Collisionless Stellar Dynamics + Vlasov / Jeans Equation

February 23

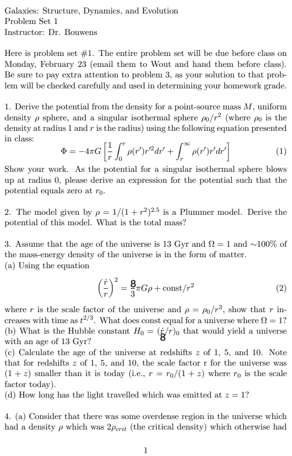
Layout of the Course

Lectures

- Feb 2: Course Introduction, Overview, and Galaxy Formation Basics
- Feb 9: Disk Galaxies (I)
- Feb 12: Disk Galaxies (II)
- Feb 16: Disk Galaxies (III) / Collisionless Stellar Dynamics
- **Feb 23: Collisionless Stellar Dynamics + Vlasov/Jeans Equations**
- Feb 26: Vlasov/Jeans Equations / Elliptical Galaxies (I)
- Mar 9: Elliptical Galaxies (II)
- Mar 23: Elliptical Galaxies (III)
- Mar 30: Dark Matter Halos
- Apr 13: Large Scale Structure
- Apr 20: Galaxy Stellar Populations
- Apr 23: Lessons from Large Galaxy Samples at z<0.2
- May 4: Evolution of Galaxies with Redshift
- May 11: Galaxy Evolution at z>1.5 / Review for Final Exam

Problem Set 1

(Distributed last week, due today before class)



Galaxies: Structure, Dynamics, and Evolution
Problem Set 1
Instructor: Dr. Bouwens

Here is problem set #1. The entire problem set will be due before class on Monday, February 23 (email them to Wout and hand them before class). Be sure to pay extra attention to problem 3, as your solution to that problem will be checked carefully and used in determining your homework grade.

1. Derive the potential from the density for a point-source mass M, uniform density ρ sphere, and a singular isothermal sphere ρ_0/r^2 (where ρ_0 is the density at radius 1 and r is the radius) using the following equation presented in class:

$$\Phi = -4\pi G\left[\frac{1}{r}\int_0^r \rho(r')r'^2dr' + \int_r^\infty \rho(r')r'dr'\right] \quad (1)$$

Show your work. As the potential for a singular isothermal sphere blows up at radius 0, please derive an expression for the potential such that the potential equals zero at r_0.

2. The model given by $\rho = 1/(1+r^2)^{2.5}$ is a Plummer model. Derive the potential of this model. What is the total mass?

3. Assume that the age of the universe is 13 Gyr and $\Omega = 1$ and ~100% of the mass-energy density of the universe is in the form of matter. ← This will be the graded problem
(a) Using the equation

$$\left(\frac{\dot{r}}{r}\right)^2 = \frac{8}{3}\pi G\rho + \text{const}/r^2 \quad (2)$$

where r is the scale factor of the universe and $\rho = \rho_0/r^3$, show that r increases with time as $t^{2/3}$. What does const equal for a universe where $\Omega = 1$?
(b) What is the Hubble constant $H_0 = (\dot{r}/r)_0$ that would yield a universe with an age of 13 Gyr?
(c) Calculate the age of the universe at redshifts z of 1, 5, and 10. Note that for redshifts z of 1, 5, and 10, the scale factor r for the universe was $(1+z)$ smaller than it is today (i.e., $r = r_0/(1+z)$ where r_0 is the scale factor today).
(d) How long has the light travelled which was emitted at $z = 1$?

4. (a) Consider that there was some overdense region in the universe which had a density ρ which was $2\rho_{crit}$ (the critical density) which otherwise had

1

Problem Set 2

(To be distributed today, due on March 16)

Layout of the Course

Lectures

Note lecture this Thursday at 15:15

- Feb 2: Course Introduction, Overview, and Galaxy Formation Basics
- Feb 9: Disk Galaxies (I)
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- **Feb 26: Vlasov/Jeans Equations / Elliptical Galaxies (I)**
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- Mar 23: Elliptical Galaxies (III)
- Mar 30: Dark Matter Halos
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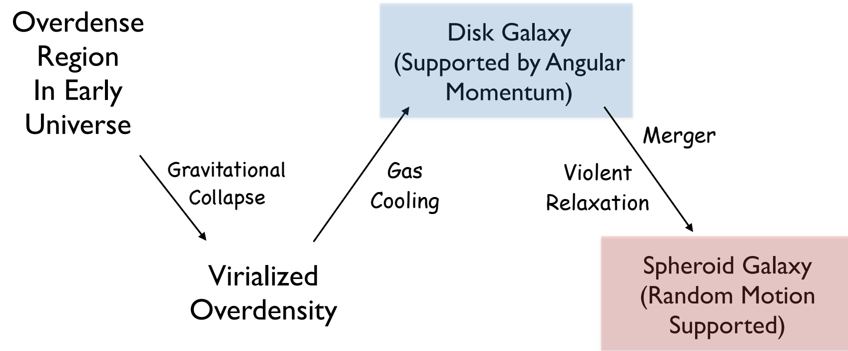
- May 11: Galaxy Evolution at z>1.5 / Review for Final Exam

No lecture or practical classes the following week!!!

First, let's review the important material from last week

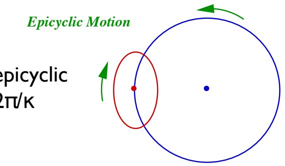
Galaxy Formation: Major Steps



Stars in Spiral Galaxies are on Epicyclic orbits

The motion can be approximately described as the combination of orbital motion around a disk galaxy and an epicyclic motion in radius:

Frequency of epicyclic motion = $2\pi/\kappa$



Orbital Frequency around Disk Galaxy = $2\pi/\Omega$

The orbital frequency of a star $\Omega(R)$ can be written as follows

$$\Omega^2(R) = \frac{1}{R}\left(\frac{\partial\Phi}{\partial R}\right)_{(R,0)} = \frac{L_z^2}{R^4},$$

Meanwhile, the frequency of epicyclic motion $\kappa(R)$ can be written as follows:

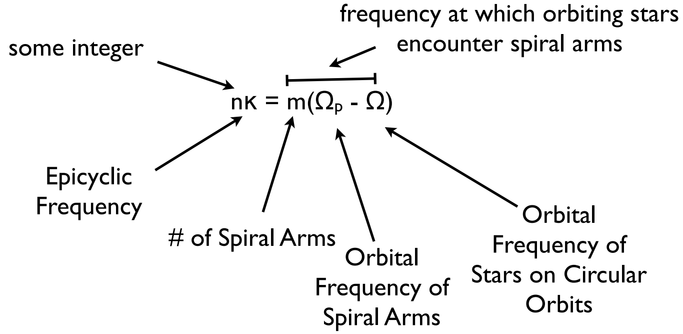
$$\kappa^2(R_g) = \left(R\frac{d\Omega^2}{dR} + 4\Omega^2\right)_{R_g}.$$

The frequency of epicycle motion is very similar to the orbital frequency:

In general, $\Omega < \kappa < 2\,\Omega$

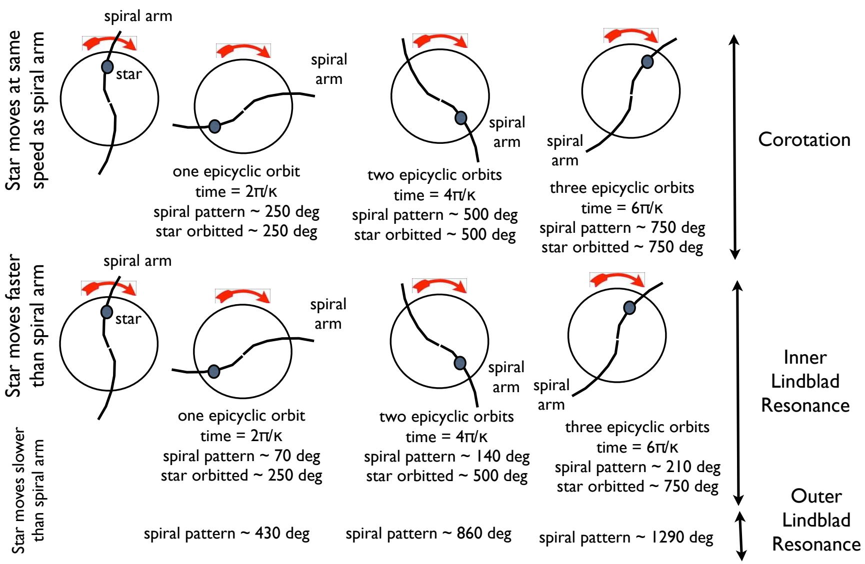
Near the solar system, the epicycle frequency $\kappa \sim 1.3\,\Omega$

Resonances can occur to reinforce structure in spiral arms of galaxies, if the epicyclic frequency of a stellar orbit is similar to the frequency at which a star orbitting around the galaxy encounters a spiral arm.



The only integers n for this relation that are interesting are 0, +1, -1.

Let's consider snapshots in time where the star completes an entire epicyclic orbit. Typically a star must complete 70% of a revolution around a galaxy before this happens.



Density Wave Theory

This results in a number of well known resonances:

Inner Lindblad resonance:	$\Omega_p = \Omega - \kappa/m$	Most relevant cases: $\Omega_p = \Omega - \kappa/2$
Outer Lindblad resonance:	$\Omega_p = \Omega + \kappa/m$	$\Omega_p = \Omega + \kappa/2$
Corotational radius:	$\Omega_p = \Omega$	$\Omega_p = \Omega$

In most cases, the only relevant case is that of two spiral arms, i.e., m = 2

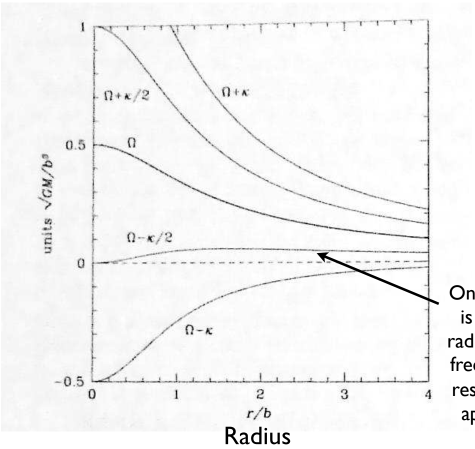
Density Wave Theory

At what rotational frequencies for the spiral arms are these resonances relevant?

Compute $\Omega_p = \Omega - \kappa/2, \Omega, \Omega + \kappa/2$

Ω_p

Orbital frequency for spiral arms at which these resonances become important



isochrone potential

One thing you should note is the extended range in radius where the rotational frequency for one of these resonances, i.e., $\Omega - \kappa/2$ is approximately constant.

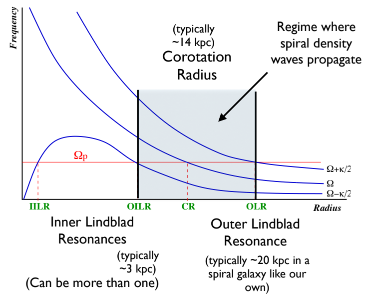
Density Wave Theory

What are the typical physical radii where these resonances apply?

highly relevant m = 2 case

Rotational frequency for spiral arms

(pattern speed for spiral arms Ω_p ~15 km/s / kpc)



What is necessary for instabilities to grow in disk galaxies?

Disks are unstable if Toomre Q parameter < 1

$$Q = \frac{\sigma_R \kappa}{3.36\, G\, \Sigma} < 1$$

Q parameter > 1 and < 2: best spiral structure

The Toomre Q parameter indicates the relative size of terms in WKB dispersion relation

$$(\omega - m\Omega)^2 = \kappa^2 + \sigma_R^2 k^2 - 2\pi G \Sigma |k| \mathcal{F}.$$

if Q large, this term large

if Q small, this term large

Do disk galaxies tend to self regulate and what is impact on Q?

If Q is small, gravitational collapse results in star formation and injects energy and Q increases

If Q is large, eventually gas cools and increases in surface density, resulting in a decrease of Q.

This self regulation drives the Q parameter towards ~1-2

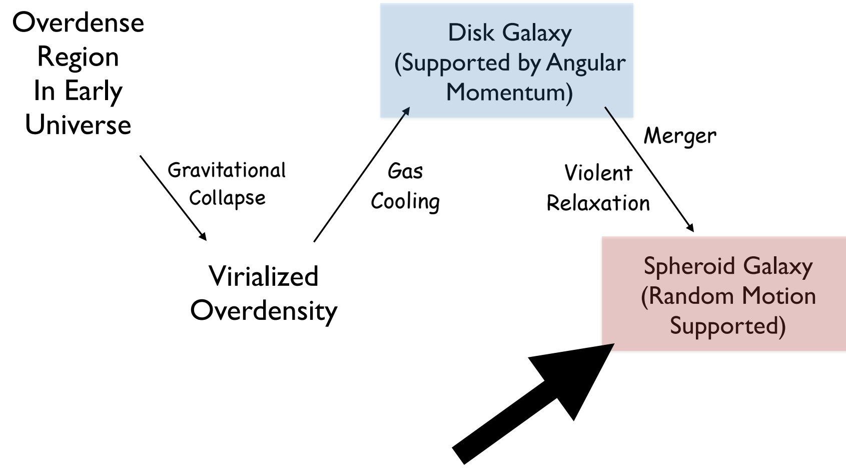
Next topic is elliptical galaxies...

Elliptical galaxies consist of large numbers of stars on diverse orbits.

While spiral galaxies are rotation supported, elliptical galaxies are supported by the random motions of stars they contain

Their behavior can largely be described using collisionless dynamics.

Galaxy Formation: Major Steps



New Material

REVIEW Point from Bachelor Course: The time scale for the relaxation time of individual stars to collisions will other stars is very high, i.e., 10^{16} years, and thus can be ignored in modeling the dynamics of stars in a galaxy. Consequently, it is possible to model the potential and phase space as smoothly varying.

First let's look at velocity perturbation created by one star passing by another.

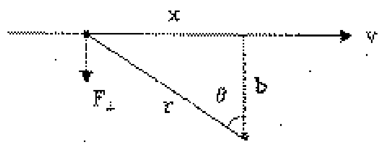


Figure 4-2. A field star approaches the test star at speed v and impact parameter b. We estimate the resulting impulse to the test star by approximating the field star's trajectory as a straight line.

BT4: pages 187-190

Relevant variables:

m = mass of "stationary" star
b = impact parameter
v = velocity of moving star
$v_\perp$ = velocity perturbation

We begin by showing that the velocity perturbation is the following:

$\Delta v_\perp = 2Gm / bv$

In this problem, we make the assumption that the impact parameter undergoes no meaningful change during the encounter.

However, this will not be true if the velocity kick is on order of the original velocity of the star.

So, we can show that our derivation breaks down if $\Delta v = Gm / v$

Which is equivalent to

$b_{min} = Gm / v^2$

But, each star is perturbed by not just one star, but many stars along the line of sight. Each perturbation is in a random direction.

While on average the perturbations cancel each other out,

$\langle v_\perp \rangle = 0$

the many perturbations introduces a spread in the overall distribution through a random walk

$\langle v_\perp^2 \rangle > 0$

In calculating the dispersion in the velocity distribution caused by perturbations by other stars, we add velocity kicks in quadrature

Let's first considering the effect from an interaction with some random star in a star and then let's consider all N stars in a galaxy.

Fraction of Stars with impact parameter b = $2\pi b db / \pi R^2$ ← Radius of galaxy

During derivation, we make use of the virial theorem

Total mass $v^2 = GM/R = (GNm)/R$

N = # of stars per galaxy

Here is the result, i.e., dispersion in velocity kicks divided by typical velocity in system is just a function of the number of particles in a dynamical system....

$\langle v_\perp^2 \rangle / v^2 = 8 (\ln N) / N$ (per crossing time)

Number of times star must cross galaxies such that the dispersion in its velocity kick equals the typical velocity gives us the relaxation time.

$n_{cross,relax} \langle v_\perp^2 \rangle = v^2$

$t_{relax} = t_{cross}\, n_{cross,relax} = t_{cross}(N / (8 \ln N))$

Example of Time Scales

System	Mass $M_\odot$	Radius kpc	Velocity $\mathrm{km\,s^{-1}}$	N	t_{cross} yr	t_{relax} yr
Galaxy	10^{10}	10	100	10^{10}	10^8	$> 10^{15}$
DM Halo	10^{12}	200	200	$> 10^{50}$	10^9	$> 10^{60}$
Cluster	10^{14}	1000	1000	10^3	10^9	$\sim 10^{10}$
Globular	10^4	0.01	2	10^4	5×10^6	5×10^8

- **Dark Matter Haloes and Galaxies are collisionless**
- **Collisions may or may not be important in clusters of galaxies**
- **Relaxation is expected to have occured in (some) globular clusters**

Credit: van den Bosch

MOTIVATION:

We have this very complicated situation:
how can we model the orbits of all the stars in a galaxy simultaneously

Seems difficult!

Before even thinking about how to solve it, how shall we even try to model it?

Ignore the fact that stars are discrete sources and assume that we treat them as a fluid with each star individually having an infinitesimal mass.

We will use a 7-dimension distribution function to describe where they are in 6-dimensional phase space at some time t.

REVIEW point from Bachelor course: The time evolution of the distribution function is defined by the distribution function at that time, spatial derivatives, and the gradients of the potential (Vlasov-Equation). This follows directly from a conservation equation on the stars.

At any time t, one can describe the collective positions and velocities for stars in a dynamical system by a distribution function f(**x**,**v**,t)

To describe the time evolution, we define a six dimensional vector
w = (**x**,**v**)

The flow of stars in the six dimensional phase can be described as d**w**/dt = (**v**,-**∇**Φ)

The flow d**w**/dt conserves stars...

At any time t, one can describe the collective positions and velocities for stars in a dynamical system by a distribution function f(**x**,**v**,t)

$$\int_S (f\dot{\vec{w}}) \cdot d^2S = - \int_V \frac{\partial f}{\partial t}$$

"stars moving in and out of volume through some surface" — "change in the total number of stars in volume"

equal to each other from divergence theorem from vector calculus

$$\int_V \vec{\nabla} \cdot (f\dot{\vec{w}}) = - \int_V \frac{\partial f}{\partial t}$$

"divergence of flow inside volume" — "change in the total number of stars in volume"

$$\frac{\partial f}{\partial t} + \vec{\nabla} \cdot (f\dot{\vec{w}}) = 0$$

At any time t, one can describe the collective positions and velocities for stars in a dynamical system by a distribution function f(**x**,**v**,t)

$$\frac{\partial f}{\partial t} + \vec{\nabla} \cdot (f\dot{\vec{w}}) = 0$$

Collisionless Boltzmann Equation "Vlasov Equation"

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0$$

or

$$\frac{\partial f}{\partial t} + \vec{v}.\vec{\nabla} f - \vec{\nabla}\Phi.\frac{\partial f}{\partial \vec{v}} = 0$$

also can write as $\frac{df}{dt} = 0$

REVIEW Point from Bachelor Course: Integrals of motion are functions of x and v which are constant along an orbit. They are not explicit functions of time. Examples: energy, angular momentum. Most 3D densities allow for 3 integrals of motion, 2 of which are non-classical.

Integrals of motion can be a very useful concept for characterizing the orbits of stars in a galaxy.

They are useful in the case that they are isolating integrals of motion since the reduce the dimensionality of the phase space in which a star travels during its orbit.

There can be no more than 6 integrals of motion. Typically there is at least one integral of motion (energy).

What are some examples of isolating integrals of motion?

- Energy is always an integral of motion for a star in a static potential.

The energy per unit mass for a star remains constant throughout its orbit: E(**x**,**v**) = (1/2) **v**^{2} + Φ(**x**)

- L_z: angular momentum in the z direction (for an axisymmetric potential)

- **L**: all three components of the angular momentum in spherically symmetric potential

Integrals of motion tend to arise from some symmetry in the system.

However, dynamical systems can also have other isolating integrals of motion outside of the classical ones (i.e., energy, angular momentum)

How can integrals of motion reduce the phase space explored by an orbit?

Consider a spherically symmetric potential:

- 1. Spherical potentials:

E, L_x, L_y, L_z are integrals of motion, but also $E, |L|$ and the direction of $\vec{L}$ (given by the unit vector $\vec{n}$, which is defined by two independent numbers). $\vec{n}$ defines the plane in which $\vec{x}$ and $\vec{v}$ must lie. Define coordinate system with z axis along $\vec{n}$

$x_3 = 0$

$v_3 = 0$

$$\vec{x} = (x_1, x_2, 0)$$

$$\vec{v} = (v_1, v_2, 0)$$

$\rightarrow$ $\vec{x}$ and $\vec{v}$ constrained to 4D region of the 6D phase space. In this 4 dimensional space, $|L|$ and E are conserved. This constrains the orbit to a 2 dimensional space. Hence the velocity is uniquely defined for a given $\vec{x}$

Energy

because of L_z conservation

$$v_r = \pm\sqrt{2(E-\Phi) - L^2/r^2}$$

$$v_\psi = \pm L/r$$

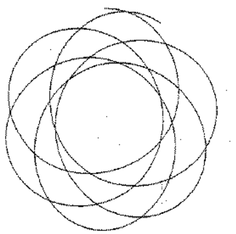


Figure 3-1. A typical orbit in a spherical potential forms a rosette.

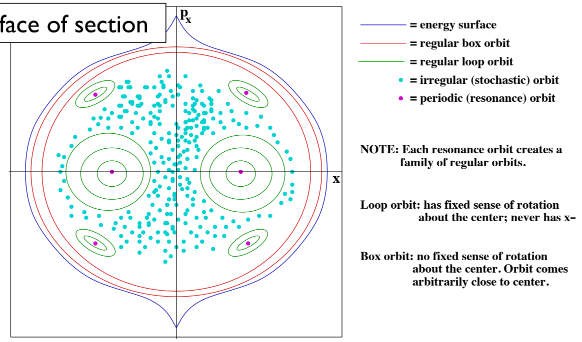
>=4 integrals of motion:
Energy
L_x
L_y
L_z

One alternate way of determining how restricted the orbital manifold of galaxies are is to construct poincare surfaces of section:

To investigate whether the orbits admit any additional (hidden) isolating integrals of motion, Poincaré introduced the surface-of-section (SOS)

Consider the intersection of $\mathcal{M}_3$ with the surface $y = 0$. Integrate the orbit, and everytime it crosses the surface $y = 0$ with $\dot{y} > 0$, record the position in the (x, p_x)-plane. After many orbital periods. the accumulated points begin to show some topology that allows one to discriminate between regular, irregular and resonance orbits.

Example surface of section



Credit: van den Bosch

Setting up equilibrium models for a collisionless system.

It is not necessarily an easy thing to do

We must set up a self-consistent system whereby each of the following steps imply the next:

(1) given density distribution ρ(r), calculate the potential Φ(r) the density distribution would imply

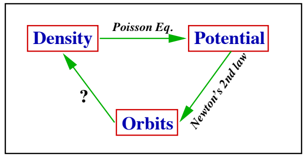
(2) given some potential Φ, determine the set of orbits that stars would undergo

(3) calculate the density distribution that would result from the collective orbits of all the stars in a system

The Density Distribution derived in step #3 must be the same as assumed in step #1

The relevant equations are:

$$\rho(\vec{x}) = \int f(\vec{x}, \vec{v})\, d^3\vec{v}$$
$$\nabla^2\Phi(\vec{x}) = 4\pi G \rho(\vec{x})$$
$$\frac{df}{dt} = 0$$

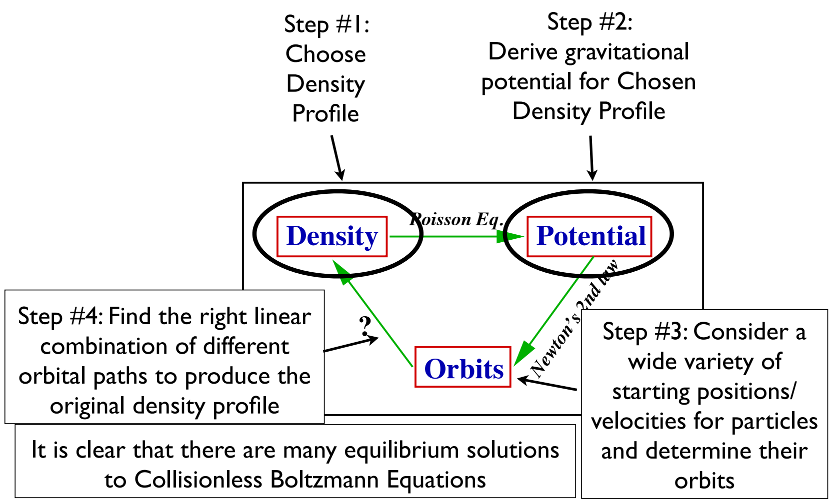


REVIEW Point from Bachelor Studies. Another way of constructing equilibrium models is the Schwardschild method. It involves making a library of orbits, calculate their spatial densities, and calculate the weight function which reproduces the density distribution for which the orbits were calculated.

What is the Schwarzschild method for constructing equilibrium models?

1. Define density
2. Derive gravitational potential
3. Derive orbit families for this gravitational potential
   Determine the densities that result from different orbits
4. Combine the different orbital families in such a way to produce the defined density.

We can illustrate the Schwarzschild Method using the following diagram illustrating the basic problem we must solve in setting up an a stable equilibrium system in stellar dynamics:



It is clear that there are many equilibrium solutions to Collisionless Boltzmann Equations

In considering different starting positions in phase space (spatial position and velocities), can get a wide variety of different orbits, e.g.,

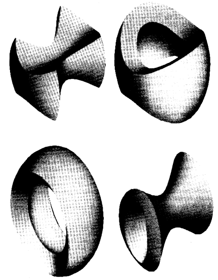


Figure 3-20. Orbits in a nonrotating triaxial potential. Clockwise from top left: (a) box orbit; (b) short-axis tube orbit; (c) inner long-axis tube orbit; (d) outer long-axis tube orbit. [Courtesy of T. Statler; see Statler (1986).]

BT 3.4: page 155

It is clear that one would need to include a contribution from particles on a wide variety of different orbits to construct a density profile with no significant holes!

Using the Schwarzschild method, these are analogous to basis functions in linear algebra

REVIEW (from Bachelor course)

JEANS EQUATIONS

Jeans Equations build on the material just presented and provide us with another means to estimate the masses and mass profiles in galaxies...

The Jeans equations are derived by taking various velocity moments of the Collision Boltzmann Equation.

This is a useful approach because velocity moments of the stars are straightforward to derive from the observations.

REVIEW point from Bachelor course. An important method for deriving the total mass of individual galaxy involves the Jeans equations and moments of the stellar velocity.

First define several velocity moments of the distribution function:

0. Spatial density of stars (0th moment):

$$\nu(\vec{x}) = \int f(\vec{x},\vec{v})\,\mathrm{d}^3\vec{v}$$

1. Mean velocity of stars (1st moment):

$$\overline{v_i}\,(\vec{x}) \equiv \frac{1}{\nu}\int v_i f(\vec{x},\vec{v})\,\mathrm{d}^3\vec{v}, \quad i=1,2,3$$

2. Second moment of stars (2nd moment):

$$\overline{v_i v_j}\,(\vec{x}) \equiv \frac{1}{\nu}\int v_i v_j f(\vec{x},\vec{v})\,\mathrm{d}^3\vec{v}, \quad j=1,2,3$$

Velocity Dispersion Tensor:

$$\sigma_{ij}^2 \equiv \overline{(v_i - \overline{v_i}\,)(v_j - \overline{v_j}\,)} = \overline{v_i v_j} - \overline{v_i}\;\overline{v_j}$$

Jeans Equation 1 (Continuity equation):

$$\frac{\partial \nu}{\partial t} + \sum_{i=1}^3 \frac{\partial}{\partial x_i}\nu\,\overline{v_i} = 0$$

Jeans Equation 2 (The Force Equation):

$$\frac{\partial(\nu\,\overline{v_j}\,)}{\partial t} + \sum_{i=1}^3 \frac{\partial}{\partial x_i}(\nu\,\overline{v_i v_j}\,) + \nu\frac{\partial \Phi}{\partial x_j} = 0$$

Jeans Equation 3 (Rewrite of Equation 2)

$$\nu\frac{\partial\,\overline{v_j}}{\partial t} + \sum_{i=1}^3 \nu\,\overline{v_i}\,\frac{\partial\,\overline{v_j}}{\partial x_i} = -\nu\frac{\partial \Phi}{\partial x_j} - \sum_{i=1}^3 \frac{\partial}{\partial x_i}\nu\sigma_{ij}^2$$

Let us start out by deriving the first Jeans Equation from the collisionless Boltzmann Equation:

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0$$

Derive ↓

$$\frac{\partial \nu}{\partial t} + \sum_{i=1}^3 \frac{\partial}{\partial x_i}\nu\,\overline{v_i} = 0$$

This is the continuity equation and is anologous to the following expression from fluid mechanics:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

Let us start out by deriving the second Jeans Equation from the collisionless Boltzmann Equation:

$$\frac{\partial f}{\partial t} + \sum_{i=1}^{3} v_i \frac{\partial f}{\partial x_i} - \frac{\partial \Phi}{\partial x_i}\frac{\partial f}{\partial v_i} = 0$$

Derive

$$\frac{\partial(\nu\,\overline{v_j}\,)}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\nu\,\overline{v_i v_j}\,) + \nu\frac{\partial \Phi}{\partial x_j} = 0$$

Finally let us derive the third Jeans Equation from the first two Jeans Equations:

$$\frac{\partial \nu}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\nu\,\overline{v_i} = 0 \qquad \frac{\partial(\nu\,\overline{v_j}\,)}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\nu\,\overline{v_i v_j}\,) + \nu\frac{\partial \Phi}{\partial x_j} = 0$$

Derive

$$\nu\frac{\partial\,\overline{v_j}}{\partial t} + \sum_{i=1}^{3} \nu\,\overline{v_i}\,\frac{\partial\,\overline{v_j}}{\partial x_i} = -\nu\frac{\partial \Phi}{\partial x_j} - \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\nu\sigma_{ij}^2$$

This is the "force" equation and is analogous to the Euler equation from fluid mechanics:

$$\frac{\rho\partial\vec{v}}{\partial t} + \rho(\vec{v}\cdot\vec{\nabla})\vec{v} = -\rho\vec{\nabla}\Phi - \vec{\nabla}p = 0$$

$$\frac{\partial \nu}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\nu\,\overline{v_i} = 0$$

$$\frac{\partial(\nu\,\overline{v_j}\,)}{\partial t} + \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(\nu\,\overline{v_i v_j}\,) + \nu\frac{\partial \Phi}{\partial x_j} = 0$$

$$\nu\frac{\partial\,\overline{v_j}}{\partial t} + \sum_{i=1}^{3} \nu\,\overline{v_i}\,\frac{\partial\,\overline{v_j}}{\partial x_i} = -\nu\frac{\partial \Phi}{\partial x_j} - \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\nu\sigma_{ij}^2$$

- These are the Jeans Equations. There are three equations and six unknowns. Possible to find many solutions to these equations.

- For a stationary problem (where the time derivatives are zero and net streaming motion is zero, i.e., $\overline{v_i}$ = 0), the left terms disappear completely. We then have the stellar velocity dispersion counteracting the force of gravity, in the same way that gas pressure counteracts the force of gravity in a star. Note however that there is no equation of state.

- Velocity dispersion tensor σ_{ij}^2 is symmetric and so an orthogonal coordinate system can be found where it is diagonal, i.e., σ_{ij}^2 = 0 if i does not equal j.

Jeans Equations for Spherically Symmetric Models

Adopt a spherical coordinate system (r, θ, ϕ). Assume invariant under rotations about center and there are no streaming motions. Hence,

$$\overline{v_r} = \overline{v_\theta} = \overline{v_\phi} = 0$$

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$$\overline{v_\theta^2} = \overline{v_\phi^2}$$

The second Jeans equation (in spherical coordinates) reduces to the following:

$$\frac{d(\nu\,\overline{v_r^2}\,)}{dr} + \frac{\nu}{r}\left[2\,\overline{v_r^2} - 2\,\overline{v_\theta^2}\right] = -\nu\frac{d\Phi}{dr}$$

Define anisotropy function

$$\beta(r) = 1 - \overline{v_\theta^2}\,/\,\overline{v_r^2}\,.$$

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Clearly, β <= 1. Manipulating the previous expression, we find

$$\frac{1}{\nu}\frac{d}{dr}\nu\,\overline{v_r^2} + 2\frac{\beta}{r}\overline{v_r^2} = -\frac{d\Phi}{dr}$$

Total Enclosed Mass and Rotation Curve

Assuming a particle is on a circular orbit at radius r, there is a relationship between the circular velocity v_c and $\frac{d\Phi}{dr}$

$$\frac{d\Phi}{dr} = \frac{GM(<r)}{r^2} = \frac{v_c^2}{r}$$

Using this relation, the Jeans Equation can be written as

$$v_c^2 = \frac{GM(<r)}{r} = -\overline{v_r^2}\left(\frac{d\ln\nu}{d\ln r} + \frac{d\ln\overline{v_r^2}}{d\ln r} + 2\beta\right)$$

From this expression, we see that if we can measure the density of stars ν, the velocity dispersion in the radial direction, and anisotropy function, we can determine the enclosed mass inside some radius

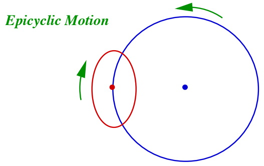
What other applications do the Jeans Equations have?

A few examples:

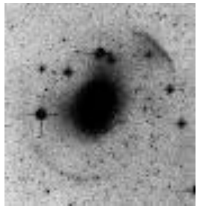
1. Estimating the surface mass density of our own Milky Way Galaxy based on the velocities of stars perpendicular to the plane.

2. Understanding quantitatively why stars in spiral galaxies have lower tangential velocities than gas (i.e., why their tangential velocities are less than the circular velocity)

3. Understanding quantitatively the small epicyclical motion that stars rotating around a spiral galaxy experience.



What is the nature of elliptical galaxies?



End state of galaxy formation!

Dominated by the random motions of its component parts (stars, hot gas, dark matter)

While progenitors to elliptical galaxies experienced lots of star formation, elliptical galaxies themselves experience almost no star formation

Only way for an elliptical galaxy to transform into another type of galaxy (e.g., spiral) is if lots of cold gas cools onto it (but this may not happen!)