^{1.67}} = 0.8\log_{10}T \quad (1)$$

The time dependence of colors of a population with constant star formation is the same as that of a simple stellar population.

3. An important assumption in the analysis of unresolved stellar population is that of a universal initial mass function. What would be the impact if this assumption were not true? Consider two cases: the first being a Salpeter IMF with cut-offs at 0.1 $M_\odot$ and 100 $M_\odot$ and the second being a Salpeter IMF with cut-offs at 0.1 $M_\odot$ and 1 $M_\odot$.
(a) Assume that a galaxy formed stars acording to the two IMFs described. Very qualitatively, what would the SEDs of galaxies look like like 10 Myr later and 11 Gyr later? How similar are the SEDs of galaxies in the two cases at the later time?
(b) How do the SEDs of galaxies evolve in the case of the first IMF vs. the second IMF? How accurately could one determine the time since the instantaneous burst of star formation in the two cases?
(c) Let's suppose that the true IMF of a galaxy corresponded to the second case, but let's suppose one assumed it was the first case. How might it impact one's estimates of the total mass locked up in stars based on the observed SED? How might it impact one's estimates on the total metals ejected as a result of supernovae in the formed stars? Describe each case.

(a) Stellar lifetime τ varies like $\frac{M}{L}$ (amount of fuel divided by energy output rate). Given the mass-luminosity relation of L $\propto$ M$^{3.5}$, $\tau \propto M^{-2.5}$. The Sun's lifetime is ~10^9 years.

At 10 Myr, stars that are more massive than ~10 M$_\odot$ have evolved off the main sequence. Still, there are more massive stars in case 1 than in case 2, since the case 2 IMF has an upper cut-off of 1 M$_\odot$. As a result, the case 1 SED will be significantly bluer than that of case 2. (1)

At 11 Gyr, stars that are more massive than ~1 M$_\odot$ have evolved off the main sequence. The most massive stars in the two cases then have the same mass. As a result, their SEDs look similar. (1)

(b) The case 1 SED will evolve quickly during the 11 Gyr as the IMF extends to 100 M$_\odot$, and that massive stars evolve more quickly than low-mass stars. It will also have more spectral features that can be used in breaking the age-metallicity degeneracy. Thus the case 1 age can be determined accurately. (1)

The case 2 SED will not change much during the 11 Gyr as the most massive stars are only 1 M$_\odot$, which have only just started to evolve off the main sequence. Therefore the case 2 age cannot be determined accurately. (1)

(c) If case 2 is the underlying IMF but case 1 is assumed instead, and that the true age is less than 11 Gyr, the age will be (over-)estimated to be 11 Gyr based on the color, which can be explained by stars that are more massive than 1 M$_\odot$ have evolved off the main sequence. However, the mass still locked up in main sequence stars will be quite accurate if one assumes evolved stars have mostly returned their mass to the ISM. (1)

The efficiency of metals ejected by massive stars will be over-estimated because there were never that many massive stars in the case 2 IMF. (1)

4. Use a modern stellar population synthesis code to predict galaxy spectra. In this problem you will use the Flexible Stellar Population Synthesis (FSPS) code through its Python interface (`python-fsps`). Start early, as installation and setup may take some time.

(a) Consider a simple stellar population (SSP), in which all stars form instantaneously at $t = 0$. Using FSPS, generate spectra for a population with the following parameters:

- Star formation history: instantaneous burst (SSP)
- Metallicity: $Z \approx 0.004$ (subsolar)
- Initial mass function: Salpeter
- No dust attenuation and no nebular emission

Compute spectra over a range of ages from 10^6 to 10^{10} years, using logarithmically spaced time steps (e.g., ~ 100 steps). Plot the spectrum at approximately 10^7, 10^8, 10^9, and 10^{10} years on the same wavelength range.

Briefly describe how the spectral shape evolves with time.

(b) Using the same model, compute the absolute magnitudes in the V and I bands as a function of time over the range 10^6 to 10^{10} years.

From these results, determine the absolute magnitude in the I band, M_I, either directly or using

$$M_I = M_V - (V - I).$$

Plot M_I as a function of $\log_{10}(t/\mathrm{yr})$.

Using the stellar mass of the population and the I-band luminosity, determine how the mass-to-light ratio evolves with time. Assume a power-law form

$$\frac{M}{L_I} \propto t^{\alpha}.$$

Estimate the value of α over the range where a power-law provides a reasonable approximation.

How does your result compare with the analytic scaling derived in class under the assumption that the light is dominated by red giant stars and the initial mass function is Salpeter?

TBD