Building physical intuition for eigenvalues and eigenvectors

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Abstract

The goal of this article is to motivate why the study of eigenvalues and eigenvectors are very important in many systems.

1 Introduction

Eigen analysis, also known as spectral analysis, play an important role in many fields. This is seen mostly as a calculational tool without any physical intuition into it. After having defined the eigenvalues and eigenvectors for the sake of completeness, we try to throw some light on how to interpret them from examples from coordinate geometry and dynamical systems. Very basic prior knowledge in these fields are assumed.

1.1 Definition

Given a (square) matrix M, if $\exists$ a vector $\mathbf{v} \neq \mathbf{0}$ that satisfies the equation

$$H\mathbf{v} = \lambda\mathbf{v} \tag{1}$$

for some $\lambda \in \mathbb{C}$, then $\mathbf{v}$ is said to be an eigenvector of H and λ is said to be the corresponding eigenvalue of M.

A few properties of eigenvalues and eigenvectors:

- If $\mathbf{v}$ is an eigenvector, then $\alpha\mathbf{v}$ is also an eigenvector with the same eigenvalue $\forall\alpha \in \mathbb{C}\setminus\{0\}$ i.e. the norm of the eigenvector is unfixed. Usually, we normalize it to unity. By convention, $\mathbf{v}^T\mathbf{v} = 1$.
- If M has a set of eigenvalues denoted by λ_i $(i = 1, \dots)$, then αM has eigenvalues $\alpha\lambda_i$ $(i = 1, \dots)$ $\forall\alpha \in \mathbb{C}\setminus\{0\}$. i.e. the eigenvalues scale with the matrix M.

1.2 Procedure

[label:Procedure]

Without providing any motivation, we just explain the procedure to find the eigenvalues and eigenvectors.

The eigenvalues are found out by solving the secular equation

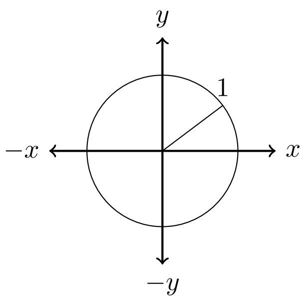
$$det\,(M - \lambda I) = 0 \tag{2}$$

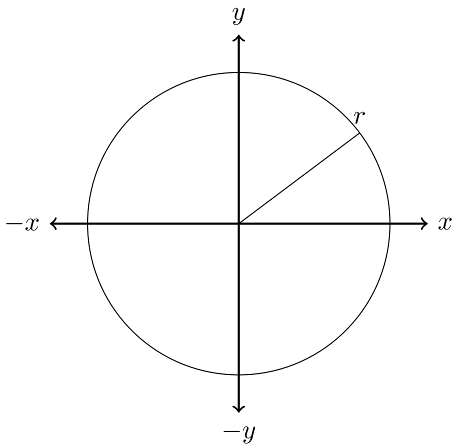
n roots are guaranteed, where n is the number of columns in M. There could be roots with multiplicity greater than one, i.e. the eigenvalues can be degenerate. For every eigenvalue of degeneracy g, we get $n - g$ (or lesser) linearly independent equations in n variables from which we come up with the eigenvectors.

2 Motivation

2.1 Coordinate Geometry

Consider the equation $x^2+y^2 = 1$. The LHS, x^2+y^2 is the square of the distance of the point (x, y) from the origin in the $x - y$ plane. If we were to draw the curve corresponding to this equation, then it would be the locus of all such points whose distance squared from the orgin is unity and hence the distance itself being unity. Obviously, by definition, we get the unit circle centered at the origin.





Now consider $x^2 + y^2 = r^2$, where $r > 0$ without loss of generality. We could re-write this as

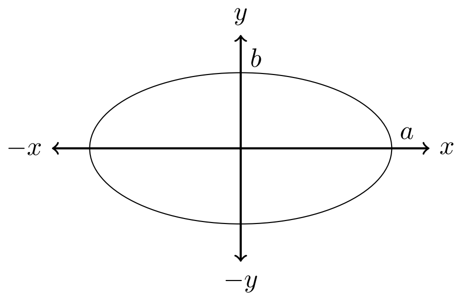
$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

Defining $u = x/r$ and $v = y/r$, we get the unit circle in the $u - v$ plane. The variables x and y are related to u and v by the scale factor r. So in $x - y$ plane, it is a circle scaled by r in both horizontal and vertical directions. Or in otherwords, it is a circle of radius r. That was easy.

How about this -

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1?$$

Well, it's a circle that is scaled by different amounts in different directions. So it should look like in ellipse as follows:



All this was fairly intuitive. Any student who has studied basic coordinate geometry would have drawn these without a second thought. If there are any linear terms in x or in y, then we could complete the squares in appropriate variable(s) and it reflects as a shift in the center.

Now comes the real demon. Try drawing

$$3x^2 - 2xy + 3y^2 = 1 \tag{3}$$

Clueless? OK, can you at least tell if it is going to be a circle or ellipse? In fact a general quadratic equation in x and y can represent an ellipse, a parabola, a hyperbola or a pair of two straight lines in general. It is not at all obvious, even for someone who is well-versed with the subject. Why did it suddenly become so complicated?

Nobody panics when there are squared and linear terms. But I introduce one little cross term xy, well then everyone loses their minds

The appearance of the cross term seems to have eaten our intuition immediately. If not for the $-2xy$, then it is clear that it is a circle with radius $1/\sqrt{3}$. How do we deal with the cross terms?

Let's try to bring together the easier and the harder examples within a single formalism. An expression of the form $ax^2 + by^2$ can be expressed as

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{4}$$

This is known as the 'quadratic form'. So we try to write $3x^2 - 2xy + 3y^2$ in a similar fashion.

$$3x^2 - 2xy + 3y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 3 & c \\ d & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{5}$$

where $c + d = -2$. Nothing fixes c and d uniquely, so we may as well choose $c = d = -1$. This makes the matrix symmetric and symmetric matrices are nice.

Therefore,

$$3x^2 - 2xy + 3y^2 = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{6}$$

Thus, we see that the appearance of the cross term is equivalent to the appearance of the off-diagonal terms. Diagonal matrices were easy to work with. We wish this matrix was diagonal too, but it's not.

Suppose we have a genie which would take this matrix and gives us back

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{7}$$

It is easy to verify that the product of these three matrices is equal to the matrix of interest. Our genie has managed to convert the matrix with off-diagonal elements to a product of three matrices, with the middle one being in diagonal.

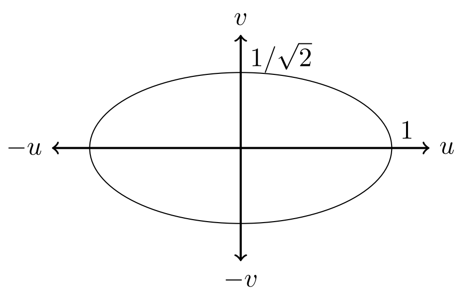
Let's try to see if this helps us in understanding how the curve looks like. Substituting,

$$\begin{aligned} 3x^2 - 2xy + 3y^2 &= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

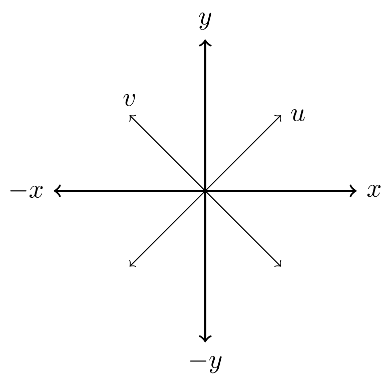
Ideally, we would like to have only that diagonal matrix in the center. Let's get rid of the other two matrices by multiplying them with the adjust row & column vectors respectively. Defining $u := x + y$ and $v := x - y$, we get

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u & v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{8}$$

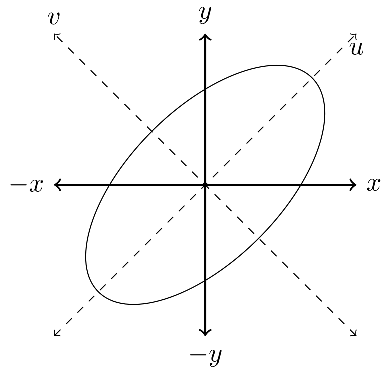
Instantly we recognize this as an ellipse in the $u - v$ plane. We can draw it immediately as follows:



But we want the curve in the $x - y$ plane. For that, we must first draw the u and the v axes on the $x - y$ plane. Note that u axis is given by the equation $v = 0$ and vice-versa. So u axis is the line corresponding to $x - y = 0$ or the line $y = x$. Similarly, v axis is the line $y = -x$.



We know how the curve looks like in $u-v$ plane. So we redraw it now where u and v axes are drawn on the $x-y$ plane. But we don't want the u and the v axes at the end of the day. Erasing them, we get $3x^2 - 2xy + 3y^2$ in the $x-y$ plane.



We now see that it is an ellipse with its semi-minor axis and semi-major axis along 45^0 and -45^0 lines and the lengths being 1 and $1/\sqrt{2}$.

Cool! But we have to learn to do this without the Genie because we are not Aladdins. And even Aladdin gets only three wishes. It's not that difficult to get rid of the Genie. Following the procedure explained in Sec **??**, we see that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{9}$$

is nothing but just the set of eigenvectors expressed as columns. (The other instance of this matrix is actually its inverse. It is an accident that this matrix happened to be its own inverse) . Our next task will be to motivate the procedure for finding the eigenvalues and eigenvectors and then explain why eigenvectors happened to be the special directions.

2.2 Dynamical systems

It is very common in the field of dynamical systems to encounter equations of the form

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

where the dot on the variables x and y denote the time derivatives of x and y respectively. This can be written as a differential equation of matrices:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \dot{\begin{pmatrix} x \\ y \end{pmatrix}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = L \begin{pmatrix} x \\ y \end{pmatrix} \tag{10}$$

Solutions to this set of simultaneous coupled first order differential equations are of interest. If $b = c = 0$, then the solution is immediately obvious. The first equation reduces to $\dot{x} = ax$ and given an initial condition x_0, $x(t) = x_0 e^{at}$ is the solution. Similarly $y(t) = y_0 e^{dt}$. Once again note that this was possible because the off-diagonal terms in the matrix $L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ vanished.

As a matrix equation, if we denote $\begin{pmatrix} x \\ y \end{pmatrix}$ by $\mathbf{x}$, then the solution to the equation $\dot{\mathbf{x}} = L\mathbf{x}$ is given by $\mathbf{x}(t) = e^{Lt}\mathbf{x}(0)$. When $b = d = 0$, then

$$e^{Lt} = e^{\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} t} = \begin{pmatrix} e^{at} & 0 \\ 0 & e^{dt} \end{pmatrix} \tag{11}$$

When $b \neq 0$ or $d \neq 0$, then we have evaluate $e^{Lt} = \mathbf{1} + Lt + (Lt)^2/2! + \ldots$ which is not an easy task when L is not diagonal.

But we can assert $e^{Lt} = M(t)$ where $M(t)$ is some unknown 2×2 matrix, which is a function of t. Say

$$M = \begin{pmatrix} c_{11}(t) & c_{12}(t) \\ c_{21}(t) & c_{22}(t) \end{pmatrix} \tag{12}$$

Then the solutions are of the form

$$\begin{aligned} x(t) &= c_{11}(t)x_0 + c_{12}(t)y_0 \\ y(t) &= c_{21}(t)x_0 + c_{22}(t)y_0 \end{aligned}$$