Work and Energy

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June 12, 2013

1 Work-Kinetic Energy theorem

The statement of the theorem is
The work done by the net force on an object is equal to the change in the kinetic energy of the object.

Mathematically, it is written as $W_{net} = \Delta K = K_f - K_i$

1.1 What's all this about?

In the theorem, we stated atleast two terms that were probably familiar but not so intuitive. Let's first define those terms and then see how we were motivated to defined such (wierd?) quantitites.

1.1.1 Definition of Work

The work done W by a force $\vec{F}$ on a particle that moves from an initial point A to a final point B along a path P is given as

$$W = \int_A^B \vec{F}.\mathrm{d}\vec{r} \tag{1}$$

where the integration has to be done along the path P i.e. the $\vec{dr}$ must change along the path from point to point.

1.1.2 Definition of Kinetic Energy

A particle of mass m moving with a velocity $\vec{v}$ is said to have an kinetic energy K which is given by

$$K = \frac{1}{2}m\vec{v}^2 = \frac{1}{2}m\vec{v}.\vec{v} = \frac{1}{2}m|\vec{v}|^2 \tag{2}$$

Therefore, ΔK would be $\frac{1}{2}m{v_B}^2 - \frac{1}{2}m{v_A}^2$. And by the statement of the Work-Kinetic Energy theorem,

$$W_{net} = \int_A^B \vec{F_{net}}.\mathrm{d}\vec{r} = \frac{1}{2}m{v_B}^2 - \frac{1}{2}m{v_A}^2 \tag{3}$$

1.2 Motivation

Loosely speaking, the work is the product of force and displacement. Why on earth would you do something like that? Before we answer that in the following paragraph, it is worthwhile to understand that the work done characterizes the effect of force on an object. The product of force and time, which should sound more natural to the reader, would give you the change in momentum, by definition. I said it is natural because the longer the force acts, the more the effect. Similarly, the product of force and displacement tells you that, the longer the object moves under the influence of the force, the larger the effect. One might expect that both these effects that I talked about should be equivalent, but apparently they are not.

Let's start with acceleration. By definition, acceleration is the rate of change of velocity. For uniform acceleration,

$$a = \frac{\Delta v}{\Delta t} = \frac{v_2 - v_1}{t_2 - t_1} \tag{4}$$

where v_2 is the velocity at time t_2 and v_1 is the velocity at time t_1. Reaaranging, we get $v_2 = v_1 + a\Delta t$, the first one of Newton's equations of motion.

Now, again by definition, the velocity $v = \frac{\Delta x}{\Delta t}$ or equivalently, $\Delta x = v\Delta t$. But if the velocity is not constant (which is the case when we have a non-zero uniform acceleration), then we replace the velocity with the average velocity in the above expression. In particular, if v_1 is the initial velocity and v_2 is the final velocity, then [1]

$$\Delta x = v_{avg}\Delta t = \left(\frac{v_1 + v_2}{2}\right)\Delta t \tag{5}$$

Combining the equations (4) and (5), we get

$$a\Delta x = (v_2 - v_1)\left(\frac{v_2 + v_1}{2}\right) = \frac{v_2^2 - v_1^2}{2} \tag{6}$$

So far these have followed from the mathematical definitions of various quantitites. Let's introduce some Physics by multiplying both sides of the equation by the mass of the particle m on both sides of this equation. Therefore,

$$ma\Delta x = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \tag{7}$$

We can immediately recognize the term ma on the left hand side of the equation with

[1] The average velocity is just that sum of the initial and final velocity divided by 2, only if the acceleration is constant

the net or the toal force. The right hand side of the equation is a difference of two similar looking terms. Each of the term contains the properties of the particle and hence the term itself must be a property of that particle. Let's call it K. Thus, the effect of the force carrying along the particle by a distance Δx is change in the property K of the system. We just name the L.H.S as the *work done by the force on the particle* and the property $K = \frac{1}{2}mv^2$ to be the kinetic energy of the particle.

Thus we get the Work-Energy theorem $W_{net} = F_{net}\Delta x = \Delta K$.

1.3 Let's get realistic

In the above discussion, we just talked about a particle moving in 1-D, say along the x-axis. But the physical world is three dimensional and we are interested in problems that are in more than one dimensions. So, how to we generalize the expression for the work and the kinetic in 3-D?

In 3-D, the force and displacement should be denoted as vectors, like $\vec{F}$ and $\vec{\Delta x}$. We know of two ways to multiply two vectors - i) Dot/Scalar product ii) Cross product. When $\vec{F}$ and $\vec{\Delta x}$ are along the same direction, we want to retrive back our old simple formula. A cross product would give zero if the vectors are parallel to each other ($sin 0 = 0$). Hence, the natural way to do this is to take the dot product. This can also be seen if you made the vector nature of the acceleration, velocity and displacement explicit in the previous discussion. You could argue - what about just multiplying the magnitude of the force and the displacement. This is unnatural, because, the details of the directions are completely lost and a more pathological reason is the loss of linearity. To see this, if you consider a mass moving by a distance with a constant force $\vec{F}$ and come back to its original position by another constant force $-\vec{F}$, then we expect the total work to be zero, but $|\vec{F}|S + |-\vec{F}|S \neq 0$.

What about the kinetic energy? Well, we conclude that the left hand side of the equation is a scalar and so should be the right hand side. What do we mean by v^2 when v is really a vector, like $\vec{v}$. It really means $\vec{v}.\vec{v} = v_x^2 + v_y^2 + v_z^2 = |\vec{v}|^2 \equiv v^2$. This is consistent with our definition in the 1-D case, and we get a scalar. Again, it can be seen if you made the vector nature of the acceleration, velocity and displacement explicit in the earlier discussion.

Modification 1:

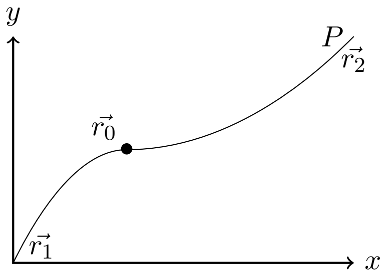
$$W_{net} = \vec{F}.\Delta\vec{r} = \frac{1}{2}m\vec{v_2}^2 - \frac{1}{2}m\vec{v_1}^2 \tag{8}$$

is the new statement of the Work-Energy theorem. We have denoted the displacement by $\Delta\vec{r}$ instead of $\Delta\vec{x}$ to avoid confusing the position vector with its x-component.

1.4 Meh, not challenging enough

Fair! Let's complicate it further. Suppose that the force was not constant which would mean that the acceleration was not uniform as the particle travelled by Δr. What then?

Which force would you choose to multiply with Δr? You could say some kind of average force, like $\frac{\vec{F}(t_2)+\vec{F}(t_1)}{2}$. But how would you know how I varied the force between the time t_1 and t_2? What if I applied no force at all? This must tell you that such a method won't work.



A more systematic way to do this is as follows. Consider a simpler problem. Let $\vec{r_0}$ be an intermediate point between the points $\vec{r_1}$ and $\vec{r_2}$ along the path P of the particle. Let $\vec{F_1}$ be the force from $\vec{r_1}$ to $\vec{r_0}$ and $\vec{F_2}$ be the force from $\vec{r_0}$ to $\vec{r_2}$. If $\vec{v_0}$ is the velocity of the particle at $\vec{r_0}$, then by the Work-Kinetic energy theorem,

$$\vec{F_1}.\left(\vec{r_0} - \vec{r_1}\right) = \frac{1}{2}m\left(v_0^2 - v_1^2\right) \tag{9a}$$

$$\vec{F_2}.\left(\vec{r_1} - \vec{r_0}\right) = \frac{1}{2}m\left(v_2^2 - v_0^2\right) \tag{9b}$$

Adding both gives you,

$$\vec{F_1}.\left(\vec{r_0} - \vec{r_1}\right) + \vec{F_2}.\left(\vec{r_2} - \vec{r_0}\right) = \frac{1}{2}m\left(v_2^2 - v_1^2\right) \tag{10}$$

The L.H.S of the theorem now has a sum of work terms equal to the change in the kinetic energy. If the force changed N number of times along the path, then we would naturally have a sum of N terms on the R.H.S. What if the force changes at every point along the path? What if it is a function of the position of the particle? Then, we would convert this sum of N terms to an integral from the initial point to the final point along the path P.

Modification 2:

$$\int_{\vec{r_i}}^{\vec{r_f}} \vec{F}.\mathrm{d}\vec{r} \equiv W_{net} = \Delta K = \frac{1}{2}m\left(v_f^2 - v_i^2\right) \tag{11}$$

is the final statement of the Work-Kinetic energy theorem. Here we refer to the initial and final conditions by the subscripts i and f. Note that the integral has to be done along the path P along which the particle moves.

Exercise

Calculate the work done by the force $F = -kx$ as the particle moves from x_1 to x_2 in the following cases

(i) $x_2 > x_1$
(ii) $x_2 < x_1$.

Exercise

Any general point in the 3-D space is represented by its position vector $\vec{r}$ which could be like $\langle x, y, z\rangle$ that tells you the coordinates of the point. Calculate the work done by the force $\vec{F}(\vec{r}) = \frac{K}{r^2}\hat{r}$ as the particle moves from a point $\vec{r_i}$ to $\vec{r_f}$. Here r is the distance between the origin and the point represented by $\vec{r}$ and $\hat{r}$ is the unit vector pointing along the direction joining the origin to the point represented by $\vec{r}$. Assume that the particle moves in a straight line, for simplicity.

Advanced exercise 1

Prove that this statement of the Work-Kinetic energy theorem is true for another observer moving at a constant velocity from you.

1.5 What path?

If you noticed carefully, all along the discussion, I have been saying *along the path* whenever I mention integrating. What's the big deal? Why can't I just do the normal integration and use $\vec{r_f}$ and $\vec{r_i}$ as my limits? Is this any different from the definite integral that I've known so far?
The answer to the last question is YES. These kind of integrals are called Line Integrals. Meh, who cares what its name is. Let me explain to you by an example what goes wrong if you try to do it the normal way.

Example:

Q: Consider a bead of mass m moving on a circular ring. The ring is rough and exerts a frictional force of constant magnitude f opposite to the direction of instantaneous velocity. Let $K(0)$ be the kinetic energy at $\theta = 0$ and $K(\pi/2)$ be the kinetic energy at $\theta = \pi/2$. Find the difference between $K(0)$ and $K(\pi/2)$.

A: Since the normal force, which acts as the centripetal force is perpendicular to the velocity (direction of infinitesimal displacement, $\vec{v} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t} \; ||^{ll} \; \mathrm{d}\vec{x}$), it does no work. Therefore, the difference in kinetic energy has to be due to the friction. Depending on whether the particle is moving clockwise or anti-clockwise, we have $K(\pi/2) > K(0)$ or $K(0) > K(\pi/2)$. Also, we need to specify whether we are talking about the particle taking the shortest path from 0 to $\pi/2$ or the longer path. The work done by the friction is $fR\pi/2$ and $fR3\pi/2$ Also depending on whether it has made a complete turn or not, the answer would differ. We would have to add $f2\pi R$ for every complete turn.
This simple example must clearly indicate the fact that the work done by the force simply

doesn't depend on the initial and final points alone. I should be frank here. The problem is ill-posed - I shouldn't even say something like K at 0 since it is not well-defined.

Advanced exercise 2

The viscous drag in fluids on a solid object is given by $\vec{f} = -k\vec{v}\,(k > 0)$.

a) Find the velocity $\vec{v}$ as a function of time given some initial velocity $\vec{v}_0$ at time $t = 0$.
b) Calculate the distance travelled by the object before it comes to a complete stop.
c) Calculate the work done by the friction from the definition of work and see that it is equal to the change in the kinetic energy i.e. $-\frac{1}{2}m{v_0}^2$

2 Conservative forces

Okay, we have established the fact that to calculate the work done by the force in a process, it is not enough to specify the initial and the final points but also need to specify the path, in general, by which the object travelled. But for some special kinds of forces, the work done depends only on the initial and final points and doesn't depend on the path taken by the object. Such forces are called conservative forces. We'll see why they get their name by the end of this section.

Definition 2(a):

If the work done by a force $\vec{F}$ is same along any path joining any arbitrary initial and final points, then $\vec{F}$ is said to be a conservative force.

We (should) have already seen a few examples of conservative forces. The forces given in earlier exercises: $F = -kx$ and $\vec{F} = \frac{k}{r^2}\hat{r}$ are the ones. If you had done the math correctly, you would have obtained the work to be $\frac{1}{2}k\left(x_2^2 - x_1^2\right)$ and $k\left(\frac{1}{|\vec{r}_f|} - \frac{1}{|\vec{r}_i|}\right)$ whatever path you had chosen.

2.1 Path independence & Closed loops

There is an interesting relation between path independence of the work done by a force and the work done along a closed loop, which leads to another definition of a conservative force.

Definition 2(b):

If the work done by a force $\vec{F}$ is zero along *any* closed loop, then the force is conservative.

Let us first see how these two definitions are equivalent.

2(a) implies **2(b)**:
Let i and f denote the initial and final points respectively. Consider an arbitrarily long

path from i to f. Suppose that the initial and final points were very close to each other i.e. almost coinciding. Then I could simply say that the work done by the force along the shortest path (straight line) joining i and f is infinitesimally small because the distance between them is almost zero. In the limit that f coincides with i, we can conclude that the work done is zero. And the arbitrarily long path we considered earlier forms a closed loop. By definition, the work done is independent of the path. Therefore, the work done along the arbitrarily closed loop is zero *if* the work done by the force is independent of the path for a given set of initial and final points.

2(b) implies **2(a)**:
Consider a particle moving along some closed loop C starting and ending at point A. The work done by the force in this process is zero. Consider an arbitrary point B in the closed loop. Then, we know that

$$W_{P1}\left(A \to B\right) + W_{P2}\left(B \to A\right) = 0 \tag{12}$$

where $P1 + P2 = C$.
Instead, if you came from B to A by a different path $P3$, we still have a closed loop and by definition, the work done is zero. Therefore,

$$W_{P1}\left(A \to B\right) + W_{P3}\left(B \to A\right) = 0 \tag{13}$$

This means that $W_{P3}\left(B \to A\right) = W_{P2}\left(B \to A\right) = -W_{P1}\left(A \to B\right)$. Since the paths $P2$ and $P3$ are arbitrary, we could say that the work done along any path P joining B to A has to be the same. Thus, we arrive at a conclusion that the work done by the force is the same given the initial and final points *if* the work done by the force along any closed loop is zero.

2.2 Vector identities

2.2.1 ***Conservative fields and Curls

When the force experienced by an object is given as a function of its position alone (and possibly time), then we called it a force field, denoted by $\vec{F}\left(x, y, z; t\right)$. If we know the force field, then there is an easy way to figure out whether it is a conservative force or not.

From Green's theorem, for any vector field $\vec{E}$, we have,

$$\oint_C \vec{E}.\mathrm{d}\vec{r} = \int_S \vec{\nabla} \times \vec{E}.\mathrm{d}\vec{S} \tag{14}$$

If the field is conservative, then the left hand side of this equation is zero. The right hand side of the equation $\int_S \vec{\nabla} \times \vec{E}.\mathrm{d}\vec{S}$ is bounded by $|\vec{\nabla} \times \vec{E}|_{max}S$. Since S is non-zero for a closed loop, $|\vec{\nabla} \times \vec{E}|_{max}$ has to be equal to zero which means $\vec{\nabla} \times \vec{E} = 0$.

Thus we make the statement that a force field $\vec{E}$ is conservative *iff* its curl, $\vec{\nabla} \times \vec{E}$ vanishes.

Advanced exercise 3:

Prove that the curl of the force field $\vec{F} = \frac{k}{r^2}\hat{r}$ vanishes. You may use any coordinate system that you like.

2.2.2 Conservative field as a gradient

Let's turn off the time dependence for a while, since pure mathematics doesn't know that nothing travels faster than light. A calculation on Vector Analysis tells us that if the curl of a vector field $\vec{E}(x, y, z)$ is identically zero i.e zero everywhere, then the field can be expressed as a gradient of a scalar function. For reasons that we will see later, we shall denote the scalar function that gives $\vec{E}$ as $-V_{\vec{E}}\,(x, y, z)$.

$$\vec{E}\,(x, y, z) = -\vec{\nabla} V_{\vec{E}}\,(x, y, z) \tag{15}$$

The subscript $\vec{E}$ for V tells you that the scalar function is specific to the vector field at hand. Let me remind you that the gradient of a scalar function V is

$$\vec{\nabla} V\,(x, y, z) = \frac{\partial V}{\partial x}\hat{e}_x + \frac{\partial V}{\partial y}\hat{e}_y + \frac{\partial V}{\partial z}\hat{e}_z$$

or equivalently

$$\vec{\nabla} V\,(x, y, z) = \frac{\partial V}{\partial n}\hat{n}$$

Here $\hat{e}_x$, $\hat{e}_y$ and $\hat{e}_z$ denote the unit vectors along x, y and z axes respectively and $\hat{n}$ denotes a unit vector along the direction of the gradient itself (or the vector field it produces). This might look like an ambiguous definition - how do we know which way the gradient points to? Without proof, I claim that $\hat{n}$ points that direction along which the rate of the increase of the scalar function is the highest.

2.3 Potential function

The previous section was a little too mathematical and I promise to keep this section simpler. The result that we need from the previous section is that a conservative force $\vec{F}$ can be written as a gradient of some scalar function $-U$ (minus for reasons yet to be seen), where I am not explicitly denoting with a subscript that U is specific to the $\vec{F}$ at hand.

$$\vec{F} = -\vec{\nabla} U$$

Let me try to motivate this from a physical perspective. When we say $\vec{F}$ is conservative, the work done by the force in moving a mass from a point $A\,(x_1, y_1, z_1)$ to another point $B\,(x_2, y_2, z_2)$ is independent of the path taken by the object. So the work done, at the best, could only be a function of the position of the initial and final points, like $W_{A\to B}\,(x_1, y_1, z_1; x_2, y_2, z_2)$. Now, if I make a shift of the origin from to another point (a, b, c) in the current coordinate system, then the work done by the force in moving from the same physical initla point A to the same physical final point B becomes

$W_{A\to B}\left(x_1+a, y_2+b, z_1+c; x_2+a, y_2+b, z_2+c\right)$. But the origin and the coordinate system is something that I construct for my calculation and the real physics should not depend on the values of a, b and c. This suggests that the work done is not any general function of the initial and the final points but only depends on the difference between the final and the initial points. So it takes the form

$$W_{A\to B}\left(x_2-x_1, y_2-y_1, z_2-z_1\right)$$

1. We'll list out a few properties that $W_{A\to B}$ must obey based on a bunch of logical conclusions. If $C\left(x_3, y_3, z_3\right)$ a point on a path from A to B, then the work done in going from A to B is the sum of the work done in going from A to C and from C to B. This translates into

$$\begin{aligned} W_{A\to B}\left(x_2-x_1, y_2-y_1, z_2-z_1\right) = & W_{C\to B}\left(x_2-x_3, y_2-y_3, z_2-z_3\right) \\ & + W_{A\to C}\left(x_3-x_1, y_3-y_1, z_3-z_1\right) \end{aligned}$$

   The only possibility for this property to hold for arbitrary $\left(x_3, y_3, z_3\right)$ is to have the following form for $W_{i\to f}$

$$W_{i\to f}\left(x_f-x_i, y_f-y_i, z_f-z_i\right) = U\left(x_i, y_i, z_i\right) - U\left(x_f, y_f, z_f\right)$$

   Then we immediately see that the above property is satisfied for any intermediate point C. This is an important result. The work done by the force in moving an object from an initial point to the final point is given by the difference of a single scalar function evaluated at the initial and the final points. Note that this also takes care of the fact that the work done is independent of the origin chosen.

   Typically, in physics, we choose to write down in 'final' minus 'initial' form. But this form is easier as we'll see in the next paragrah. This should also suggest why we had a minus sign in our earlier definitions of potential.

2. So we have,

$$W_{A\to B}\left(x_2-x_1, y_2-y_1, z_2-z_1\right) = U\left(x_1, y_1, z_1\right) - U\left(x_2, y_2, z_2\right)$$

   The work done by a constant force in 3-D, $\vec{F}.\Delta\vec{r}$ takes the form $F_x\Delta x + F_y\Delta y + F_z\Delta z$. . Again based on path independence, I could chose to go from A to B first along the x-axis, then along the y-axis and finally along the z-axis.

$$\begin{aligned} W_{A\to C}\left(x_2-x_1, y_1-y_1, z_1-z_1\right) &= F_x\Delta x = U(x_1, y_1, z_1) - U(x_2, y_1, z_1) \\ W_{C\to D}\left(x_2-x_2, y_2-y_1, z_1-z_1\right) &= F_y\Delta y = U(x_2, y_1, z_1| - U(x_2, y_2, z_1) \\ W_{D\to B}\left(x_2-x_2, y_2-y_2, z_2-z_1\right) &= F_z\Delta z = U(x_2, y_2, z_1| - U(x_2, y_2, z_2) \end{aligned}$$

which we some to get the total work. Notice that we get the components of the force in terms of the work done by the force i.e.,

$$
\begin{aligned}
F_x &= \frac{U(x_1,y_1,z_1)-U(x_2,y_1,z_1)}{x_2-x_1} = -\left(\frac{\Delta U}{\Delta x}\right)_{y,z} \\
F_x &= \frac{U(x_2,y_1,z_1)-U(x_2,y_2,z_1)}{y_2-y_1} = -\left(\frac{\Delta U}{\Delta y}\right)_{z,x} \\
F_x &= \frac{U(x_2,y_2,z_1)-U(x_2,y_2,z_2)}{z_2-z_1} = -\left(\frac{\Delta U}{\Delta z}\right)_{x,y}
\end{aligned}
$$

If the force is not a constant one, then we need to do this for small displacements and the Δ s are replaced by the partial derivatives ∂ and thus we get

$$\vec{F} = F_x\hat{e}_x + F_y\hat{e}_y + F_z\hat{e}_z = -\frac{\partial U}{\partial x}\hat{e}_x - \frac{\partial U}{\partial y}\hat{e}_y - \frac{\partial U}{\partial z}\hat{e}_z = -\vec{\nabla}U$$

Thus, we have arrived at the same result as we arrived in the previous subsection, hopefully with a better physical intuition.

Remember that we also have $W\left(A \to B\right) = \int_A^B \vec{F}.\mathrm{d}\vec{r} = -\Delta U = -\left(U(B) - U(A)\right)$

3 Energy Conservatiion

I have been postponing the reason for naming these special forces/fields as 'conservative' and the reason for having a negative sign for the scalar function. These two questions are going to be answered in this section before concluding.

Consider the Work-Kinetic energy theorem again. If the system is subjected to only these 'special' forces, then we should be able to write the work done by the net force as a difference of a scalar function between two different points. For multiple 'special fields' acting on the object, the work done by the net force is the difference of the sum of the scalar functions associated with each of those forces. Thus,

$$W\left(A \to B\right) = -\Delta U = \Delta K \tag{16}$$

or equivalently,

$$\Delta K + \Delta U = \Delta\left(K + U\right) = 0 \tag{17}$$

This means that the quantity $K + U$ stays constant. Since K has the units of energy, so should U. This is nothing but the potential energy associated between the system and the force(s). The total energy is defined as the *sum* of the kinetic and the potential energy. This is precisely why we have been keeping the negative sign explicitly without absorbing them into the function itself. Since, the total energy is conserved throught, these 'special' fields are known as 'conservative' fields; not because they are orthodox and conservative but because they *conserve* the total energy.

Let's try to build in some more intuition on the negative sign associated with the scalar function. A positive work done *by* the conservative force helps to increase the kinetic energy of the system. By the conservation of energy, the kinetic energy term can increase only at the cost of the potential energy. Thus a positive work *by* the force corresponds to the change in the potential energy in magnitude, but has opposite signs. Hence, their equality is accompanied by an extra negative sign.

If we have a mixture of conservative and non-conservative fields, then the work done by the net force can be written as a sum of the work done by all the non-conservative forces and the work done by all the conservative forces.

$$W_{net} = W_{NC} + W_C$$

But $W_C = -\Delta U$ for some u. In this case, the energy is no longer conserved since

$$W_{NC} = \Delta K + \Delta U = \Delta \left(K + U \right) \tag{18}$$

Thus, if a non-conservative force (dissipative force) does some work, then the energy is no longer conserved. Note that this requires only the work done by the dissipative force to be zero, not the force itself! This is precisely the reason why we never consider the normal force or the centripetal accelaration for energy conservation priciples.