Black hole physics

``The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time. And since the general theory of relativity provides only a single unique family of solutions for their descriptions, they are the simplest objects as well."

Chandrasekhar (1983)

Central engine in AGN is likely to be super-massive black hole

Black holes form a two-parameter family of near-magical solutions of the Einstein equations of General Relativity
- Parameters are mass and angular momentum
- Non-rotating solutions: Schwarzschild (1916)
- Full family: Kerr (1963)

Here derive some basic properties
- Provide foundation for understanding physics of accretion
- Quantify theoretical efficiency of energy generation

Literature

Khembavi & Narlikar § 5.3

Background

Black Holes, White Dwarfs, and Neutron Stars
Shapiro S.A., Teukolsky S.L., 1983, § 5, § 12 (ST)

Gravitation
Misner C.W., Thorne K.S., Wheeler J.A., 1973, § 25, § 33 (MTW)

The Mathematical Theory of Black Holes
Chandrasekhar S., 1983, § 3, § 7 (C83)

The Classical Theory of Fields
Landau L.D., Lifshitz E.M., 1978, § 81-84, 102, 104

Natural units

Choose units l_0 of length and t_0 of time so that c=G=1

- 1 second $= 3 \times 10^{10}$ cm $= 2 \times 10^{-3}$ AU $= 0.4 R_\odot$
- 1 gram $= 0.7425 \times 10^{-28}$ cm
- $l_0 = 1.47 \frac{M}{M_\odot}$ km $= t_0 = 4.9 \times 10^{-6} \frac{M}{M_\odot}$ s

Natural length scale is the gravitational radius: $r_g = \frac{GM}{c^2} = M$

Newtonian approximation

Particle moving in gravitational field of black hole of mass M:

$$\frac{\dot{\mathbf{r}}^2}{2} - \frac{GM}{|\mathbf{r}|} = E$$ with E total energy of the particle

h: angular momentum of the particle per unit mass, then the Newtonian laws of motion give

$$\frac{\dot{r}^2}{2} + V(r) = E$$

where the effective potential is defined as: $$V(r) = \frac{h^2}{2r^2} - \frac{GM}{r}$$

h=0: effective potential reduces to the Newtonian
h≠0: repulsive centrifugal forces active
E<0: particle cannot escape: $r \to \infty$ then $V(r) \to 0$ and $\dot{r}^2$ becomes negative
and move in ellipsoidal orbits
E>0 particle can escape

A relativistic stationary black hole

The Schwarzschild (1919) line element *ds* describes space-time near a stationary mass M:

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2 (d\theta^2 + \sin^2\theta d\phi^2)$$

Divergence at $r=2MG/c^2$, the *Schwarzschild radius*, is an artifact of the coordinate choice. No locally measured physical quantity diverges here. Also called event horizon

Local time τ

A static observer with fixed r,θ,ϕ

Observes his proper local time $d\tau$ which is related to the time *dt* at infinity by

$$d\tau = (1 - \frac{2GM}{c^2 r})^{1/2} dt$$

They are identical when r>>2M, but $dt \gg d\tau$ when $r \downarrow 2M$

Relation not defined inside the *event horizon* $r=r_+= 2MG/c^2$

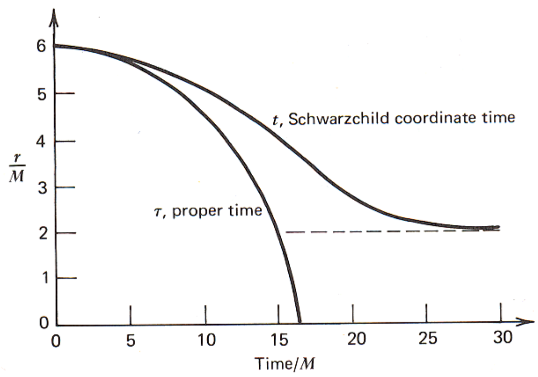

Figure 12.1 Fall from rest toward a Schwarzschild black hole as described (a) by a comoving observer (proper time τ) and (b) by a distant observer (Schwarzschild coordinate time t). In the one description, the point $r = 0$ is attained, and quickly [see Eq. (12.4.23)]. In the other description, $r = 0$ is never reached and even $r = 2M$ is attained only asymptotically [Eq. (12.4.24)]. [From *Gravitation* by Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler, W. H. Freeman and Company. Copyright © 1973.]

Gravitational redshift

Consider a photon emitted at radius r, observed at infinity:

Emitter $\nu_{\text{em}} = \frac{1}{d\tau_{\text{em}}}$ $d\tau_{\text{em}} = (1 - \frac{2M}{r})^{1/2} dt$

Receiver $\nu_{\text{re}} = \frac{1}{d\tau_{\text{re}}}$ $d\tau_{\text{em}} = dt$

Gravitational redshift

$$z = \frac{\lambda_{\text{re}}}{\lambda_{\text{em}}} - 1 = \frac{\nu_{\text{em}}}{\nu_{\text{re}}} - 1 = \frac{1}{(1-\frac{2M}{r})^{1/2}} - 1$$

No information can reach us from inside the event horizon: the singularity is hidden, and hence the name black hole

A relativistic stationary black hole

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From *ds* we can derive a Lagragian and then the equations of motions (ST p. 340):

θ = constant, (can be taken to be $\theta = \pi/2$): particle moves in a plane

$r^2 \frac{d\phi}{ds} = constant = h$ conservations of angular momentum

$\frac{dt}{ds} = \frac{E}{c}\left(1 - \frac{2GM}{c^2 r}\right)^{-1}$ conservations of energy (KN 5.8)

- These first integrals can be used together with the Schwarzschild line element:

$$\left(\frac{dr}{ds}\right)^2 + V^2(r) = E^2 \qquad \text{(KN 5.9)}$$

  - With as relativistic effective potential: $V^2(r) = \left(1 - \frac{2GM}{c^2 r}\right)\left(1 + \frac{h^2}{r^2}\right)$
- Define normalized angular momentum per unit mass/

$$H \equiv c^2 h/(GM)$$

- When $H > \sqrt{12}$
  - V(r) has a maximum at r_{max} : unstable circular orbits
  - and a minimum at r_{min} : stable circular orbits
- When $H < \sqrt{12}$ no minimum in potential: no stable orbit and particle always moves into the BH. This is unlike the Newtonian case.

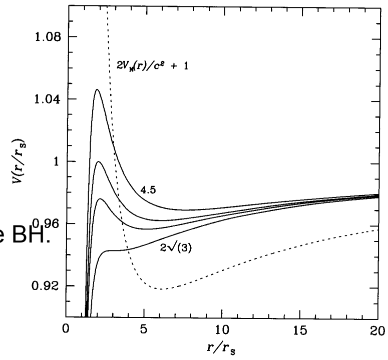
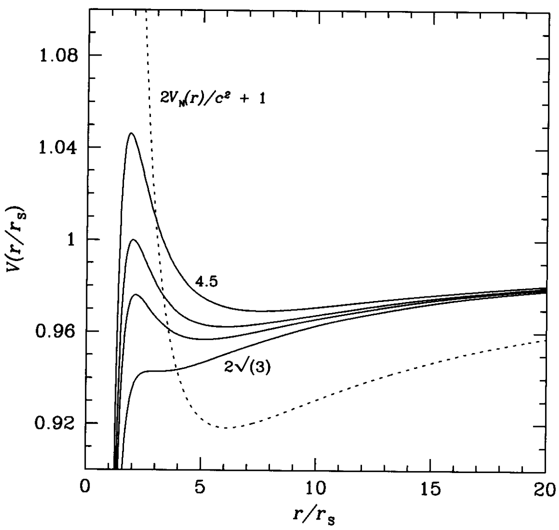


Fig. 5.2. The general relativistic effective potential as a function of the radial distance from a black hole. Potentials for $H = 4.5$ and $H = 2 \times 3^{1/2}$ are labelled. The other two curves are for $H = 4$ and $H = 3.8$ respectively. The dotted line is a Newtonian potential well for an angular momentum per unit mass of $12.25GM/c$, where M is the mass of the black hole. The function actually plotted is indicated by the label.



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Or:

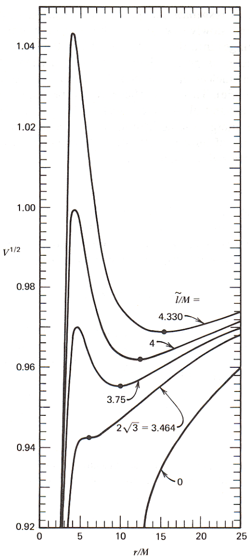

Figure 12.3 The effective potential profile for *nonzero* rest-mass particles of various angular momenta $\tilde{l}$ orbiting a Schwarzschild black hole of mass M. The dots at local minima locate radii of stable circular orbits. Such orbits exist only for $\tilde{l} > 2\sqrt{3}\,M$. [From *Gravitation* by Charles W. Misner, Kip S. Thorne, and John Archibald Wheeler, W. H. Freeman and Company. Copyright © 1973.]

Stable circular orbits (sco) occur at extrema of effective potential: $V_{eff}(r) = 0$ and $V_{eff}'(r)=0$

$$r_{\mathrm{sco}} = \frac{GM}{2c^2}\left[H^2 + (H^4 - 12H^2)^{1/2}\right]$$

On a circular orbit, the specific angular momentum is:

$$l = hc = \sqrt{\frac{GMr^2}{r - 1.5r_s}}$$

For a circular orbit, *dr/ds=0,* (using KN5.9), the energy is given as:

$$E_{\mathrm{SCO}} = \mu c^2 \frac{r - r_s}{\sqrt{r(r - 1.5r_s)}}$$

The efficiency of energy conversion, with μc^2, the rest mass energy of the particle:

$$\epsilon = (\mu c^2 - E_{\mathrm{SCO}})/\mu c^2$$

For the innermost stable orbit ($H^2 = 12$) situated at $r_{\mathrm{sco,min}} = 6GM/c^2$

$$\epsilon_{\mathrm{max}} = \left(1 - \frac{\sqrt{8}}{3}\right)\frac{\mu c^2}{\mu c^2} = 0.057$$

- This indicates that the maximum energy extraction from a Schwarzschild black hole is 0.057
  - Particles move in nearly circular orbits in an accretion disk until they reach the last stable orbit
  - In this process, heat is generated through viscosity in an accretion disk/flow and radiated way.
  - After the last stable orbit, the the particle quickly disappears and has no time to radiate away its gained energy

Assignment

- Starting from KN 5.9 verify that stable orbit occur at:

$$r_{\rm sco} = \frac{GM}{2c^2}\left[H^2 + (H^4 - 12H^2)^{1/2}\right]$$

- Verify that $\epsilon_{\rm max} = \left(1 - \frac{\sqrt{8}}{3}\right)\frac{\mu c^2}{\mu c^2} = 0.057$

Overview

- Last week:
  - Stationary BH
    - Schwarzschild metric
    - Effective potential for particles with mass
    - Stable and unstable orbits
    - Efficiency : 5.7 %
- This week
    - Radial orbits for particles with mass
    - Effective potential for photons
    - Criteria for photon escape capture cross section
  - Rotating BH
    - Effective potential
    - Orbits
    - Efficiencies
    - Ergosphere
  - Introduction to General relativity
    - Metrics and transformation rules
    - From a metric to a Lagrangian to the equation of motions

Last week

- These first integrals can be used together with the Schwarzschild line element:

$$\left(\frac{dr}{ds}\right)^2 + V^2(r) = E^2$$ (KN 5.9)

  - With as relativistic effective potential: $V^2(r) = \left(1 - \frac{2GM}{c^2 r}\right)\left(1 + \frac{h^2}{r^2}\right)$
- Define normalized angular momentum per unit mass/
- When $H \equiv c^2 h/(GM)$
  - V(r)$H > \sqrt{12}$n at r_{max} : unstable circular orbits
  - and a minimum at r_{min} : stable circular orbits
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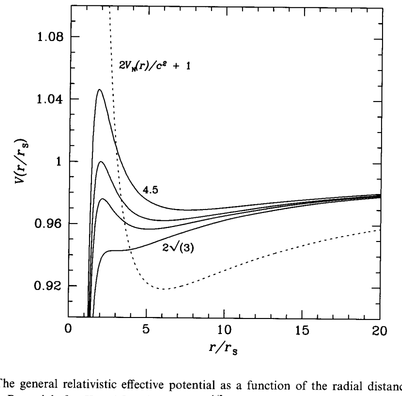

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Radial orbits

Start from KN 5-9, and take c=G=1

Then equation for purely radial motion (h=0): $\left(\frac{dr}{ds}\right)^2 = \frac{2M}{r} - (1 - E^2)$

Drop particle from rest at $r = r_i$, then $r_i = \frac{2M}{(1-E^2)}$, and $E \rightarrow 1$ when $r_i \rightarrow \infty$ (restmass)

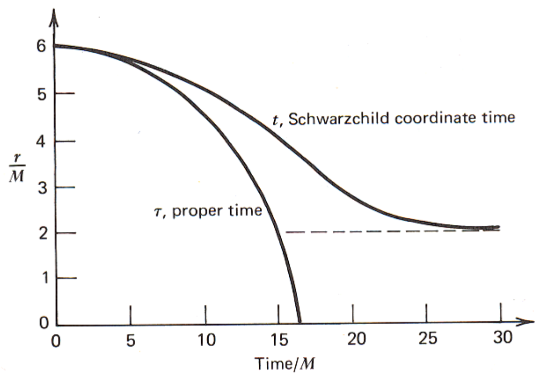
General solution is conveniently given in parameterized form (ST p. 343):

$$\tau = \sqrt{\frac{r_i^3}{8M}}(\eta + \sin\eta) \qquad r = \frac{r_i}{2}(1+\cos\eta), \qquad (0 \leq \eta \leq \pi)$$

Proper time τ_0 required to fall from r_i to $r=0$ is finite: $\tau_0 = \pi\sqrt{\frac{r_i^3}{8M}}$

Drop to $r=r_s$ (=2M: event horizon) takes infinite coordinate time:

$$\frac{t}{2M} = \ln\left|\frac{(R/2M-1)^{1/2}+\tan(\eta/2)}{(R/2M-1)^{1/2}-\tan(\eta/2)}\right| + \left(\frac{R}{2M} - 1\right)^{1/2}\left[\eta + \frac{R}{4M}(\eta + \sin\eta)\right]$$



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Mass-less particle orbits in a Schwarzschild metric

For a photon we can also derive a Lagrangian and then the equations of motions (ST p. 340):

$$\frac{dt}{d\lambda} = \frac{1}{b(1-2M/r)}$$
$$\frac{d\phi}{d\lambda} = \frac{1}{r^2}$$
$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)$$

The worldline only depends on the parameter, $b \equiv \frac{l}{E}$ which is the particles impact parameter.

In terms of an effective potential:

$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - V_{\text{phot}}(r)$ with $V_{\text{phot}} = \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)$

V_{phot} has a maximum of $1/(27M^2)$ at r = 3M

- If $b > 3\sqrt{3}M$ then there is a turning point at r>3M
- If $b < 3\sqrt{3}M$ then capture

-The capture cross section for photons from infinity:

$$\sigma_{\text{phot}} = \pi(3\sqrt{(3)}M)^2 = 27\pi M^2$$

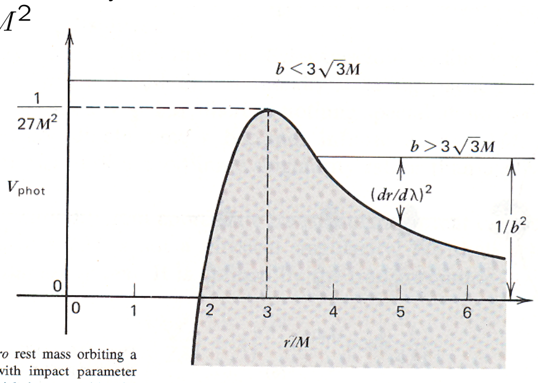


Figure 12.5 Sketch of the effective potential profile for a particle with *zero* rest mass orbiting a Schwarzschild black hole of mass M. If the particle falls from $r = \infty$ with impact parameter $b > 3\sqrt{3}\,M$ it is scattered back out to $r = \infty$. If, however, $b < 3\sqrt{3}\,M$ the particle is captured by the black hole.

The observed emission from gas near a black hole

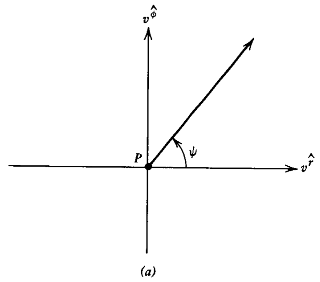
A photon escapes at r > 3M escapes only if

(i) $v^{\hat{r}} > 0$

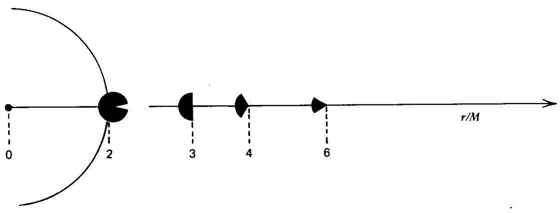
(ii) $v^{\hat{r}} < 0$ and $b > 3\sqrt{3}M$

An inward moving photon escapes the black hole if

$$\sin\psi \sim \frac{3\sqrt{3}M}{r}\left(1 - \frac{2M}{r}\right)^{1/2}$$



At r = 6M escape requires ψ < 150°
At r = 3M escape requires ψ < 90°: 50 % is captured



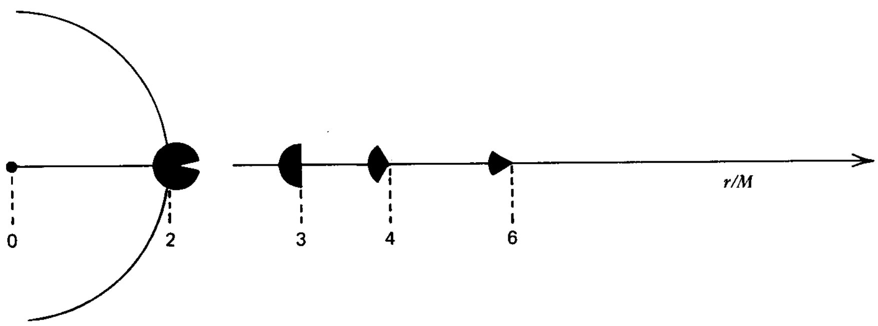


Figure 12.6 (a) The angle ψ between the propagation direction of a photon and the radial direction at a given point P. (b) Gravitational capture of radiation by a Schwarzschild black hole. Rays emitted from each point into the interior of the *shaded* conical cavity are captured. The indicated capture cavities are those measured in the orthonormal frame of a local static observer.

Table 5.1: The Criteria for Photon Escape

Direction	$r < 3M$	$r > 3M$
Inward	Certain capture	$\sin\Psi > \frac{3\sqrt{3}M}{r}\sqrt{1-\frac{2M}{r}}$
Outward	$\sin\Psi < \frac{3\sqrt{3}M}{r}\sqrt{1-\frac{2M}{r}}$	Certain escape

General orbits for particles with non-zero mass

Equation for $\dot{r}$ can be integrated to give $r = r(\tau)$ in terms of elliptic integrals for general orbits C83 § 3, and figures on p. 116—121.

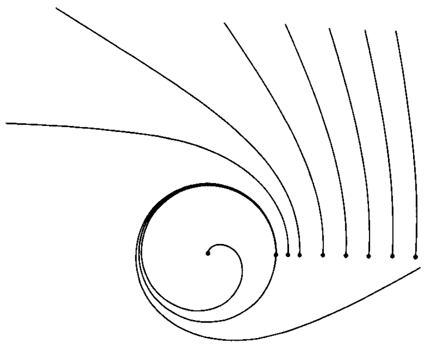


Figure 25.6.
The orbit of a photon in the "equatorial plane" of a black hole, plotted in terms of the Schwarzschild coordinates r and ϕ, for selected values of the turning point of the orbit, $r_{TP}/M = 2.99$, 3.00 (unstable circular orbit), 3.01, 3.5, 4, 5, 6, 7, 8, 9. The impact parameter is given by the formula $b = r_{TP}(1 - 2M/r_{TP})^{-1/2}$. In none of the cases shown, even for the inward plunging spiral, is the impact parameter less than $b_{crit} = (27)^{1/2} M$, nor are any of these orbits able to cross the circle $r = 3M$. That only happens for orbits with b less than b_{crit}. For such orbits there is no turning point; the photon comes in from infinity and ends up at $r = 0$: straight in for $b = 0$ (head-on impact); only after many loops near $r = 3M$, when $b/M = (27)^{1/2} - \varepsilon$, where ε is a very small quantity. Appreciation is expressed to Prof. R. H. Dicke for permission to publish these curves, which he had a digital calculator compute and plot out directly from the formula $d^2u/d\phi^2 = 3u^2 - u$, where $u = M/r$.

A rotating black hole

A Kerr line element in Boyer-Lindquist coordinates:

$$ds^2 = \frac{\Delta}{\rho^2}(cdt - a\sin^2\theta d\phi)^2 - \frac{\sin^2\theta}{\rho^2}[(r^2+a^2)d\phi - acdt]^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2$$

with *ac*, the angular momentum of the black hole about the polar axis

and $\Delta = r^2 + a^2 - 2GMr, \quad \rho^2 = r^2 + a^2\cos^2\theta$

and a=J/M is the angular momentum per unit mass (Kerr 1963)

The metric is *stationary* and *axisymmetric*, and describes a black hole rotating in the ϕ direction

The effective potential in the equatorial plane is defined as the minimum energy per unit mass which is required for possible motion at that point:

$$V(r) \equiv E_{\text{min}}(r)$$
$$= \frac{(r^2-2mr+a^2)^{1/2}\{r^2h^2+[r(r^2+a^2)+2a^2m]r\}^{1/2}+2ahm}{[r(r^2+a^2)+2a^2m]}$$

with $m=GM/c^2$

- The event horizon inside which particles can not escape:

$$r_+ = m + \sqrt{m^2 - a^2}$$

- a=0 : Schwarzschild radius
- a=m: the angular momentum per unit mass = mc:
  - the event horizon then disappears and we would witness a black hole with a ``naked singularity''
  - A major unsolved problem in general relativity is Penrose's Cosmic Censorship Conjecture that states that gravitational collapse from well behaved initial conditions never give rise to a naked singularity (ST p. 358)
  - Hence an event is required to exist, hence a<m
  - The case a → m is called *extreme Kerr* or *maximally rotating* black hole

The innermost stable orbit is:

$$r_{\text{sco,min}} = m\left[3 + B \mp \sqrt{(3-A)(3+A+2B)}\right]$$

with

$$A = 1 + (1-x^2)^{1/3}\left[(1+x)^{1/3} + (1-x)^{1/3}\right]$$
$$B = (3x^2 + A^2)^{1/2}$$

Minus sign: particle co-rotates with black hole: Kerr BH: $r_{\text{sco,min}} = m$
Plus sign: particle counter-rotates with black hole: Kerr BH: $r_{\text{sco,min}} = 9\ m$

The last stable orbit corresponds to a **maximum efficiency of energy extraction**:

$$\epsilon_{\text{max}} = 1 - \frac{r_{\text{sco,min}} - 2m \pm a(m/r_{\text{sco,min}})^{1/2}}{\sqrt{r_{\text{sco,min}}[r_{\text{sco,min}} - 3m \pm 2a(m/r_{\text{sco,min}})^{1/2}]}}$$

For a Kerr BH this is 42 % (compare to 5.7 % for Schwarzschild BH)

Thorne (1974): include physics of radiation and losses to BH, then $a_{\text{max}} = 0.998M$ and $\varepsilon \sim 30\%$ at maximum

- The Kerr black hole has a critical radius, called the *static limit*.
- Within this limit the light cones always point in the ϕ direction, because of inertial frame dragging by the spinning black hole
- Particles within this limit are always in orbit wrt non-rotating observers at infinity: stationary observers are impossible, and local coordinate frames are dragged by the hole: every realizable frame of motion must rotate.
- The static limit is given by

$$r_E = m + \sqrt{m^2 - a^2\cos^2\theta}$$

- The region between the event horizon and the static limit is called the *ergosphere*

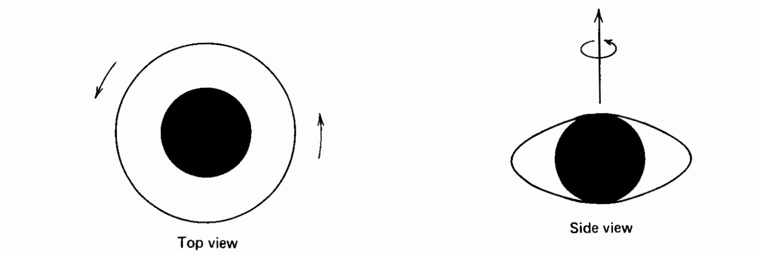


Figure 12.10 Ergosphere of a Kerr black hole: the region between the static limit [the flattened outer surface $r = M + (M^2 - a^2\cos^2\theta)^{1/2}$] and the event horizon [inner sphere $r = M + (M^2 - a^2)^{1/2}$].

General relativity

- General relativity is a relativistic theory of gravitation
- Newtonian gravitation a field theory for a scalar field ϕ following Poisson's equations:

$$\nabla^2\phi = 4\pi G\rho_0$$

  – With gravitational acceleration : $-\nabla\phi$

- General relativity is a geometric theory of gravitation.

- Let's start with special relativity:
- The distance or interval between two nearby events in space time is given by:

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

  – Note that *ds* does not depend on which frame is used to evaluate it: it is Lorentz invariant
  – Ds^2=-cdt^2 elapsed time in particles restframe

- Use $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$
- To write: $ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta$ with a 4x4 diagonal matrix:
  – Note: summation convention
  – $\eta_{\alpha\beta}$ metric tensor, gives a geometric description of space time: *Minkowski space* $\eta_{\alpha\beta} = \text{diag}(-1,1,1,1)$

- One could use another (non-inertial) coordinate system y^α such as:
  – Polar coordinates
  – Coordinate system of an accelerated observer
- Relation of the two coordinate systems:

$$x^\alpha = x^\alpha(y^\gamma)$$

- In the new coordinate system the 'interval' is now:

$$ds^2 = g_{\alpha\beta}(y^\gamma)dy^\alpha dy^\beta$$

  – With (using the chain rule): $g_{\alpha\beta}(y^\gamma) = \frac{\partial x^\lambda}{\partial y^\alpha}\frac{\partial x^\sigma}{\partial y^\beta}\eta_{\lambda\sigma}$

- Although this looks complicated, space time is still flat: there exist transformation such that the metric takes the form:
- In general relativity, an interval is given by: $ds^2 = g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta$
- But now there is not transformation that can reduce the metric to the simple form: $\eta_{\alpha\beta} = \text{diag}(-1,1,1,1)$
  Space time is curved
- $g_{\alpha\beta}$ gives a geometric description of gravity: GR metric tensor

- *ds* is still invariant, so the transformation of $g_{\alpha\beta}$ from coordinates $\overline{x}^\alpha$ to coordinates x^α is given by:

$$g_{\alpha\beta} = \frac{\partial \overline{x}^\lambda}{\partial x^\alpha}\frac{\partial \overline{x}^\sigma}{\partial x^\beta}\overline{g}_{\lambda\sigma}$$

As in special relativity, *ds* measures the proper time $d\tau$ along a worldline of a particle: $ds^2 = -c^2\, d\tau^2$

$ds^2 = g_{\alpha\beta}(x^\gamma)dx^\alpha dx^\beta$ can be written as a dot product between two vectors:

$$ds^2 = d\vec{\mathbf{x}} \cdot d\vec{\mathbf{x}}$$

Where the dot product is defined as: $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} \equiv g_{\alpha\beta}A^\alpha B^\alpha$

Writing conventions: $A^\alpha \equiv g^{\alpha\beta}A_\beta$ and $A_\alpha \equiv g_{\alpha\beta}A^\beta$

with $g^{\alpha\beta}$ is the matrix inverse of $g_{\alpha\beta}$

Then: $\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = A_\alpha B^\beta = A^\alpha B_\beta = A_\alpha B_\beta g^{\alpha\beta}$

- A local inertial frame is given by a coordinate system such that:

$$ds^2 = \left[\eta_{\alpha\beta} + \mathcal{O}(|x|^2)\right] dx^\alpha dx^\beta$$

- Measurements in the local inertial frames are carried out as in special relativity.
- Principle of equivalence: all (non-gravitational) laws of physics are the same in local inertial frames as they are in special relativity
- Exemplified by the famous elevator thought experiment
  - Accelerated observer = stationary in a gravitational field
  - Free falling in uniform field = no observable effects of the gravitational field

Intermezzo: the principle of least action

- Non-relativisitic Langragian
  - L = Kinetic Energy - Potential energy
- Action $\int_{\text{trajectory}} \mathcal{L} dt$
- Correct path is found by maximizing the action which is equivalent to solving the Euler-Lagrange equations:

$$\frac{dp_i}{dt} \equiv \frac{d}{dt}(\frac{\partial \mathcal{L}}{\partial \dot{x}_i}) = \frac{\partial \mathcal{L}}{\partial x_i}$$

- Where $i = 1, 2, 3, ...$ labels the systems degree of freedom and the coordinate of each is x_i

Motions of test particles

- In the case of special relativity, one can get motions of particles, from a variational principle that extremizes the distance (interval) along a world line:

$$\delta \int ds = 0$$

- Rewrite: $ds = (-\eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta)^{1/2} d\lambda$
- Where: $\dot{x}^\alpha \equiv \frac{dx^\alpha}{d\lambda}$
- With λ any parameter along the world line
- The Lagrangian then is: $L = (\eta_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta)^{1/2}$
- And the Euler-Lagrangian equations: $\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) = \frac{\partial L}{\partial x^\alpha}$
- Define conjugate momentum four vector: $p_\alpha = \frac{\partial L}{\partial \dot{x}^\alpha}$
- Leading to the EL equations in the form: $\frac{d}{d\lambda} p_\alpha = \frac{\partial L}{\partial \dot{x}^\alpha}$

- The GR Lagrangian now becomes:

$$L = \left[-g_{\alpha\beta}(x^\gamma)\dot{x}^\alpha\dot{x}^\beta\right]^{1/2}$$

Choose λ such that L is constant along a world line. Then (see :

$$L = \frac{1}{2}g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$$

- Recall the Schwarzschild metric (take c = G = 1):

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

- Then we find as Lagrangian:

$$2L = -\left(1-\frac{2M}{r}\right)\dot{t}^2 + \left(1-\frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2$$

- The Euler Lagrange equations are:

$$\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0, \quad x^\alpha = (t, r, \theta, \phi)$$

- Which gives

$$\frac{d}{d\lambda}(r^2\dot{\theta}) = r^2\sin\theta\cos\theta\dot{\phi}^2$$

$$\frac{d}{d\lambda}(r^2\sin^2\theta\dot{\phi}) = 0$$

$$\frac{d}{d\lambda}\left[\left(1-\frac{2M}{r}\right)\dot{t}\right] = 0$$

  – Where λ = τ/m and $\dot{t} \equiv dt/d\lambda = p^t$ is the t-component of the 4-momentum, etc

Assignment

- The Lagrangian for a certain metric can found using (ST 5.2.21):
$$L = \frac{1}{2} g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta$$
- Check that for the Schwarzschild metric (ST 12.3.1):
$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$
- One finds (ST 12.4.1): $2L = -\left(1 - \frac{2M}{r}\right)\dot{t}^2 + \left(1 - \frac{2M}{r}\right)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2$
- Given the EL equation (ST 12.4.2): $\frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = 0, \quad x^\alpha = (t, r, \theta, \phi)$

- Find the equations of motion:
$$\frac{d}{d\lambda}(r^2\dot{\theta}) = r^2 \sin\theta\cos\theta\dot{\phi}^2$$
$$\frac{d}{d\lambda}(r^2\sin^2\theta\dot{\phi}) = 0$$
$$\frac{d}{d\lambda}\left[\left(1 - \frac{2M}{r}\right)\dot{t}\right] = 0$$

Relation to the starting point KN (page 8)

The Schwarzschild line element *ds* describes space-time near a stationary mass M:
$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

From *ds* we just derived a Lagrangian and then the equations of motions (ST p. 340):

θ = constant, (can be taken to be $\theta = \pi/2$): particle moves in a plane

$$r^2\frac{d\phi}{ds} = constant = h$$ conservations of angular momentum

$$\frac{dt}{ds} = \frac{E}{c}\left(1 - \frac{2GM}{c^2 r}\right)^{-1}$$ conservations of energy (KN 5.8)

Summarizing

- Choose metric
  - Schwarzschild for a stationary BH
  - Kerr for a rotating BH
- Calculate Lagrangian
- Derive equation of motion
- Determine trajectories
  - Mass-less particles
  - Massive particles
- Efficiency of energy generation

Literature

- ST: § 5.1, 5.2 until eq. 5.2.12
- ST: §12.1, 12.2, 12.3, 12.4, 12.5, 12.7
- KN: p.101-109 (skip Lens-Thirring precession)