Subtleties in the Laws of Indices

Arun Kannawadi Jayaraman

June 9, 2013

1 Law(s) of indices

For two positive integers m and n for some any number a (real or complex), the following is obvious:

$$a^m \times a^n = a^{m+n} \tag{1}$$

We begin with positive integer valued m and n because we understand what a^m and a^n means - it is just the number obtained by multiplying a s m number of times. But we often find people talking about $a^{1/2} = \sqrt{a}$. We understand what $\sqrt{a}$ is but what the heck is $a^{1/2}$ if we go by the definition? How can we multiply a 0.5 number of times ? It doesn't make *any* sense!

The reason for such a definition is that it will satisfy (1). Let $a^{1/2}$ mean whatever, but when I multiply it twice and demand that (1) holds true, then

$$a^{1/2} \times a^{1/2} = a^1 = a \tag{2}$$

Thus $a^{1/2}$ is that number which when multiplied with itself gives a, which is precisely the definition of $\sqrt{a}$. Thus, we define $a^{1/2} \equiv \sqrt{a}$. The law of indices, proved when $m, n \in \mathbb{N}$ are used to define a^m when m is not a natural number.

So, when we wonder what a^0 is, we should not think what it means to multiply a 0 number of times. We should apply it in (1) and see what it gives. For any m,

$$a^m \times a^0 = a^{m+0} = a^m \tag{3}$$

Thus, a^0 is something that when multiplies any number leaves the number unchanged. And 1 is the only number (identity) that has that property. Thus, defining $a^0 \equiv 1$ is consistent with the laws of indices and hence defined so.

2 Order of limits

For some positive m, it is again obvious that $0^m = 0$. Setting $a = 0$ in (1) will tell you that $0^m = 0$ for (almost) all ms. We find something interesting when we ask what happens when $m = 0$. Is 0^0 the identity element i.e. 1 because it is something raised to the power 0 or is it 0 because it is 0 raised to the power something?

The diplomatic way to avoid this uncomfortable situation is to leave 0^0 undefined. If we come across it in any physical problem, a limit process is applied usually and the order of limits need to be carefully chosen. This is something that is not taken very seriously, except by mathematicians. In Physics, we let the nature of the problem set the order of the limits.

To frame this beautiful result mathematically,

$$\lim_{a\to 0} \lim_{m\to 0} a^m = 1$$

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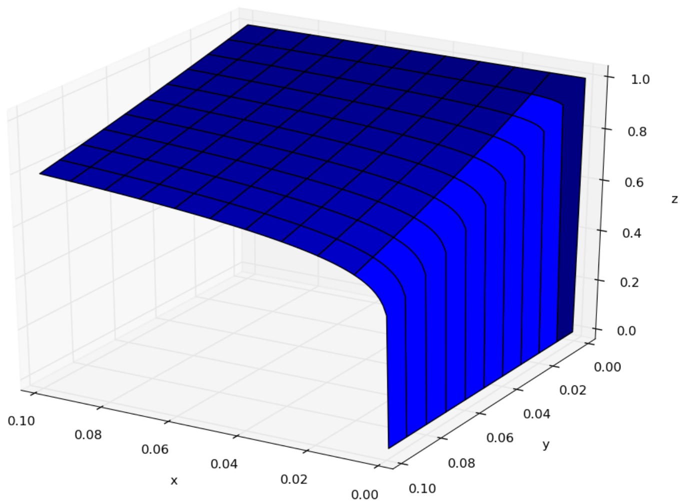
And coming back to our main question of interest, we say that

$$a^0 \equiv 1 \qquad \forall a \neq 0 \tag{4}$$

The following is a surface plot of $z = x^y$. You can see that the surface is distorted near $(0, 0, 1)$.



In the above case, we let x and y go to zero independently i.e. we let one variable go to zero after evaluating the limit of the other going to zero. What if the base and the exponent goes to zero simultaneously? I leave it to you as an exercise.

Exercise

Prove that

$$\lim_{x\to 0} x^{mx} = 1 \tag{5}$$

(Hint: Use L'Hopital rule after taking logarithm)