Some notes on the ideal fermion gas

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Abstract. These are notes on the physics of the ideal fermion gas collected from various sources. The text provides a deeper understanding of the behaviour of degenerate fermion gases in order to apply this to white dwarfs and neutron stars.

The discussion in this note concerns the academic case of a pure (consisting of one type of particle) ideal fermion gas. What is neglected are the interactions between the particles (e.g., the Coulomb force in the case of electrons) and the fact that in practice the fermions will be embedded in a gas consisting of a number of different kinds of particles (think of the carbon, oxygen, etc nuclei in white dwarfs and the neutrons and protons in neutron stars). This will introduce the important force of gravity of the heavier particles and alter the equation of state. Lastly, processes arising from nuclear physics are completely ignored.

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## References


1 The ideal gas

In this section the properties of ideal gases are reviewed. The ideal gas is often also called a perfect gas and is defined in Mandl (1988, chapter 7) as follows: The perfect gas represents an idealization in which the potential energy of interaction between gas particles is negligible compared to their kinetic energy of motion. This definition implies that one can write down a set of private energies $\varepsilon_i$ for each particle, which simplifies the application of statistical physics to gases enormously. In practice one can indeed often neglect the interaction energies of particles.

However, quantum effects cannot be neglected so readily and one needs to take the particle concentration into account. The ideal gas usually treated in elementary textbooks is therefore the classical ideal gas for which the condition is that the probability is very small that any single-particle state is occupied by more than one particle. An ideal gas for which this last condition does not hold is an ideal quantal gas. These kinds of gases have to be treated with quantum statistics. In the book by Mandl (1988) this is done extensively for the non-relativistic regime, while in Phillips (1998) the ideal quantal gas is treated in both the relativistic and non-relativistic regimes. The next section summarizes the results for the non-relativistic regime from Mandl (1988).

1.1 The ideal non-relativistic quantal gas

Chapters 9 and 11 in Mandl (1988) discuss statistical physics of quantal gases, with chapter 11 focusing on fermions and bosons. For these particles the mean occupation number $f_i$ for the $i$-th single-particle state, with energy $\varepsilon_i$, is given by:

$$f_i = \frac{1}{e^{\beta(\varepsilon_i - \mu)} + 1},$$

(1)

where the plus sign is for fermions, giving the Fermi-Dirac (FD) distribution, and the minus sign is for bosons, for which the Bose-Einstein (BE) distribution holds. In this equation $\beta = 1/kT$ and $\mu$ is the chemical potential. The latter is a thermodynamic quantity relevant for systems with a variable number of particles (e.g., chemical reactions). From here on we concentrate on fermions for which equation (1) shows that $f_i$ is never larger than 1. This reflects the Pauli exclusion principle which states that no more than one fermion can occupy a given quantum state.

In order to obtain the distribution of particles as a function of energy the mean occupation number has to be multiplied by the so-called density of states, which basically describes in how many ways a particular energy state can be attained for a particle. In terms of momentum $p$ the density of states $g(p)$ is (see Phillips 1998, chapter 2):

$$g(p)dp = g_s \frac{4\pi V}{h^3} p^2 dp,$$

(2)

where $g_s$ is the degeneracy of each state (for example $g_s = 2$ for particles with two possible polarizations or spins for the same momentum). The particles are confined to the volume $V$ and $h$ is the Planck constant. The number of particles at a certain energy is then given by $N(\varepsilon_i) \propto f_i g(p_i)$, where $p_i = p(\varepsilon_i)$. The distribution of the particles over the different energies can then be written in terms of the momentum $p$ as:

$$N(p) dp = g_s \frac{4\pi V}{h^3} \frac{p^2}{e^{\beta(\varepsilon_p - \mu)} + 1} dp,$$

(3)

where the function $f_i$ is now written in its continuum form $f(\varepsilon)$. This expression is entirely general for ideal fermion gases as we have not yet specified the relation between momentum and energy.

In the non-relativistic regime considered here one can write:

$$\varepsilon = \frac{p^2}{2m},$$

(4)

where $m$ is the mass of the fermion. The density of states in terms of energy then becomes:

$$g(\varepsilon) d\varepsilon = g_s \frac{2\pi V}{h^3} (2m)^{3/2} \varepsilon^{1/2} d\varepsilon.$$

(5)
The particle number $N(\varepsilon)$ per energy interval now becomes:

$$N(\varepsilon)\, d\varepsilon = f(\varepsilon)\, g(\varepsilon)\, d\varepsilon = g_s \frac{2\pi V}{\hbar^3} (2m)^{3/2} \frac{\varepsilon^{1/2}}{e^{\beta(\varepsilon-\mu)} + 1} \, d\varepsilon. \quad (6)$$

The total number of fermions is now obtained by integrating (6) which results in:

$$N = g_s \frac{2\pi V}{\hbar^3} (2m)^{3/2} \int_0^\infty \frac{\varepsilon^{1/2}}{e^{\beta(\varepsilon-\mu)} + 1} \, d\varepsilon. \quad (7)$$

In terms of the particle concentration, or number density, $n$ the last two equations can be written as:

$$n(\varepsilon)\, d\varepsilon = f(\varepsilon)\, g(\varepsilon)\, d\varepsilon = g_s \frac{2\pi}{\hbar^3} (2m)^{3/2} \frac{\varepsilon^{1/2}}{e^{\beta(\varepsilon-\mu)} + 1} \, d\varepsilon, \quad (8)$$

and

$$n = g_s \frac{2\pi}{\hbar^3} (2m)^{3/2} \int_0^\infty \frac{\varepsilon^{1/2}}{e^{\beta(\varepsilon-\mu)} + 1} \, d\varepsilon. \quad (9)$$

Equations (7) and (9) provide an implicit definition of the chemical potential for a fixed number of fermions in a fixed volume (eq. 6), or for a fixed number density (eq. 9). For a fixed number and volume the chemical potential is a function of temperature only $\mu = \mu(T)$ (cf. Mandl, 1988, section 11.5.1).

### 1.2 The classical dilute gas

If the condition

$$e^{-\beta \mu} \gg 1 \quad (10)$$

is satisfied the term $+1$ in $f_i$ in equation (1) can be ignored and the average occupation of each state is much lower than 1. In this case the fermion gas obeys Maxwell-Boltzmann statistics and is said to behave like a dilute classical gas. This can be verified by starting from equation (8) and using the assumption above in which case the following expression for $n(\varepsilon)$ holds:

$$n(\varepsilon)\, d\varepsilon = e^{\mu/kT} g_s \frac{2\pi}{\hbar^3} (2m)^{3/2} e^{-\varepsilon/2kT} \varepsilon^{1/2} d\varepsilon, \quad (11)$$

which in terms of momentum (using expression 4) becomes:

$$n(p)\, dp = e^{\mu/kT} g_s \frac{1}{\hbar^3} e^{-p^2/(2mkT)} 4\pi p^2 \, dp, \quad (12)$$

or in terms of particle velocities:

$$n(p)\, dp = e^{\mu/kT} g_s \left(\frac{m}{\hbar}\right)^3 e^{-mv^2/(2kT)} 4\pi v^2 \, dv. \quad (13)$$

Equations (11)–(13) represent of course the familiar Maxwell-Boltzmann distribution of particle energies and momenta for the ideal classical gas. In this case the chemical potential can be determined from:

$$n = \int_0^\infty n(\varepsilon)\, d\varepsilon \quad \text{and} \quad \int_0^\infty e^{-x}\, x^{1/2} \, dx = \frac{\sqrt{\pi}}{2}.$$  

The resulting chemical potential is:

$$\mu = kT \ln \left( \frac{n}{g_s n_Q} \right),$$

where $n_Q$ is the so-called quantum concentration:

$$n_Q = \left( \frac{2\pi mkT}{\hbar^2} \right)^{3/2}. $$
1.3 The degenerate non-relativistic fermion gas

We return now to the quantal gas and define the Fermi energy $\varepsilon_F$ as the value of the chemical potential at the absolute zero temperature:

$$\varepsilon_F \equiv \mu(0).$$

(14)

It is shown in [Mandl (1988)] that $\varepsilon_F$ must be positive. As $T \to 0$, $\beta \to \infty$ which means that $\exp[\beta(\varepsilon - \mu)]$ tends to $\exp(\mp \infty)$ for $\varepsilon < \varepsilon_F$ and $\varepsilon > \varepsilon_F$, respectively. That is, $f(\varepsilon)$ tends to 1 or 0 for $\varepsilon < \varepsilon_F$ and $\varepsilon > \varepsilon_F$ respectively. The fermion gas in this state is said to be completely degenerate.

The distribution of particles over the momenta then becomes:

$$n(p) \, dp = \begin{cases} 
  g_s \frac{4\pi}{h^3} p^2 \, dp & \text{for } p \leq p_F \\
  0 & \text{for } p > p_F 
\end{cases},$$

(15)

while the distribution over energies becomes:

$$n(\varepsilon) \, d\varepsilon = \begin{cases} 
  g_s \frac{2\pi}{h^3} (2m)^{3/2} \varepsilon^{1/2} \, d\varepsilon & \text{for } \varepsilon \leq \varepsilon_F \\
  0 & \text{for } \varepsilon > \varepsilon_F 
\end{cases}.$$  

(16)

Figure 1: The mean occupation number $f(\varepsilon)$ (left panel) and the corresponding number density of particles $n(\varepsilon)$ (right panel) as a function of energy for a nitrogen (N$_2$) gas at room temperature and atmospheric pressure. The energy is expressed in units of $kT$.

N$_2$ gas at $T = 300$ K, $P = 10^5$ N m$^{-2}$

Substituting this value of the chemical potential in (11) one correctly obtains the expression:

$$n(\varepsilon) \, d\varepsilon = \frac{2}{\pi^{1/2}} \frac{1}{(kT)^{3/2}} e^{-\varepsilon/kT} \varepsilon^{1/2} \, d\varepsilon.$$
1.3 The degenerate non-relativistic fermion gas

The quantity \( p_F \) is the Fermi momentum corresponding to the Fermi energy. The interpretation of this particle distribution is as follows. At \( T = 0 \) K the gas is in its lowest energy state but because of the Pauli exclusion principle not all \( N \) fermions can crowd in to the single-particle state of lowest energy. The Fermi energy then is the topmost energy level occupied at \( T = 0 \) K. The energy distribution at \( T = 0 \) K will be proportional to \( \varepsilon^{1/2} \) with a sharp cutoff at \( \varepsilon = \varepsilon_F \). Figures 2 shows the occupation number and energy distribution for the completely degenerate fermion gas.

The Fermi energy can now be derived from equation (16):

\[
n = g_s \frac{2\pi}{h^3} (2m)^{3/2} \int_0^{\varepsilon_F} \varepsilon^{1/2} d\varepsilon = g_s \frac{2\pi}{h^3} (2m)^{3/2} \frac{2}{3} \varepsilon_F^{3/2},
\]

(17)

which leads to:

\[
\varepsilon_F = \frac{\hbar^2}{2m} \left( \frac{3}{g_s 4 \pi} \right)^{2/3} n^{2/3} = \frac{\hbar}{2m} \left( \frac{6\pi^2}{g_s} \right)^{2/3} n^{2/3}.
\]

(18)

From this result it is clear that \( \varepsilon_F \) depends on the fermion mass \( m \) and the particle concentration \( n \). The Fermi temperature is defined as:

\[
\varepsilon_F = kT_F.
\]

(19)

In terms of momentum one can write for the degenerate case:

\[
n = g_s \frac{4\pi}{h^3} \int_0^{p_F} p^2 dp,
\]

(20)

where \( p_F \) is now the Fermi-momentum. Working out the integral above and solving for \( p_F \) gives:

\[
p_F = \hbar \left( \frac{3}{g_s 4 \pi} \right)^{1/3} n^{1/3} = \hbar \left( \frac{3}{g_s 2 \pi^2} \right)^{1/3} n^{1/3}.
\]

(21)

So far the (ideal) fermion gas has been treated in the non-relativistic regime. However, for the application to stars the possibility of relativistic energies has to be taken into account and this is done in Phillips (1998). The following material is extracted from that text and further worked out. It will become clear that the energy distribution for a fermion gas will be slightly different when the full relativistic formulation for particle energies is taken into account.
2 Particle energies and momenta in special relativity

I first recall the full expressions for energy and momentum for the particles in a gas according to special relativity. The energy $\epsilon_p$ of a particle of mass $m$ in quantum state with momentum $p$ is then given by:

$$\epsilon_p^2 = p^2 c^2 + m^2 c^4,$$

(22)

where $c$ is the speed of light. The speed of the particle $v_p$ is given by:

$$v_p = \frac{pc}{\epsilon_p},$$

(23)

which is consistent with the familiar form:

$$p = \frac{mv_p}{\sqrt{1 - (v_p/c)^2}}.$$  

(24)

For later use the following expressions for the energy and momentum are useful:

$$\frac{\epsilon_p}{mc^2} = \left[\left(\frac{p}{mc}\right)^2 + 1\right]^{1/2},$$

(25)

and

$$\frac{p}{mc} = \left[\left(\frac{\epsilon_p}{mc^2}\right)^2 - 1\right]^{1/2}.$$  

(26)

These expressions give the energy and momentum in a form normalized to the rest-mass energy and the dividing line between non-relativistic and relativistic momenta. In addition the following relation between energy and momentum is useful.

$$\frac{d\epsilon_p}{dp} = \frac{pc}{\epsilon_p} = v_p \quad \text{or} \quad dp = \frac{\epsilon_p}{pc^2} d\epsilon_p = \frac{1}{v_p} d\epsilon_p,$$

(27)

For most applications it is the kinetic energy of the gas that is important as only gas particles in motion can exert a pressure, transport energy, etc. The kinetic energy $\epsilon_{\text{kin}}$ is simply given by:

$$\epsilon_{\text{kin}} = \epsilon_p - mc^2.$$  

(28)

Similar useful relations to the ones for $\epsilon_p$ and $p$ can be derived in this case:

$$\frac{\epsilon_{\text{kin}}}{mc^2} = \left[\left(\frac{p}{mc}\right)^2 + 1\right]^{1/2} - 1,$$

(29)

and

$$\frac{p}{mc} = \left[\left(\frac{\epsilon_{\text{kin}}}{mc^2} + 1\right)^2 - 1\right]^{1/2},$$

(30)

and

$$\frac{d\epsilon_{\text{kin}}}{dp} = \frac{pc^2}{\epsilon_{\text{kin}} + mc^2} \quad \text{or} \quad dp = \frac{\epsilon_{\text{kin}} + mc^2}{pc^2} d\epsilon_{\text{kin}}.$$  

(31)

\(^3^)\text{The variable $\epsilon$ rather than $\varepsilon$ is used here for energy in order emphasise that the relativistic formulation is used.}\)
3 The relativistic description of the Fermion gas

With the energy of the fermions $\epsilon_p$ defined in the previous section we can now write for the mean occupation number:

$$f(\epsilon_p) = \frac{1}{e^{(\epsilon_p - \mu)/kT} + 1} = \frac{1}{e^{(\epsilon_{\text{kin}} - (\mu - mc^2))/kT} + 1}, \quad (32)$$

so that the condition for a classical gas now becomes:

$$e^{(mc^2 - \mu)/kT} \gg 1 \quad (33)$$

The density of states is still given by equation (2) so for the energy distribution we can write:

$$n(p) \, dp = f(\epsilon_p) g(p) \, dp, \quad (34)$$

and

$$n = \int_0^\infty f(\epsilon_p) g(p) \, dp. \quad (35)$$

In addition to classical or quantal the gas can now be ultra-relativistic, in which case for a significant fraction of the particles $p \gg mc$, or non-relativistic, if for most of the particles $p \ll mc$. This leads to four possible extreme states of the fermion gas: classical and non-relativistic, classical and ultra-relativistic, degenerate and non-relativistic, degenerate and ultra-relativistic. At this point it should be realized that when labelling the energies of the states that the fermions can occupy there is no point in counting the rest mass energy $mc^2$. This is the same for all particles and cannot be used to distinguish quantum states. Therefore the equation for $f$ and $n$ can be written in terms of $\epsilon_{\text{kin}}$, the kinetic energy of the fermions (i.e., the energy over and above the rest mass energy).

The way to formally account for the rest mass energy is to define a modified chemical potential:

$$\bar{\mu} = \mu - mc^2. \quad (36)$$

The basic equations are straightforward to write down as we are simply relabelling the energy levels from $\epsilon_p$ to $\epsilon_{\text{kin}}$ and the chemical potential from $\mu$ to $\bar{\mu}$:

$$f(\epsilon_{\text{kin}}) = \frac{1}{e^{(\epsilon_{\text{kin}} - \bar{\mu})/kT} + 1}, \quad (37)$$

and

$$n(p) \, dp = f(\epsilon_{\text{kin}}) g(p) \, dp. \quad (38)$$

In order to derive the distribution of particles over kinetic energy I now work out equation (38) and I make use of:

$$n(p) \, dp = n(\epsilon_{\text{kin}}) \, d\epsilon_{\text{kin}} \quad \text{or} \quad n(\epsilon_{\text{kin}}) = n(p) \frac{dp}{d\epsilon_{\text{kin}}}. \quad (39)$$

Starting from:

$$n(p) \, dp = f(\epsilon_{\text{kin}}) g_s \frac{4\pi}{h^3} p^2 \, dp \quad (40)$$

the distribution in energy can be found by using the relation above and eq. (31). The expression for the number density then becomes:

$$n(\epsilon_{\text{kin}}) \, d\epsilon_{\text{kin}} = f(\epsilon_{\text{kin}}) g_s \frac{4\pi}{h^3} p^2 \frac{dp}{d\epsilon_{\text{kin}}} \, d\epsilon_{\text{kin}}$$

$$= f(\epsilon_{\text{kin}}) g_s \frac{4\pi}{h^3} \frac{\epsilon_{\text{kin}} + mc^2}{c^2} \, d\epsilon_{\text{kin}}$$

$$= f(\epsilon_{\text{kin}}) g_s \frac{4\pi}{h^3} \frac{p}{mc} \frac{mc^2(\epsilon_{\text{kin}}/mc^2 + 1)}{c} \, d\epsilon_{\text{kin}}. \quad (41)$$
Now the value of \( p/mc \) can be substituted by using eq. (30) and at the same time I express the energies in terms of \( mc^2 \).

\[
n \left( \frac{\epsilon_{\text{kin}}}{mc^2} \right) d \left( \frac{\epsilon_{\text{kin}}}{mc^2} \right) = f(\epsilon_{\text{kin}}) g_s \frac{4\pi m}{h^3} e \left( \left( \frac{\epsilon_{\text{kin}}}{mc^2} + 1 \right)^2 - 1 \right)^{1/2} mc^2 \left( \frac{\epsilon_{\text{kin}} + mc^2}{mc^2} \right) mc^2 d \left( \frac{\epsilon_{\text{kin}}}{mc^2} \right).
\]

Note the extra factor \( mc^2 \) coming from \( d(\epsilon_{\text{kin}}/mc^2) \). The final result then is:

\[
n(\epsilon'_{\text{kin}}) = f(\epsilon'_{\text{kin}}) g_s \frac{4\pi}{h} \left( \frac{mc}{h} \right)^3 (\epsilon'_{\text{kin}} + 1) \left( (\epsilon'_{\text{kin}} + 1)^2 - 1 \right)^{1/2},
\]  

(42)

where \( \epsilon'_{\text{kin}} = \epsilon_{\text{kin}}/mc^2 \). Note that \( f(\epsilon_{\text{kin}}) = f(\epsilon'_{\text{kin}}) \) because \( \bar{\mu} \) and \( kT \) are also expressed in terms of \( mc^2 \). The units of \( mc/h \) are \( m^{-1} \) which means that \( n(\epsilon'_{\text{kin}}) \) is indeed a number density (for a given energy interval).

The Fermi energy now becomes (in analogy to equation (17)):

\[
n = g_s \frac{4\pi}{h} \left( \frac{mc}{h} \right)^3 \int_0^{\epsilon_F} (\epsilon'_{\text{kin}} + 1) \left( (\epsilon'_{\text{kin}} + 1)^2 - 1 \right)^{1/2} d\epsilon'_{\text{kin}} = g_s \frac{4\pi}{h} \left( \frac{mc}{h} \right)^3 \frac{1}{3} \left( (\epsilon_F' + 1)^2 - 1 \right)^{3/2},
\]  

(43)

where \( \epsilon_F' = \epsilon_F/mc^2 \). In the limit \( \epsilon_F/mc^2 \ll 1 \) the expression (17) is recovered. Thus the expression for the Fermi energy now becomes:

\[
\frac{\epsilon_F}{mc^2} = \left( \frac{3}{4\pi} \frac{n}{g_s} \right)^{2/3} \left( \frac{h}{mc} \right)^2 + 1 \right]^{1/2} - 1,
\]  

(44)

or with the particle density \( n \) normalized to \( g_s 4\pi (mc/h)^3 \):

\[
\frac{\epsilon_F}{mc^2} = \left( \frac{3n}{g_s 4\pi (mc/h)^3} \right)^{2/3} \left( \frac{h}{mc} \right)^2 + 1 \right]^{1/2} - 1.
\]  

(45)

The Fermi-momentum can again be derived using equation (20) which leads to the same expression for \( p_F \). Normalizing to \( mc \) we obtain:

\[
\frac{p_F}{mc} = \left( \frac{3n}{g_s 4\pi (mc/h)^3} \right)^{1/3}.
\]  

(46)

Note that the expression for the Fermi-energy could thus have been obtained more easily by starting from \( p_F \) and then using the now different relation between energy and momentum as given by equation (25). Keep in mind the subtraction of the rest-mass energy.

### 4 The fermion gas phases

With the expressions for the Fermi energy in the non-relativistic case (eq. (18)) and the relativistic expression (eq. (44)) one can proceed to draw the \( T \) vs \( n \) diagram and broadly indicate where the gas is degenerate/non-degenerate and/or relativistic or non-relativistic. I first write the expressions for the Fermi energies (where NR stands for non-relativistic) as follows:

\[
\frac{\epsilon_{F,NR}}{mc^2} = \frac{1}{2} \left( \frac{3n}{g_s 4\pi (mc/h)^3} \right)^{2/3} = \frac{1}{2} \left( \frac{n}{n_0} \right)^{2/3},
\]  

(47)

and

\[
\frac{\epsilon_F}{mc^2} = \left( \frac{n}{n_0} \right)^{2/3} + 1 \right]^{1/2} - 1,
\]  

(48)

where the density normalization is:

\[
n_0 = g_s \frac{4\pi}{3} \left( \frac{mc}{h} \right)^3.
\]  

(49)
Figure 3: Phase diagram for fermions in the $\log T$ vs. $\log n$ plane, with $T$ expressed in units of $mc^2/k$ and $n$ expressed in units of $n_0$. The equations of state for the four extreme phases are also shown.

Figure 4: Phase diagram for electrons in the $\log T$ vs. $\log n$ plane.
For the Fermi-momentum the expression becomes:

$$\frac{p_F}{mc} = \left( \frac{n}{n_0} \right)^{1/3}. \quad (50)$$

The $T$ vs. $n$ plane can now be divided in four regions containing degenerate (D) or non-degenerate (ND) gas and non-relativistic (NR) or ultra-relativistic (UR) gas. These regions overlap leading to the combinations D-NR, D-UR, ND-NR, and ND-UR. The dividing line between the NR and UR cases can be found by realizing that the fermion gas becomes ultra-relativistic when $(3/2)kT \gg mc^2$ or $\epsilon_F \gg mc^2$. The dividing lines are then given by:

$$\frac{3}{2}kT \approx mc^2 \quad \vee \quad \frac{\epsilon_F}{mc^2} \approx 1, \quad (51)$$

which can be written as:

$$T = \frac{2mc^2}{3k} \quad \vee \quad n = 3^{3/2}n_0. \quad (52)$$

The latter expression follows from equation (48), setting $\epsilon_F/mc^2$ to 1.

The dividing line between the degenerate and non-degenerate fermion gas can be found by setting the temperature equal to the Fermi temperature, i.e.:

$$\frac{3}{2}kT \approx \epsilon_F, \quad (53)$$

which leads to:

$$T = \frac{2mc^2}{3k} \left[ \left( \frac{n}{n_0} \right)^{2/3} + 1 \right]^{1/2} - 1. \quad (54)$$

The latter expression can be simplified in the non-relativistic ($n \ll n_0$) and ultra-relativistic regimes ($n \gg n_0$) to:

$$T_{NR} = \frac{1}{3} \frac{mc^2}{k} \left( \frac{n}{n_0} \right)^{2/3} \quad \text{and} \quad T_{UR} = \frac{2}{3} \frac{mc^2}{k} \left( \frac{n}{n_0} \right)^{1/3}. \quad (55)$$

Figure 3 shows the phase diagram for fermions in the $T$ vs. $n$ plane, with $T$ expressed in units of $mc^2/k$ and $n$ expressed in units of $n_0$. The three dividing lines derived above are indicated and the various regions are labelled. Note that setting $n/n_0 = 3^{3/2}$ in expression (54) leads to $T = (2/3)mc^2/k$, meaning that the various dividing lines meet in one point.

For fermions of a given rest mass and with known value of $g_s$, figure 3 can be translated to actual physical units. This is shown in figure 4 for electrons. For these fermions $g_s = 2$ and $m = 9.10938215 \times 10^{-31}$ kg, which leads to $\log n_0 = 35.77$ (kg m$^{-3}$) and $\log(mc^2/k) = 9.77$ (K).

### 5 The equation of state for the fermion gas

From the distribution of particles over energy one can also derive expressions for the pressure as function of the particle density and temperature, i.e. the equation of state. In chapter 2 of Phillips (1998) it is shown that for the classical regime one always obtains:

$$P = nkT. \quad (56)$$

Thus the well-known classical ideal gas equation of state holds for the non-relativistic and ultra-relativistic regimes.

In order to derive the equation of state for the degenerate fermion gas I first write down the general expression for the pressure of the gas:

$$P = \frac{1}{3} \int_0^\infty n(p)pv_p dp$$

$$= g_s \frac{4\pi}{3} \frac{1}{\hbar^3} \int_0^\infty pv_p f(\epsilon_{\text{kin}})p^2 dp. \quad (57)$$
This result is derived in Weiss et al. (2004, chapter 10) and can also be found in Ostlie & Carroll (2007, section 10.2). This equation can be worked out further using equations (23), (30), and (31):

\[ P = g_s \frac{4\pi}{3} \frac{1}{h^3} \int_0^\infty f(\epsilon_{\text{kin}}) v_p p^3 \, dp \]

\[ = g_s \frac{4\pi}{3} \frac{1}{h^3} \int_0^\infty f(\epsilon_{\text{kin}}) \frac{p c^2}{\epsilon} (mc)^3 \left( \left( \frac{\epsilon_{\text{kin}}}{mc^2} + 1 \right)^2 - 1 \right) \frac{3/2}{\epsilon_{\text{kin}} + mc^2} \, d\epsilon_{\text{kin}} \]  

(58)

In terms of the rest-mass energy normalized kinetic energy \( \epsilon'_{\text{kin}} = \epsilon_{\text{kin}} / mc^2 \) the expression for the pressure becomes:

\[ \frac{P}{n_0 mc^2} = \int_0^\infty f(\epsilon'_{\text{kin}}) \left( (\epsilon'_{\text{kin}} + 1)^2 - 1 \right)^{3/2} \, d\epsilon'_{\text{kin}}. \]  

(59)

The pressure is expressed in terms of the rest-mass energy density at particle concentration \( n_0 \). For completely degenerate fermion gases this expression reduces to:

\[ \frac{P}{n_0 mc^2} = \int_0^{\epsilon_F'} (\epsilon'_{\text{kin}} + 1)^2 - 1 \right)^{3/2} \, d\epsilon'_{\text{kin}}. \]  

(60)

This expression could in principle be worked out to obtain the equation of state for the completely degenerate fermion gas. However it is easier to proceed via the momentum distribution, writing:

\[ P = g_s \frac{4\pi}{3} \frac{1}{h^3} \int_0^{p_F} v_p p^3 \, dp. \]  

(61)

By switching to \( x = p/mc \) as integration variable and using equation (23) for \( v_p \) the integral becomes:

\[ \frac{P}{n_0 mc^2} = \int_0^{p_F/mc} \frac{x^4}{\sqrt{1 + x^2}} \, dx. \]  

(62)

The solution can be found using formula 2.273-3 from Gradshteyn & Ryzhik (2007) and realizing that \( \ln(x + \sqrt{1 + x^2}) = \text{arsinh} \, x \). The expression becomes:

\[ \frac{P}{n_0 mc^2} = \frac{1}{8} \left( z(1 + z^2)^{1/2}(2z^2 - 3) + 3 \text{arsinh} \, z \right), \]  

(63)

where \( z = p_F/mc = (n/n_0)^{1/3} \) (equation 50).

This is the general equation of state for the completely degenerate Fermion gas. Note that it does not depend on temperature, but only on density (as expected, since \( T = 0 \) K). From this general expression the equations of state for the non-relativistic and ultra-relativistic regimes can be derived. Starting with the latter, for which \( z \gg 1 \), the leading term in eq. (63) is the only one that remains. This can be seen by considering that \( \text{arsinh} \, z = \ln(z + \sqrt{z^2 + 1}) \). This means that the \( \text{arsinh} \, z \) term tends to \( \ln(2z) \) which means that the last term in eq. (63) will be negligible compared to the leading term which tends to \( 2z^4 \) as \( z \to \infty \). The expression for the pressure in the ultra-relativistic case then becomes:

\[ \frac{P_{\text{UR}}}{n_0 mc^2} = \frac{1}{4} \left( \frac{p_F}{mc} \right)^{4/3} = \frac{1}{4} \left( \frac{n}{n_0} \right)^{4/3}. \]  

(64)

For the non-relativistic limit (\( z \ll 1 \)) a Taylor expansion of the above equation for the pressure can be used, but it is easier to return to the integral (60) and note that for \( \epsilon_F = \epsilon_F/mc^2 \ll 1 \) (i.e. \( x \ll 1 \)) the integrand tends

\(^3\)The Taylor expansion should be done up to and including the 5\(^{th}\) power of \( z \).
to \((1 + 2x - 1)^{3/2} = 2^{3/2}x^{3/2}\). Thus the expression for the pressure now becomes:

\[
\frac{P_{NR}}{n_0mc^2} = \int_0^{e_F} 2^{3/2}x^{3/2} \, dx
\]

\[
= \left[ \frac{2^{3/2}x^{5/2}}{5} \right]_0^{e_F}.
\]

Using expression (47) for the non-relativistic Fermi energy the pressure in the non-relativistic case is:

\[
\frac{P_{NR}}{n_0mc^2} = \frac{1}{5} \left( \frac{n}{n_0} \right)^{5/3}.
\]

Writing out the normalized expressions for the pressure one obtains the following equations of state for the non-relativistic and ultra-relativistic degenerate fermion gas:

\[
P_{NR} = \frac{\hbar^2}{5m} \left( \frac{3}{4\pi g_s} \right)^{2/3} n^{5/3} = K_{NR} n^{5/3},
\]

\[
P_{UR} = \frac{\hbar c}{4} \left( \frac{3}{4\pi g_s} \right)^{1/3} n^{4/3} = K_{UR} n^{4/3}.
\]

For an electron gas \(\log(n_0mc^2) = 22.68\) (N m\(^{-2}\)).

In the diagrams shown in figures 3 and 4 the contours of constant pressure form vertical lines in the degenerate area, while in the classical regime the contours are lines at a 45 degree angle according to:

\[
\log \left( \frac{T}{mc^2/k} \right) = \log \left( \frac{P}{n_0mc^2} \right) - \log \left( \frac{n}{n_0} \right).
\]

The contour lines should smoothly change to vertical in the transition regime between classical and degenerate gases.

6 Density and pressure in terms of Fermi-Dirac integrals

In this note the full expressions for the density and pressure in terms of the relativistically formulated fermion energies was derived. For the density we have from eq. (42):

\[
\frac{n}{n_0} = \int_0^\infty \frac{3(\epsilon'_\text{kin} + 1)((\epsilon'_\text{kin} + 1)^2 - 1)^{1/2}}{e^{\beta'(\epsilon'_\text{kin} - \bar{\mu})} + 1} \, d\epsilon'_\text{kin},
\]

where \(\epsilon'_\text{kin} = \epsilon_{\text{kin}}/mc^2, \beta' = mc^2/kT, \bar{\mu}' = \bar{\mu}/mc^2\), and:

\[
n_0 = g_s \frac{4\pi}{3} \left( \frac{mc}{\hbar} \right)^3.
\]

For the pressure the following expression was found:

\[
\frac{P}{n_0mc^2} = \int_0^\infty \frac{((\epsilon'_\text{kin} + 1)^2 - 1)^{3/2}}{e^{\beta'(\epsilon'_\text{kin} - \bar{\mu})} + 1} \, d\epsilon'_\text{kin}.
\]

In the non-relativistic regime with \(\epsilon'_\text{kin} \ll 1\) and \(\varepsilon = mc^2\epsilon'_\text{kin}\) these expression reduce to:

\[
\frac{n}{n_0} = \frac{3\sqrt{2}}{(mc^2)^{3/2}} \int_0^\infty \frac{\varepsilon^{1/2}}{e^{\beta(\varepsilon - \bar{\mu})} + 1} \, d\varepsilon
\]

and

\[
\frac{P}{n_0mc^2} = \frac{2\sqrt{2}}{(mc^2)^{5/2}} \int_0^\infty \frac{\varepsilon^{3/2}}{e^{\beta(\varepsilon - \bar{\mu})} + 1} \, d\varepsilon
\]
In the expressions above the energy and chemical potential are normalized to $mc^2$. In the literature however, the integrals are written in terms of $x = \frac{\epsilon_{\text{kin}}}{kT}$ and the parameters $\theta = \frac{kT}{mc^2}$ and $\eta = \frac{\bar{\mu}}{kT}$. The integrals above can be rewritten in these terms by using:

$$x = \frac{\epsilon_{\text{kin}}}{kT} = \frac{mc^2 \epsilon_{\text{kin}}}{mc^2} = \frac{\epsilon'_{\text{kin}}}{\theta}. \quad (75)$$

The expression for the density then becomes:

$$\frac{n}{n_0} = \frac{3}{\theta} \int_0^\infty \frac{3(x\theta + 1)((x\theta + 1)^2 - 1)^{1/2}}{e^{x-\eta} + 1} \, dx. \quad (76)$$

Making use of $(x\theta + 1)^2 = 2\theta x + \theta^2 x^2$ the numerator of the integrand can be factored differently, resulting in:

$$\frac{n}{n_0} = 3\sqrt{2}\frac{\theta}{\theta^3/2} \int_0^\infty \frac{x^{1/2}(1 + \frac{1}{2}\theta x)^{1/2}(1 + \theta x)}{e^{x-\eta} + 1} \, dx. \quad (77)$$

Similarly the expression for the pressure becomes:

$$\frac{P}{n_0 mc^2} = 2\sqrt{2}\frac{\theta}{\theta^5/2} \int_0^\infty \frac{x^{3/2}(1 + \frac{1}{2}\theta x)^{3/2}}{e^{x-\eta} + 1} \, dx. \quad (78)$$

These expressions are consistent with equations (24.91) and (24.92) in [Weiss et al.] (2004). For the non-relativistic regime we have $x\theta \ll 1$ and then the expressions for density and pressure reduce to:

$$\frac{n}{n_0} = 3\sqrt{2}\frac{\theta}{\theta^3/2} \int_0^\infty \frac{x^{1/2}}{e^{x-\eta} + 1} \, dx, \quad (79)$$

and

$$\frac{P}{n_0 mc^2} = 2\sqrt{2}\frac{\theta}{\theta^5/2} \int_0^\infty \frac{x^{3/2}}{e^{x-\eta} + 1} \, dx. \quad (80)$$

The integrals in terms of $x$ above can all be expressed in terms of the so-called generalized Fermi-Dirac integral:

$$F_\nu(\eta, \theta) = \int_0^\infty \frac{x^{\nu}(1 + \frac{1}{2}\theta x)^{1/2}}{e^{x-\eta} + 1} \, dx, \quad (81)$$

which is discussed in, e.g., [Press et al.] (2007, section 6.10). In the non-relativistic case $\theta = 0$. Much work in the literature has gone into finding accurate approximations to these integrals. [Press et al.] (2007) present one particular algorithm for evaluating the approximation.

For the density and pressure we can write:

$$\frac{n}{n_0} = 3\sqrt{2}\frac{\theta}{\theta^3/2} (F_{1/2}(\eta, \theta) + \theta F_{3/2}(\eta, \theta)), \quad (82)$$

and

$$\frac{P}{n_0 mc^2} = 2\sqrt{2}\frac{\theta}{\theta^5/2} \left( F_{3/2}(\eta, \theta) + \frac{1}{2} \theta F_{5/2}(\eta, \theta) \right). \quad (83)$$

In the non-relativistic case we can write:

$$\frac{n}{n_0} = 3\sqrt{2}\frac{\theta}{\theta^3/2} F_{1/2}(\eta), \quad (84)$$

and

$$\frac{P}{n_0 mc^2} = 2\sqrt{2}\frac{\theta}{\theta^5/2} F_{3/2}(\eta), \quad (85)$$

where $F_\nu(\eta) = F_\nu(\eta, 0)$. 
7 The energy density of the fermion gas

So far the energy density of the fermion gas has not been considered but it is an important quantity especially when we want to understand why there is an upper limit to white dwarf and neutron star masses. The energy density \( u \) as a function of momentum or kinetic energy is simply:

\[
    u(p) \, dp = n(p) \epsilon_{\text{kin}} \, dp \quad \text{or} \quad u(\epsilon_{\text{kin}}) = n(\epsilon_{\text{kin}}) \epsilon_{\text{kin}} \, d\epsilon_{\text{kin}}.
\]

Using the normalizations introduced in the preceding sections we can write for the total energy density:

\[
    \frac{u}{n_0 mc^2} = \int_{0}^{\infty} \frac{3 \epsilon'_{\text{kin}} (\epsilon'_{\text{kin}} + 1) \left( (\epsilon'_{\text{kin}} + 1)^2 - 1 \right)^{1/2}}{e^{\beta'(\epsilon'_{\text{kin}} - \bar{\mu})} + 1} \, d\epsilon'_{\text{kin}},
\]

where equation (70) was used. In terms of Fermi-Dirac integrals this becomes:

\[
    \frac{u}{n_0 mc^2} = 3 \sqrt{2} \theta^{5/2} \int_{0}^{\infty} x^{3/2} \left( 1 + \frac{1}{2} \theta x \right)^{1/2} \left( 1 + \theta x \right) \, dx
\]

\[
    = 3 \sqrt{2} \theta^{5/2} \left( F_{3/2}(\eta, \theta) + \theta F_{5/2}(\eta, \theta) \right).
\]

In the non-relativistic limit (\( x\theta \ll 1 \)) we have:

\[
    \frac{u}{n_0 mc^2} = 3 \sqrt{2} \theta^{5/2} \int_{0}^{\infty} x^{3/2} \, dx
\]

\[
    = 3 \sqrt{2} \theta^{5/2} F_{3/2}(\eta).
\]

This last expression is the familiar result that for non-relativistic ideal gases:

\[
    P = \frac{2}{3} u.
\]

Analogous to the pressure we can write in the completely degenerate case:

\[
    u = \frac{4\pi}{\hbar^3} \int_{0}^{p_F} \epsilon_{\text{kin}} p^2 \, dp.
\]

Switching again to \( x = p_F/mc \) we obtain:

\[
    \frac{u}{n_0 mc^2} = 3 \int_{0}^{p_F/mc} \left( x^2 + 1 \right)^{1/2} - 1 \right) x^2 \, dx,
\]

for which the solution is:

\[
    \frac{u}{n_0 mc^2} = \frac{1}{8} \left( -8x^3 + 3(x^2 + 1)^{1/2}(2x^3 + x) - 3 \text{arsinh} x \right).
\]

In the non-relativistic limit (\( x \ll 1 \)) this expression tends to \( \frac{3}{10} x^5 \) (which can be derived starting from the integral for \( u \) or by expanding the \text{arsinh} term to order \( x^5 \)) and in the ultra-relativistic case (\( x \gg 1 \)) it tends to \( \frac{3}{4} x^4 \). Thus the expressions for the energy density become:

\[
    \frac{u_{\text{NR}}}{n_0 mc^2} = \frac{3}{10} \left( \frac{n}{n_0} \right)^{5/3},
\]

\[
    \frac{u_{\text{UR}}}{n_0 mc^2} = \frac{3}{4} \left( \frac{n}{n_0} \right)^{4/3}.
\]

From the latter expression we obtain the expected result for an ultra-relativistic ideal gas:

\[
    P = \frac{1}{3} u.
\]
Phase diagram for fermions, chemical potential expressed as \( \eta = (\mu - mc^2)/kT \)

Figure 5: Phase diagram for the fermion gas. The solid contours show lines of constant \( \log(P/n_0 mc^2) \) and constant \( \eta \) is shown by the dashed contours. The colour coding and the dividing lines indicate in which regions the gas is ultra/non-relativistic and classical/degenerate. Green indicates a classical, non-relativistic gas, cyan a classical ultra-relativistic gas, yellow a degenerate non-relativistic gas, and white a degenerate ultra-relativistic gas.

8 The full fermion phase diagram

The Fermi-Dirac integrals from the previous section can be evaluated using, for example, the algorithm from [Press et al., 2007] section 6.10. This means that one can map the density and pressure as a function of chemical potential and temperature. However, the phase diagram in the \( T \) vs \( n \) plane is what is usually shown (and is easier to understand). The problem is that the chemical potential cannot easily be obtained for a given choice of density and temperature. This requires solving the following equation for \( \eta \):

\[
3\sqrt{2}\theta^{3/2} \left( F_{1/2}(\eta, \theta) + \theta F_{3/2}(\eta, \theta) \right) - y/n_0 = 0,
\]

where \( y \) is the chosen density for a given temperature parameter \( \theta \). This equation has to be solved numerically and this can be done with the bisection method described in section 9.1 of [Press et al., 2007]. The root of the equation can first be bracketed with the \texttt{zbrac} function and then found with the \texttt{rtbis} function. Because of the enormous range of values of \( \eta \) (many decades) some care has to be taken in the precision demanded of the bisection method. In practice this was done by finding the root in two steps. First a rough determination was done by demanding an accuracy of 1% relative to the interval returned by \texttt{zbrac}. This rough determination was then used to scale the accuracy of \( \eta \) to \( 10^{-7} \) of the first estimate.

The resulting phase diagram in the \( \log \theta \) vs. \( \log(n/n_0) \) plane is show in figure 5. The solid contours show lines of constant \( \log(P/n_0 mc^2) \) and constant \( \eta \) is shown by the dashed contours. Note how the pressure contours indeed behave as predicted above, sloping at an angle of 45° in the classical regime and becoming...
vertical in the degenerate region of the phase diagram. The contours of constant $\eta$ show different slopes in the non-relativistic and ultra-relativistic regimes. The transition from non-degenerate to a degenerate gas occurs around the $\eta = 0$ contour. Note that large negative values of $\eta$ in the classical regime.

The degree of degeneracy $D$ and the degree to which the gas is relativistic $R$ can be judged from the values of $kT/mc^2$ and $\bar{\mu}/mc^2 = \theta\eta$ as discussed in [Weiss et al., 2004]. However, a nicer definition of these quantities is:

$$D = \frac{1}{n} \int_{0}^{\epsilon'_F} n(\epsilon'_{\text{kin}}) d\epsilon'_{\text{kin}}.$$  \hfill (98)

For a completely degenerate gas ($T \to 0$) the value of $D$ is 1. The fraction of particles with $p > mc$ measures how non-relativistic the gas is:

$$R = \frac{1}{n} \int_{\sqrt{2}-1}^{\infty} n(\epsilon'_{\text{kin}})d\epsilon'_{\text{kin}}.$$  \hfill (99)

The colour scale in figure 5 shows the values of $D$ and $R$. The degree of degeneracy of the gas is indicated with the progressive saturation of the red colour channel while the extent to which the gas is relativistic is indicated by progressive saturation of the blue channel. The green channel was set to its highest value. Where the colours mix a combination of states is indicated. Hence, green means that the gas is classical and non-relativistic, cyan means it is classical and ultra-relativistic, yellow means the gas is degenerate and non-relativistic, and white means the gas is degenerate and ultra-relativistic.

Figures 6–9 (at the back of this note) show examples of $f(\epsilon'_{\text{kin}})$ and $n(\epsilon'_{\text{kin}})$ for each of the four regions in figure 5. The number densities in these figures are expressed in units of $n_0$.

Some remarks on the average occupation of states and the energy distribution:

- As discussed for the non-relativistic case, as the gas becomes degenerate the function $f(\epsilon'_{\text{kin}})$ approaches a step function. At the limit $T = 0$ all the states with $\epsilon'_{\text{kin}} < \mu/mc^2 = \epsilon_F/mc^2$ are occupied while none with energies above this limit are. In this case the fermions only occupy the lowest possible kinetic energy states above $\epsilon'_{\text{kin}} = 0$, one fermion per single-particle state according to the Pauli exclusion principle. Note that even at $T = 0$ the gas will exert a pressure because $p \neq 0$ for all fermions!

- At this limit ($T = 0$) the energy distribution becomes: $n(\epsilon'_{\text{kin}}) \propto (\epsilon'_{\text{kin}} + 1) ((\epsilon'_{\text{kin}} + 1)^2 - 1)^{1/2}$ for $0 < \epsilon'_{\text{kin}} < \mu/mc^2$ and zero elsewhere. In this limit of the completely degenerate fermion gas the number distribution in terms of $p$ goes as $n(p) \propto p^2$. The limiting case is shown in the diagrams with the dashed lines.

- In the limit that $T \to 0$ and the gas remains non-relativistic one can use that $\epsilon'_{\text{kin}} \ll 1$ and then the expression for $n(\epsilon'_{\text{kin}})$ converges to $n(\epsilon'_{\text{kin}}) \propto \sqrt{2}(\epsilon'_{\text{kin}})^{1/2}$. Removing the normalization by $mc^2$ to get $n(\epsilon_{\text{kin}}) d\epsilon_{\text{kin}}$ then leads to the correct expression given by equation (5).

- When $T \to 0$ and the gas is ultra-relativistic the expression for $n(\epsilon'_{\text{kin}})$ is proportional to $(\epsilon'_{\text{kin}} + 1)^2$ ($\epsilon_{\text{kin}} = pc$ for an ultra-relativistic gas).

- Figures 10 and 11 show the border cases for the non-relativistic and ultra-relativistic fermion gas.
Fermion occupation of states, $\mu/mc^2 = -1.06 \times 10^{-1}$, $\log(kT/mc^2) = -2.00$

![Graph of Fermion occupation of states](image)

Fermion kinetic energy distribution: $\log(n/n_0) = -7.00$, $\log(\epsilon_F/mc^2) = -4.97$

![Graph of Fermion kinetic energy distribution](image)

Figure 6: The average of occupation of the energy states $f(\epsilon_{\text{kin}})$ (top panel) and the corresponding distribution of the number density of fermions as a function of kinetic energy $n(\epsilon_{\text{kin}})$. All energies are expressed in units of $mc^2$, including $\mu$ and $kT$. These diagrams represent a classical non-relativistic fermion gas according to figure 5.

Fermion occupation of states, $\mu/mc^2 = -1.11 \times 10^2$, $\log(kT/mc^2) = 1.00$

![Graph of Fermion occupation of states](image)

Fermion kinetic energy distribution: $\log(n/n_0) = -1.00$, $\log(\epsilon_F/mc^2) = -0.99$

![Graph of Fermion kinetic energy distribution](image)

Figure 7: The average of occupation of the energy states $f(\epsilon_{\text{kin}})$ (top panel) and the corresponding distribution of the number density of fermions as a function of kinetic energy $n(\epsilon_{\text{kin}})$. All energies are expressed in units of $mc^2$, including $\mu$ and $kT$. These diagrams represent a classical ultra-relativistic fermion gas according to figure 5.
Figure 8: The average of occupation of the energy states $f(\epsilon_{\text{kin}})$ (top panel) and the corresponding distribution of the number density of fermions as a function of kinetic energy $n(\epsilon_{\text{kin}})$. All energies are expressed in units of $mc^2$, including $\bar{\mu}$ and $kT$. These diagrams represent a degenerate non-relativistic fermion gas according to figure 5.

Figure 9: The average of occupation of the energy states $f(\epsilon_{\text{kin}})$ (top panel) and the corresponding distribution of the number density of fermions as a function of kinetic energy $n(\epsilon_{\text{kin}})$. All energies are expressed in units of $mc^2$, including $\bar{\mu}$ and $kT$. These diagrams represent a degenerate ultra-relativistic fermion gas according to figure 5.
Figure 10: The average of occupation of the energy states $f(\epsilon_{\text{kin}})$ (top panel) and the corresponding distribution of the number density of fermions as a function of kinetic energy $n(\epsilon_{\text{kin}})$. All energies are expressed in units of $mc^2$, including $\bar{\mu}$ and $kT$. These diagrams represent a partially degenerate non-relativistic fermion gas according to figure 5.

Figure 11: The average of occupation of the energy states $f(\epsilon_{\text{kin}})$ (top panel) and the corresponding distribution of the number density of fermions as a function of kinetic energy $n(\epsilon_{\text{kin}})$. All energies are expressed in units of $mc^2$, including $\bar{\mu}$ and $kT$. These diagrams represent a partially degenerate ultra-relativistic fermion gas according to figure 5.