Relevant Material from Bachelor Course where many of the key concepts are explained.

"Galaxies: Structure, Dynamics, and Evolution"

Key Point #3 from Bachelor Course: Constructing Galaxy Potentials

$$W = 1/2 \int \rho(\vec{x})\Phi(\vec{x})d\vec{x}$$

We derive this as follows. Assume that we "build" up the galaxy slowly. We have a galaxy with a density $f\rho$, with $0 < f < 1$. We add a tiny bit of density $\delta f \rho$, taking the mass from infinity to the galaxy. Ignoring the change in the potential, this costs an energy

$$\int \delta f \rho(\vec{x}) \; f\Phi(\vec{x})d\vec{x}$$

where $f\Phi$ is simply the potential of density $f\rho$, and the integral is the integral over the full galaxy volume.
We now have to add all the contributions together to derive the full energy needed to "build" the full galaxy

$$W = \int_0^1 \int \rho(\vec{x}) \; f\Phi(\vec{x})d\vec{x} \; df$$

$$= \int \rho(\vec{x})\Phi(\vec{x})d\vec{x} \int_0^1 f df$$

$$= 1/2 \int \rho(\vec{x})\Phi(\vec{x})d\vec{x}$$

3.1 Potential for spherical systems (BT 2.1, 2.2)

Newton's Theorems:

- First Theorem:
  A body inside an infinitesimally thin spherical shell of matter experiences no net gravitational force from that shell

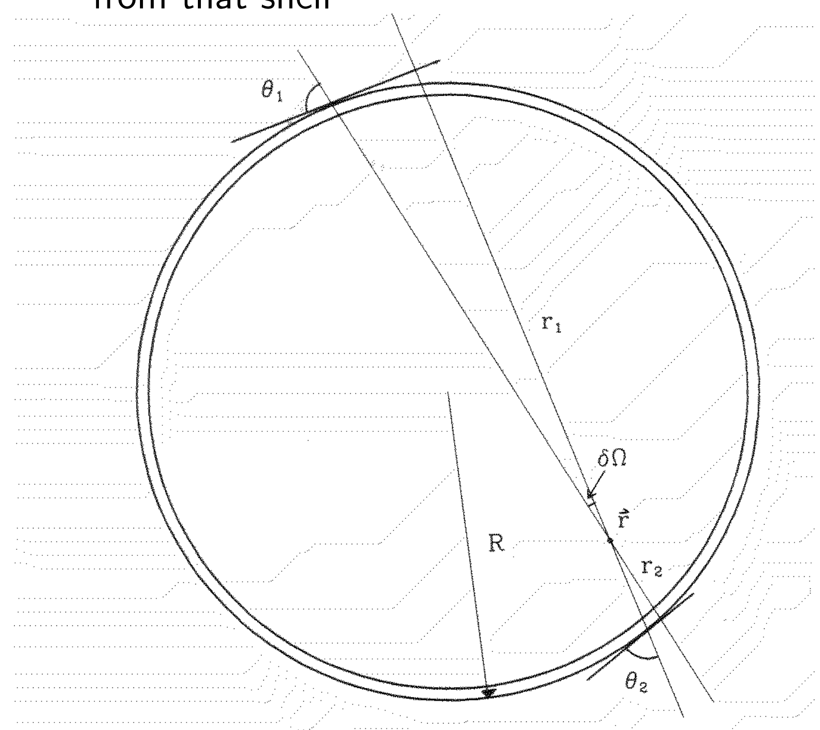


Figure 2-1. Proof of Newton's first theorem.

Consider contributions to the force at point $\vec{r}$, due to the matter in the shell in a very narrow cone $d\Omega$. The intersection angles at 1 and 2, Θ_1 and Θ_2, are equal for infinitely small $d\Omega$. The relative masses in the cone δm_1 and δm_2 satisfy $\delta m_1/\delta m_2 = (r_1/r_2)^2$. The gravitational forces are proportional to $\delta m_1/r_1^2$ and $\delta m_2/r_2^2$, and therefore equal, but of opposite sign. Hence the matter in the cone does not contribute any net force at the location $\vec{r}$. If we sum over all cones, we find no net force !

- Newton's Second Theorem:
  The gravitational force on a body outside a closed spherical shell of matter is the same as it would be if all the shell's matter were concentrated into a point at its center.

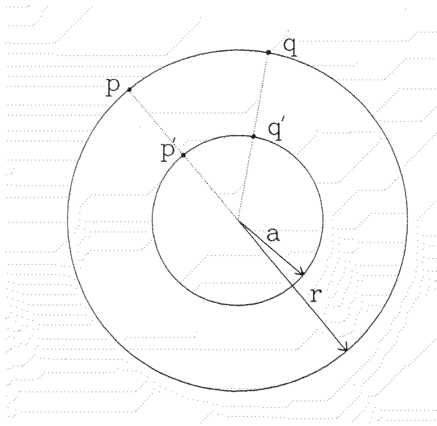


Figure 2-2. Proof of Newton's second theorem.

Calculate the potential at point $\vec{p}$ at radius r from the center of an infinitesimally thin shell with mass M and radius a. Consider the contribution from the portion of the sphere with solid angle $\delta\Omega$ at q':

$$\delta\Phi_1 = -\frac{GM}{|\vec{p} - \vec{q}'|}\frac{\delta\Omega}{4\pi}$$

Now take an infinitesimally thin shell with the same mass M, but radius r. Scale $\vec{p}$ down to $\vec{p}'$, so that it lies at a radius a inside the shell. Scale $\vec{q}'$ up, so that it lies on the shell. Calculate the potential at $\vec{p}'$. The contribution of the matter near $\vec{q}$

with the same solid angle $\delta\Omega$ is:

$$\delta\Phi_2 = -\frac{GM}{|\vec{p}' - \vec{q}|}\frac{\delta\Omega}{4\pi}$$

Since $|\vec{p} - \vec{q}'| = |\vec{p}' - \vec{q}|$, $\delta\Phi_1 = \delta\Phi_2$. Sum over all solid angles to obtain

$$\Phi_1 = \Phi_2$$

Since Φ_2 is the potential inside a sphere with mass M and radius r, it is equal to $\Phi_2 = -GM/r$, and this is equal to Φ_1. This is the same as the potential at r if all the mass is concentrated at the center.

We can now calculate potential of spherical system with density $\rho(r)$. Divide system up into shells, and add contribution from each shell. Distinguish between shells with radius r' , $r' < r$ and shells with $r' > r$:
$r' < r$: $\delta\Phi = -G\delta M/r$,
$r' > r$: $\delta\Phi = -G\delta M/r'$.

Hence total potential:

$$\Phi = -4\pi G\left[\frac{1}{r}\int_0^r \rho(r')r'^2 dr' + \int_r^\infty \rho(r')r' dr'\right].$$

Hence only single integration ! The force on the unit mass at radius r is determined by mass interior to r:

$$\vec{F}(r) = -\frac{d\Phi}{dr}\vec{e}_r = -\frac{GM(r)}{r^2}\vec{e}_r,$$

where

$$M(r) = 4\pi \int_0^r \rho(r')r'^2 dr'.$$

The circular speed $v_c(r)$ is defined as the speed of a test particle with unit mass in a circular orbit around the center, with radius r. We derive

$$v_c^2(r) = r\frac{d\Phi}{dr} = rF = \frac{GM(r)}{r}.$$

THE CIRCULAR SPEED MEASURES THE MASS INSIDE r !
And is independent of the mass outside r.
The escape speed v_e is the speed needed to escape from the system, for a star at radius r. It is given by

$$v_e(r) = \sqrt{2|\Phi(r)|}$$

Only if a star has a speed greater than that, it can escape. It is dependent on the full mass distribution.

3.2 Simple potentials

- Pointmass

$$\Phi(r) = -\frac{GM}{r}, \quad v_c(r) = \sqrt{\frac{GM}{r}}, \quad v_e(r) = \sqrt{\frac{2GM}{r}}$$

If the circular speed declines like $\frac{1}{\sqrt{r}}$ we call it "Keplerian".

- Homogeneous Sphere

density is constant ρ within radius a, outside it is 0.
For $r < a$:

$$M(r) = \tfrac{4}{3}\pi r^3 \rho, \quad v_c = r\sqrt{\tfrac{4}{3}\pi G\rho}$$

The circular velocity is proportional to the radius of the orbit. Hence the orbital period is:

$$T = \frac{2\pi r}{v_c} = \sqrt{\frac{3\pi}{G\rho}}$$

independent of radius !

release a test mass from rest at position r. Equation of motion:

Key Point #4 from Bachelor Course: Virial Theorem

3.3 Virial Theorem: (not in BT)

relation for global properties: Kinetic Energy and Potential energy.

Again consider our system of point masses m_i with positions $\vec{x}_i$.
Construct $\sum_i \vec{p}_i \vec{x}_i$ and differentiate w.r.t. time:

$$\frac{d}{dt}\sum_i \vec{p}_i\vec{x}_i = \frac{d}{dt}\sum_i m_i \frac{d\vec{x}_i}{dt}\vec{x}_i = \frac{d}{dt}\sum_i \frac{1}{2}\frac{d}{dt}(m_i x_i^2)$$

$$= \frac{1}{2}\frac{d^2 I}{dt^2}$$

where $I = \sum_i m_i x_i^2$, which is the moment of inertia.

However, we can also write:

$$\frac{d}{dt}\sum_i \vec{p}_i\vec{x}_i = \sum_i \frac{d\vec{p}_i}{dt}\vec{x}_i + \sum_i \vec{p}_i \frac{d\vec{x}_i}{dt}$$

Then

$$\sum_i \vec{p}_i \frac{d\vec{x}_i}{dt} = \sum_i m_i \vec{v}_i^2 = 2K$$

with K the kinetic energy.

Since $\frac{d\vec{p}_i}{dt} = \vec{F}_i$ we have

$$\frac{1}{2}\frac{d^2 I}{dt^2} = \sum_i \vec{F}_i\vec{x}_i + 2K.$$

Now assume the galaxy is 'quasi-static', i.e. its properties change only slowly so that $\frac{d^2 I}{dt^2} = 0$. Then the equation above implies

$$K = -\frac{1}{2}\sum_i \vec{F}_i\vec{x}_i$$

.

Then assume that the force $\vec{F}_i$ can be written as a summation over 'pairwise forces' $\vec{F}_{ij}$

$$\vec{F}_i = \sum_{j,j\neq i} \vec{F}_{ij}$$

Now realize that

$$\sum_i \vec{F}_i\vec{x}_i = \sum_i \sum_{j\neq i} \vec{F}_{ij}\vec{x}_i$$

This summation can be rewritten. We sum the terms over the full area $0 < i \leq N$, $0 < j \leq N$, $i \neq j$. However, we can also limit the summation over just half this area: $0 < i \leq N$, $i < j \leq N$, and add the (j,i) term explicitly to the (i,j) term within the summation.
Hence instead of summing over $\vec{F}_{ij}\vec{x}_i$, we sum over $\vec{F}_{ij}\vec{x}_i + \vec{F}_{ji}\vec{x}_j$

Hence

$$\sum_i \sum_{j \neq i} \vec{F}_{ij} \vec{x}_i = \sum_i \sum_{j>i} (\vec{F}_{ij} \vec{x}_i + \vec{F}_{ji} \vec{x}_j)$$

This is simply a change in how the summation is done, it does not use any special property of the force field.

Because $\vec{F}_{ij} = -\vec{F}_{ji}$ (forces are equal and opposite for pairwise forces) the last term can be rewritten

$$\sum_i \vec{F}_i \vec{x}_i = \sum_i \sum_{j>i} \vec{F}_{ij} (\vec{x}_i - \vec{x}_j)$$

For gravitational force $\vec{F}_{ij} = -\frac{G m_i m_j}{|\vec{x}_i - \vec{x}_j|^2} \frac{\vec{x}_i - \vec{x}_j}{|\vec{x}_i - \vec{x}_j|}$

$$\sum_i \vec{F}_i \vec{x}_i = -\sum_i \sum_{j>i} \frac{G m_i m_j}{|\vec{x}_i - \vec{x}_j|^3} (\vec{x}_i - \vec{x}_j)(\vec{x}_i - \vec{x}_j)$$

which equals

$$= -\sum_i \sum_{j>i} \frac{G m_i m_j}{|\vec{x}_i - \vec{x}_j|} = -\frac{1}{2} \sum_i \sum_{j \neq i} \frac{G m_i m_j}{|\vec{x}_i - \vec{x}_j|} = W$$

with W the total potential energy. Therefore for a galaxy in quasi-static equilibrium:

$$K = -\frac{1}{2} W,$$

which is the virial theorem for quasi-static systems. The more general expression for non-static systems is:

$$\frac{1}{2} \frac{d^2 I}{dt^2} = W + 2K.$$

3.4 Applications (BT pages 213, 214)

Consider a system with total mass M
Kinetic energy $K = \frac{1}{2}M\langle v^2\rangle$ with
$\langle v^2\rangle$ = mean square speed of stars (assumption: speed of star not correlated with mass of star)

Define gravitational radius r_g

$$W = -\frac{GM^2}{r_g}$$

Spitzer found for many systems that $r_g = 2.5r_h$, where r_h is the radius which contains half the mass
Virial theorom implies:

$$M\langle v^2\rangle = \frac{GM^2}{r_g}$$

$$M = \langle v^2\rangle r_g G^{-1}$$

Hence, we can estimate the mass of galaxies if we know the typical velocities in the galaxy, and its size!

Key Point #5+6: Measured Masses in Galaxies Require Presence of Dark Matter at Large Radii

7.2 Mass distribution and dark matter in spiral galaxies

Dark Matter Halos in Spiral Galaxies BT 10.1-6

measure rotation curve of cold gas
- H alpha (optical)
- CO (mm arrays)
- H I (vla, westerbork)

assume circular orbits:

$$\frac{v_c^2(r)}{r} = \frac{GM(<r)}{r^2}$$

The enclosed mass is directly given by

$$M(<r) = r v_c^2(r)/G$$

Isothermal sphere: $v_c = constant$, $\rho \propto r^{-2}$

Point Mass: $v_c \propto 1/\sqrt{r}$

Measurement in practice:

- Take edge-on system, measure maximum velocity of gas at each point
  possible problems: extinction, confusion
- Take inclined galaxies, measure velocities everywhere
  - Model by fitting a circular velocity field with unknown inclination and rotation curve

Historically, optical rotation curves like these on the following figure indicated for the first time the presence of dark matter.

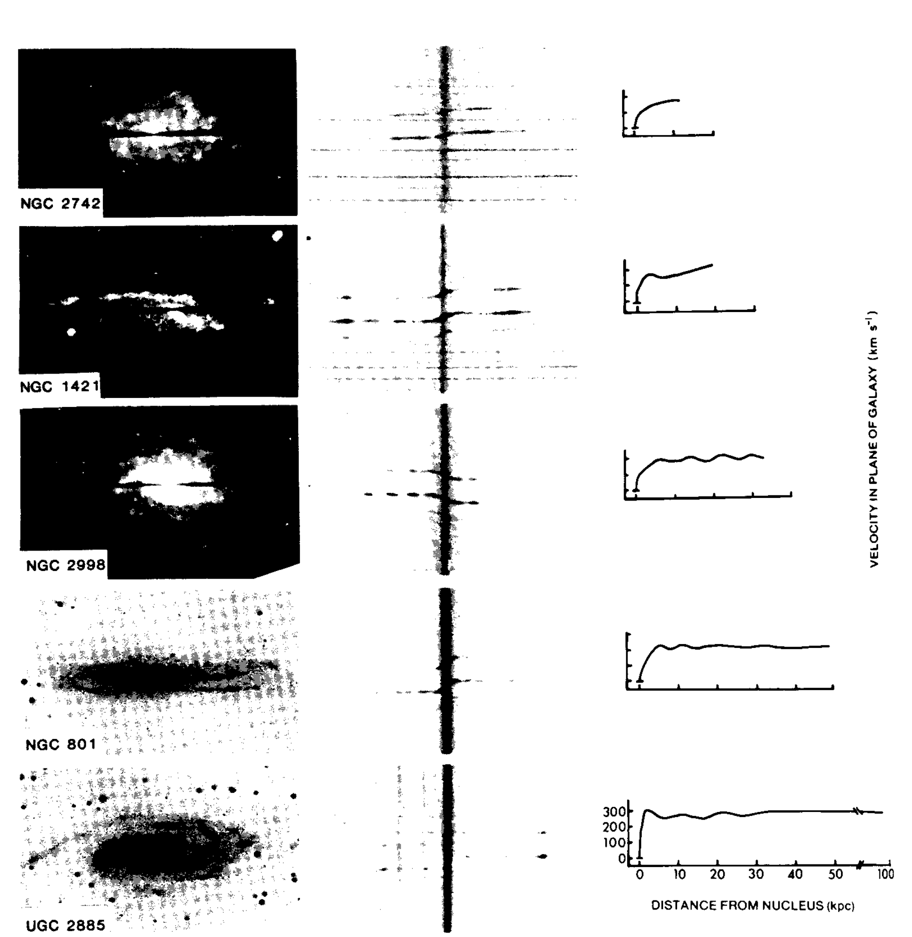


Figure 10-1. Photographs, spectra, and rotation curves for five Sc galaxies, arranged in order of increasing luminosity from top to bottom. The top three images are television pictures, in which the spectrograph slit appears as a dark line crossing the center of the galaxy. The vertical line in each spectrum is continuum emission from the nucleus. The distance scales are based on a Hubble constant $h = 0.5$. Reproduced from Rubin (1983), by permission of *Science*.

Very modern data are based on 21cm HI emission lines. The HI disks can often be followed to very large radii, and hence the rotation curve can be followed to well model looks like:

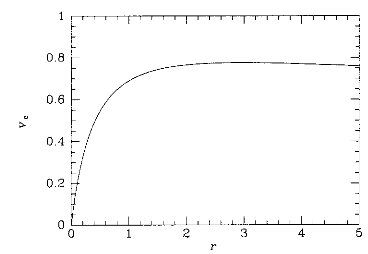


Figure 8.31 A typical galactic circular-speed curve.

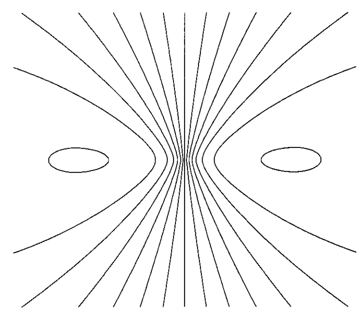


Figure 8.32 The spider diagram generated by the circular-speed curve of Figure 8.31 when the system is viewed at inclination $i = 30°$ with the apparent major axis horizontal. The area contoured is a square 10 distance units on a side.

We can now model the galaxy. Take the surface brightness profile, and calculate the rotation curve if the mass-to-light ratio were constant:

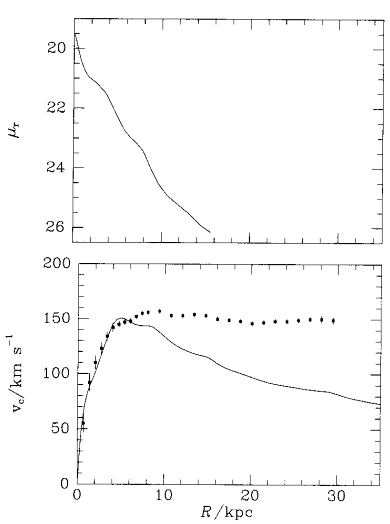


Figure 8.35 The 21-cm circular-speed curve of the Sc galaxy NGC 3198 implies that most of the galaxy's mass lies beyond R_{25}. The upper panel shows the r-band surface brightness profile from Kent's CCD photometry. The curve in the lower panel shows the circular-speed curve derived from this and the observed HI mass under the assumption that $\Upsilon_r = 3.8\Upsilon_r(\odot)$. The dots in the lower panel show the circular-speed curve derived from the 21-cm velocity field. [After Begeman (1987) using data kindly provided by K. Begeman]

Obviously, an additional mass component is necessary to explain the rotation curve.

Fit rotation curves:

Constant M/L for starlight
add halo, with $\rho = \rho_0/(1+(r/a)^2)$

Example for NGC 3198

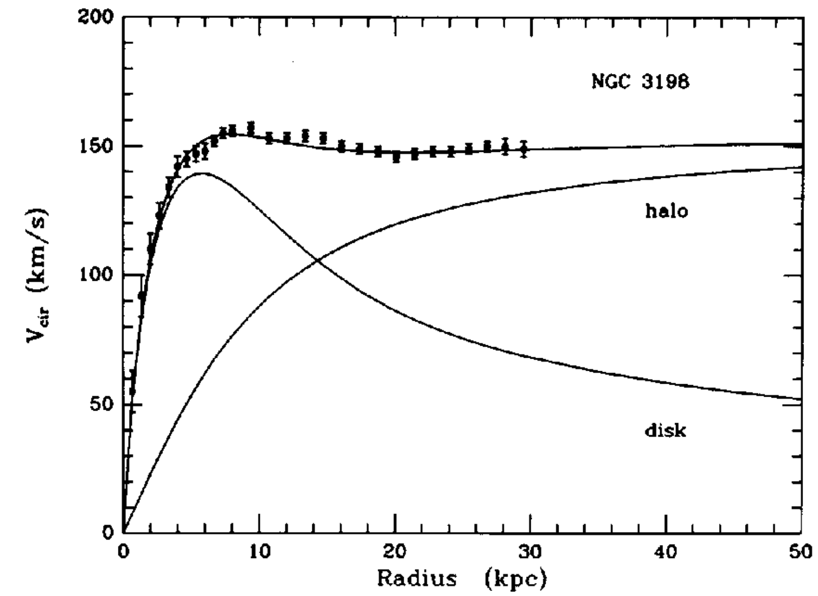


Figure 10-2. The Sc galaxy NGC 3198. Top: neutral hydrogen column density contours superimposed on an optical photograph. Bottom: circular-speed curve plus model fits using an exponential disk with constant mass-to-light ratio and the halo density profile (10-10). The model curve is for the maximum possible disk mass-to-light ratio. The horizontal scale assumes $h = 0.75$. Reprinted from van Albada et al. (1985), by permission of *The Astrophysical Journal.*

Problems: The fit is never unique. Different M/L's for the disk, and different values of a, ρ_0 for the

halo will give fits which are all good.
Hence: it is very hard to determine how much of the mass is due to the halo, within the outer most radius

Solution: determine the "minimum halo mass", by calculating "maximum disk". But this is only the minimum !

Various authors have assembled large samples of galaxies with rotation curves. An example: Casertano and van Gorkom (1991, AJ 101, 1231)

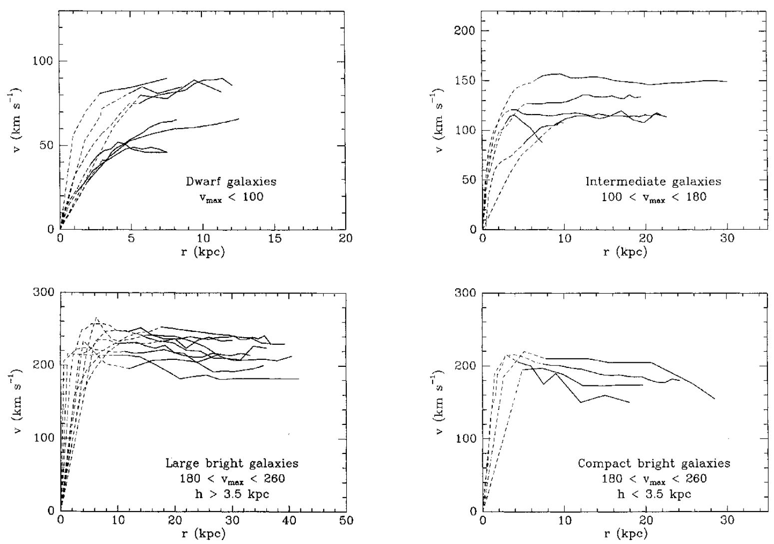


FIG. 5.Comparison of the 28 extended rotation curves shown in Fig. 4. Each panel refers to one of the four galaxy groups described in the text: dwarf, intermediate, bright and bright compact galaxies. The rotation curves are drawn with thick solid lines beyond an inner fiducial radius corresponding to two-thirds of the optical radius. A systematic change in shape is apparent: rotation curves of dwarf galaxies rise, often out to the last measured point; those of intermediate galaxies are generally flat, with some exceptions (rising for NGC 247 and declining for NGC 7793); those of bright galaxies fall, weakly for large ones, more steeply for compact galaxies.

Resulting rotation curves:

- Rapid rise near center
- nearly flat out to most distant point, with some variations:
  dwarf galaxies: still rising
  intermediate galaxies: flat

  large bright galaxies - slightly falling, compact galaxies, falling slightly more

- Never Keplerian - always a lot of dark matter !

Homework Assignment:
1) Calculate the total mass, the mass in the disk, and the mass in the halo for NGC 3198 at a radius of 30kpc. Use the decomposition into components shown in the figure above.

7.5 Clusters of galaxies BT 10.2-4

measure redshifts of cluster galaxies

velocity dispersions of 1000 km/s or higher are measured !

10.2 Dark Matter in Systems of Galaxies 615

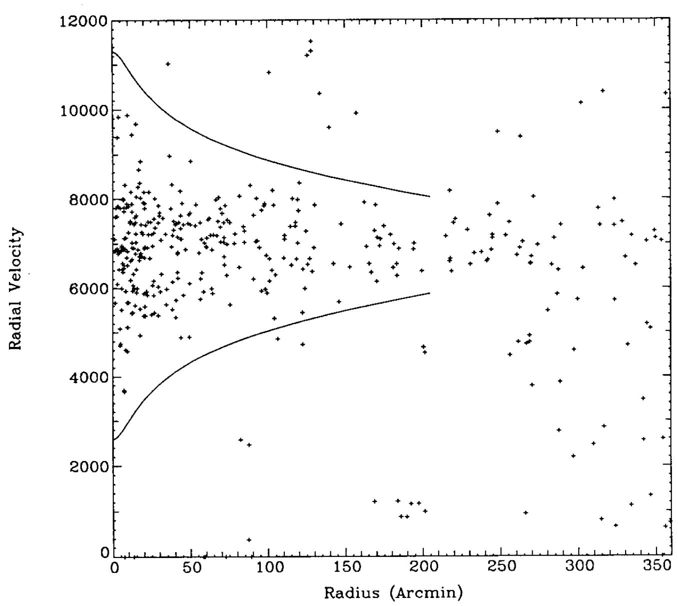


Figure 10-4. Line-of-sight velocities of galaxies in the Coma cluster (in $\mathrm{km\,s^{-1}}$) as a function of distance from the cluster center in minutes of arc (Kent & Gunn 1982). The curves mark the authors' estimate of the boundary between cluster members and interlopers. At the distance of Coma 1 arcmin = $20h^{-1}$ kpc. Reprinted by permission from *The Astronomical Journal.*

Kent and Gunn 1982

1982AJ.....87..945K

965 KENT AND GUNN: COMA CLUSTER 965

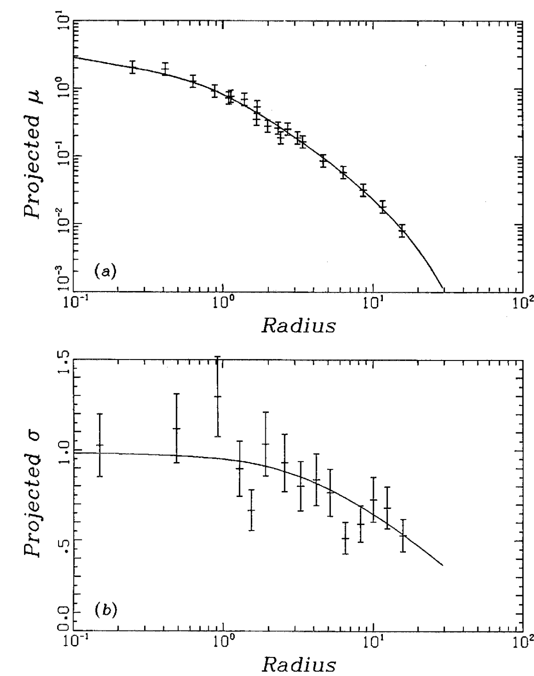


FIG. 6. Comparison of observed surface-density and velocity-dispersion profiles with predictions for a King-Michie model ($W_0 = 8$, $M_i = 0.15$, $r_t = \infty$). Radius is in units of $r_s = 8.5'$.

Careful model shows: total mass is $2 \times 10^{15} M_\odot$ (H0 = 70 km/sec/Mpc).

Total amount of light: 279 galaxies brighter than L_*, $L_* =$ 0.7 $10^{10} L_\odot$. Total amount of light is $4.33 \times L_* \times N =$ 0.8 $10^{13} L_\odot$

Hence mass to light ratio $= M/L = 210^{15}/0.810^{13} = 250\ M_\odot/L_\odot$.

Similar study (Kent and Sargent) Perseus: $M/L_V =$ 420 $M/L_\odot$

Typical masses of clusters: $10^{14} - 10^{15}\ M_\odot$

Normal galaxy: $M_{LV} < 10 M/L_\odot$, hence a lot of dark matter.

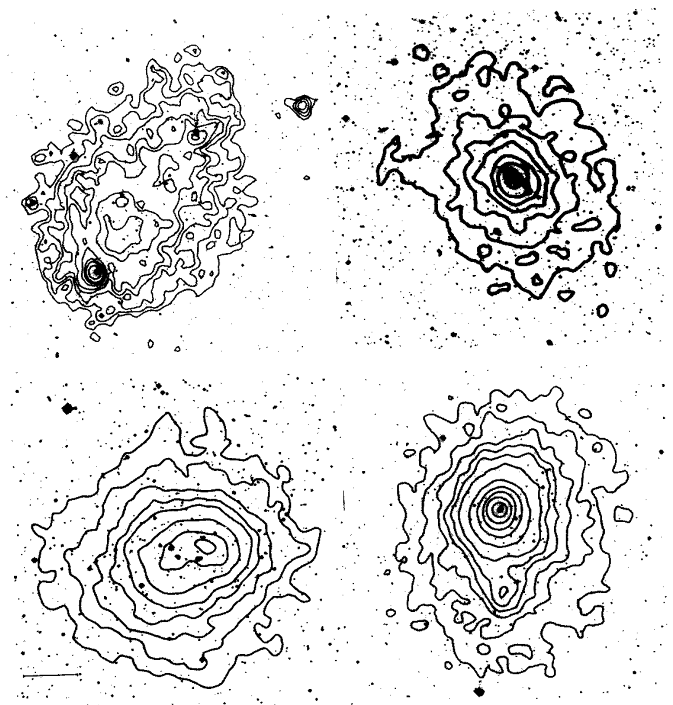


Figure 10-5. X-ray surface-brightness contours superimposed on photographs of several clusters of galaxies. Clockwise from top left, the clusters are A1367, A262, A85, and A2256 (see Jones & Forman 1984).

Similar mass determinations from gas . Temperatures can now be measured easily with X-ray satellites. Results:

- Confirms high masses

Mass budget:
- Gas mass fraction 20-30 % of total mass
- Star mass fraction < 10 % of mass
- Remaining: dark matter 60-70 % !

7.6 Dark matter in universe as a whole BT 10.3-1

The emissivity of light in the universe in the V band is

$$j_0 = 1.7 \pm 0.6 10^8 h L_\odot Mpc^{-3}$$

The critical density for the universe is:

$$\rho_c = \frac{3H_0^2}{8\pi G}$$

This density is enough to stop the expansion at $t =$ infinity

Define the density parameter $\Omega_0 = \rho_0/\rho_c$, where ρ_0 is the actual density of the universe

Express Ω_0 in terms of M/L ratio of galaxies:

$$\Omega_0 = 6.1 10^{-4} h^{-1} M/L/(M/L)_\odot$$

The critical mass-to-light ratio for $\Omega = 1$ is given by

$$M/L_c = 1600h(M/L)_\odot$$

Clusters imply $M/L_c = 300 - 600h(M/L)_\odot$, hence more realistic values are

$$\Omega_0 = 0.2 - 0.4$$

from clusters.

other indicators give:

Table 10-2. Estimates of the density parameter

Method	$\Upsilon_V/\Upsilon_\odot$	Ω_0
Solar neighborhood	5	$0.003h^{-1}$
Elliptical galaxy cores	$12h$	0.007
Local escape speed	30	$0.018h^{-1}$
Satellite galaxies	30	$0.018h^{-1}$
Magellanic Stream	> 80	$> 0.05h^{-1}$
Rotation curve of NGC 3198	$> 28h$	> 0.017
X-ray halo of M87	> 750	$> 0.46h^{-1}$
Local Group timing	100	$0.06h^{-1}$
Groups of galaxies	$260h$	0.16
Clusters of galaxies	$400h$	0.25
Virgocentric flow	–	0.25
Nucleosynthesis	–	$(0.01 - 0.05)h^{-2}$
Inflation	–	1

NOTES: All lines except the last three are based on the luminosity density (10-24). Nucleosynthesis estimate omits density in non-baryonic matter. Several methods, such as Local Group timing and X-ray halo of M87, depend on h in complicated ways, and this dependence has been suppressed. See text for further detail.

Measurements on the largest scales concern those caused by inflows around clusters (BT 10.3-2)

matter falls into clusters of galaxies. Assume that the density contrast of the cluster is the same as that of the light

$$\frac{\rho_{cluster}}{\rho_{universe}} = \frac{j_{cluster}}{j_{universe}} = \delta$$

where j is emissivity. The total mass of the cluster is proportional to Ω_0

$$Mass \propto R^3 \rho \propto \delta \rho_{universe} \propto \delta \Omega_0$$

Hence, the acceleration of galaxies outside the cluster will depend on Ω_0.

Example: determine Ω_0 from "Virgo centric infall"

Key Point #8 from Bachelor Course: Do Stars Collide?

3.1 Equations of motion

assume a collection of masses m_i at location x_i, and assume gravitational interaction. Hence the force $\vec{F}_i$ on particle i is given by

$$\vec{F}_i = m_i \frac{d^2}{dt^2}\vec{x}_i = \sum_{j \neq i} \frac{\vec{x}_j - \vec{x}_i}{|\vec{x}_j - \vec{x}_i|^3} G m_i m_j$$

- Only analytic solution for 2 point masses
- "Easy" to solve numerically (brute force) - but slow for 10^{11} particles [see http://astrogrape.org/]

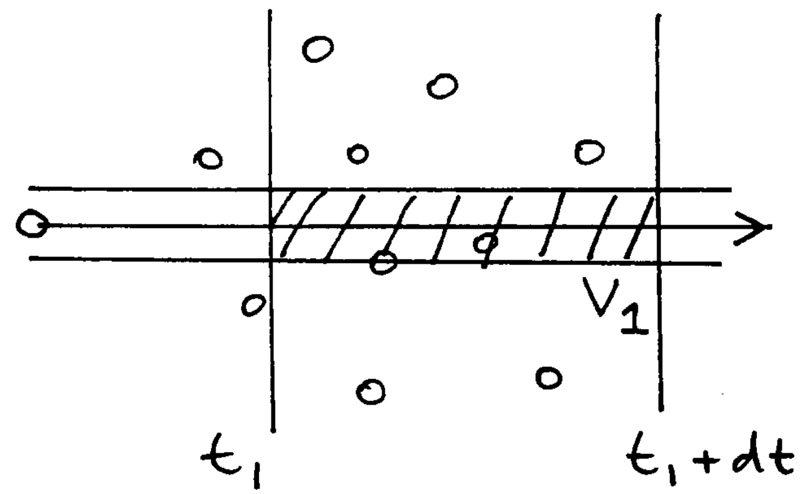
- Analytic approximations necessary for a better understanding of solution

In the following we investigate properties of gravitational systems without explicitly solving the equations of motion.

3.2 Do stars collide ? (BT 1 to 1.1)

Is it safe to ignore non-gravitational interactions ?

Calculate number of collisions between t_1 and $t_1 + dt$ of star coming in from the left, in galaxy with homogeneous number density n.



- incoming star has velocity v
- suppose all stars have radius r_*.
- all stars in volume V_1 will cause collision ->

$V_1 = \pi(2r_*)^2 \times v \times dt$

- number of stars in V_1 : $N_1 = nV_1$
- number of collisions per unit time $= 4\pi r_*^2 nv$

Typical values:

$r_\odot = 7 \times 10^{10}$ cm

$n = 10^{10}/[3kpc]^3 = 1.3 \times 10^{-56}$cm^{-3}

$v = 200$ km/sec= 2×10^7 cm/sec

which gives a collision rate of 1.6×10^{-26} sec^{-1} = 5×10^{-19} yr^{-1} -> very rare indeed

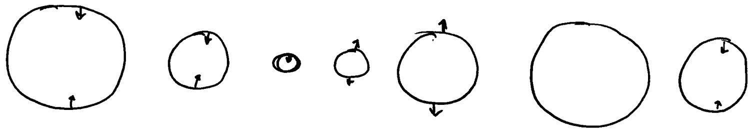
Hence we can ignore these collisions without too much trouble.

3.5 Binding Energy and Formation of Galaxies

The total energy E of a galaxy is

$$E = K + W = -K = 1/2W$$

- Bound galaxies have negative energy - cannot fall apart and dissolve into a very large homogeneous distribution
- A galaxy cannot just form from an unbound, extended smooth distribution $->$ $E_{total} = E_{start} \approx 0$, $E_{gal} = -K$, so energy must be lost or the structure keeps oscillating:



Possible energy losses through

- Ejection of stars
- Radiation (before stars would form)

Key Point #9 from Bachelor Course: Interactions with Other Stars Not Especially Important

Galactic Dynamics - Continued

3.6 Time scales (BT 4 to start 4.1)

dynamical timescale, particle interaction timescale

Is gravitational force dominated by short or long range encounters? (N.B. in a gas, only short range forces are relevant).

In a galaxy, the situation is different.
Consider force with which stars in cone attract star in apex of cone.

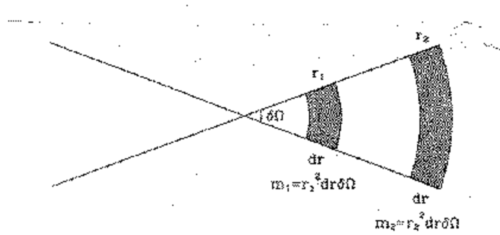


Figure 4-1. If the density of stars were everywhere the same, the stars in each of the shaded segments would make equal contributions to the net force on a star at the cone's apex. Thus the acceleration of a star at the apex is determined by the large-scale gradient in the density of stars within the galaxy.

Force $\sim 1/r^2$, with r the distance from apex. If ρ is almost constant, then the mass in a shell with width dr increases as $r^2 dr$.

Hence differential force is constant at each r, and we have to integrate all the way out to obtain the total force.

Realistic densities decrease after some radius, so that the force will be determined by the density distribution on a galactic scale (characterized by the half mass radius).

3.7 Relaxation time

Short range encounters do not dominate $\rightarrow$
Approximate force field with a smooth density $\rho(x)$ instead of point masses.

- Contrary of situation in gas: only consider long range encounters (long range $\sim$ scale of the galaxy)

Assume all stars have mass m. Analyze perturbations due to the fact that density is not smooth, but consists of individual stars. Simplify, and look first at single star-star encounter.

What is effect of a single encounter with point mass on motion of star?

- Exact: BT §7.1: hyperbolic Keplerian encounter
- Estimate: straight line trajectory past stationary perturber

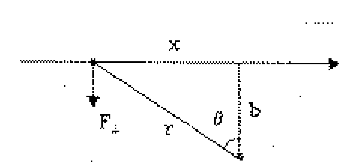


Figure 4-2. A field star approaches the test star at speed v and impact parameter b. We estimate the resulting impulse to the test star by approximating the field star's trajectory as a straight line.

The perpendicular force $\vec{F}_\perp$ gives perturbation $\delta\vec{v}_\perp$:

$$\vec{F}_\perp = \frac{Gm^2\cos\theta}{r^2} = \frac{Gm^2\cos\theta}{(b^2+x^2)} = \frac{Gm^2 b}{(b^2+x^2)^{3/2}}$$

$$\sim \frac{Gm^2}{b^2[1+(vt/b)^2]^{3/2}}$$

Newton: $\frac{d}{dt}\delta\vec{v}_\perp = \frac{\vec{F}_\perp}{m} \quad \Rightarrow$

$$\delta\vec{v}_\perp = \int \mathrm{d}t \frac{\vec{F}_\perp}{m} = \int \frac{Gm}{b^2[1+(vt/b)^2]^{3/2}}\mathrm{d}t$$

$$= \frac{Gm}{bv}\int_{-\infty}^{\infty} \frac{\mathrm{d}s}{(1+s^2)^{3/2}} = \frac{Gm}{bv}\left.\frac{s}{\sqrt{1+s^2}}\right|_{-\infty}^{\infty}$$

$$= \frac{2Gm}{bv}$$

Note: approximation fails when

$\delta\vec{v}_\perp > v \quad \Rightarrow \quad b < Gm/v^2 = b_{\min}$

Galaxy has characteristic radius R.

Define crossing time t_c as the time it takes a star to move through the galaxy $t_c = R/v$

Calculate number of perturbing encounters per crossing time t_c

In a crossing time, the star has 1 “encounter” with each other star in the galaxy

The impact parameter of each encounter can be derived by projecting each star onto a plane perpendicular to the unperturbed motion of the star

Hence “flatten” the galaxy in the plane perpendicular to the motion of the star, and assume that the stars are homogeneously distributed in that plane, out to a radius R, and no stars outside R. This is obviously a simplifying assumption, but it is reasonably accurate.

This can be used to derive the distribution of impact parameters:

N stars in total in Galaxy, distributed over total surface πR^2

per unit area: $\quad \frac{N}{\pi R^2}$

In a crossing time, the star has δn encounters with impact parameter between b and $b+db$. δn is given by the area of the annulus $2\pi b db$ times the density of stars on the surface, which is $N/(\pi R^2)$:

$\delta n = \frac{N}{\pi R^2} 2\pi b \mathrm{d}b = \frac{2N}{R^2} b\ db$

Result: $\langle \delta\vec{v}_\perp \rangle \equiv 0$

as the perturbations are randomly distributed, and will not change the average velocity

$$\langle \delta v_\perp^2 \rangle = \left(\frac{2Gm}{bv}\right)^2 \frac{2Nb}{R^2} \mathrm{d}b = 8N\left(\frac{Gm}{Rv}\right)^2 \frac{\mathrm{d}b}{b}$$

as each perturbation adds to $\langle \delta v_\perp^2 \rangle$ by an equal amount $(2Gm/bv)^2$.

The encounters do not produce an average perpendicular velocity, but they do produce an average (perpendicular velocity)^2. Hence, on average, the stars still follow their average path, but they tend to "diffuse" around it.

The total increase in rms perpendicular velocity can be calculated by integrating over all impact parameters from b_{min} to infinity:

Total rms increase:

$$\langle \Delta v_\perp^2 \rangle = \int_{b_{\min}}^{R} \langle \delta v_\perp^2 \rangle \; db = \int_{b_{\min}}^{R} 8N \left(\frac{Gm}{Rv}\right)^2 \; db/b =$$

$$= 8N\left(\frac{Gm}{Rv}\right)^2 \ln\Lambda$$

with $\ln\Lambda =$ Coulomb logarithm $= \ln \dfrac{R}{b_{\min}}$

We can rewrite this equation. Use

$b_{\min} = Gm/v^2$

From virial theorem

$v^2 = GM/R = GNm/R$

Hence $b_{min} = Gm/(GNm/R) = R/N$

$$\ln\Lambda = \ln R/b_{min} = \ln\frac{R}{R/N} = \ln N$$

Furthermore from virial theorem:

$$\frac{GM^2}{R} = Mv^2 \rightarrow \frac{GM}{R} = v^2 \rightarrow \frac{GNm}{R} = v^2 \rightarrow N = \frac{v^2 R}{Gm}$$

so that: $\dfrac{\langle \Delta v_\perp^2 \rangle}{v^2} = \dfrac{8\ln N}{N}$

This last number is the fractional change in energy per crossing time. Hence we need the inverse number of crossings $N/(8\ln N)$ to get $\langle \Delta v_\perp^2 \rangle \sim v^2$

The timescale t_{relax} is defined as the time it takes to deflect each star significantly by two body encounters, and it is therefore equal to

$$t_{relax} = \frac{N}{8\ln N} t_c$$

Conclusions

- effect of point mass perturbations decreases as N increases
- even for low N=50, $\langle \Delta v_\perp^2 \rangle / v^2 = 0.6$, hence deflections play a moderate role.
- for larger systems the effect of encounters become even less important

Notice: one derives the same equation when the exact formulas for the encounters are used. Put in another way, the encounters with $b < b_{min}$ do not dominate.

3.8 Relaxation time for large systems

$$t_{\mathrm{relax}} = \frac{0.1N}{\ln N} t_c$$

System	N	t_c (yr)	t_{relax} (yr)
globular cluster	10^5	10^5	2×10^8
galaxy	10^{11}	10^8	10^{17}
galaxy cluster	10^3	10^9	3×10^{10}

Age of Universe $\sim$ Hubble time $\sim 1.5 \times 10^{10}$ yr

$\Rightarrow$ Galaxies are collisionless systems

- motion of a star accurately described by single particle orbit in smooth gravitational field of galaxy
- no need to solve N-body problem with $N = 10^{11}$ (!)