Appendix E

Extra Problem Set: Line processes

This problem set focuses on line processes, and is a bit more challenging than the problems sets A and C. It treats the mass loss process of massive stars. No previous knowledge is really required about these objects (but may be interesting!).

E.1 P Cygni profiles

The spectra of hot, massive stars undergoing strong mass loss are characterized by the so-called P Cygni-type profiles of strong spectral lines. These lines are formed in the spherically expanding stellar wind and show an emission peak centered at the rest wavelength and a blue-shifted absorption trough.

The purpose of this exercise is to understand the formation of P Cygni profiles and how to derive information about the velocity structure of the stellar wind from these profiles.

a. Explain the shape of a P Cygni profile by indicating the regions in the stellar wind that produce absorption, respectively, emission with respect to the continuum.

b. The geometrical region from which the radiation at a given wavelength λ_1 arises is a very thin zone centered on a surface of constant velocity v_1 toward the observer. The shape of these surfaces depends on the velocity field.

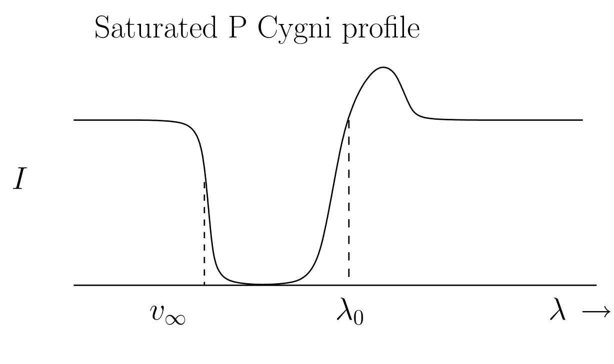

Figure E.1: P Cygni profiles

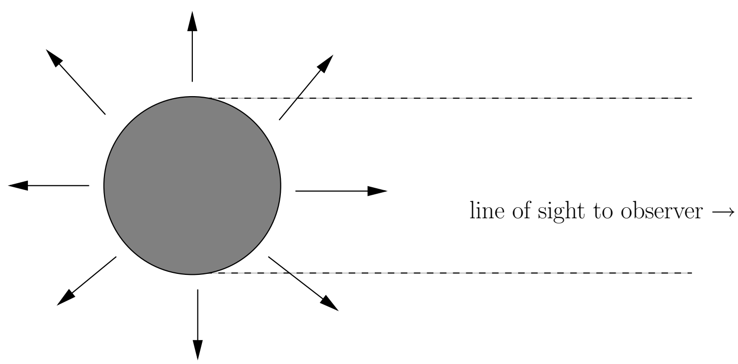

Figure E.2: Geometry of the star, its wind, and the line of sight

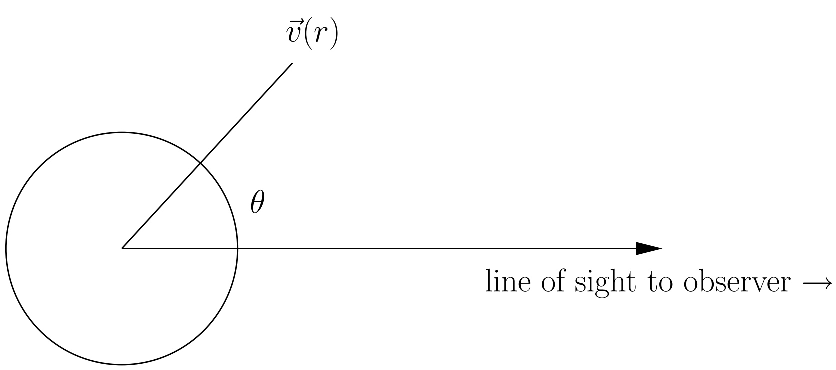

Figure E.3: Definition of the angle θ and the vector $\vec{v}$.

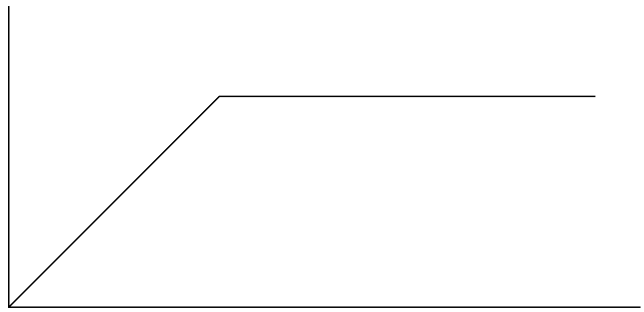

Figure E.4: Velocity as function of r.

- Draw the constant velocity surfaces for the case $v(r) = r$.
- Same, for $v(r)$ =constant.
- Same, for a velocity field sketched in Fig. E.4.

From the results derived above we can conclude that the radiation from a stellar wind at a given wavelength does *not* correspond to a specific position in the envelope.

The highest velocity reached by the stellar wind can be derived from the 'blue edge' of the P Cygni profiles. In strongly saturated P Cygni lines (see figure above) the terminal velocity v_∞ can be easily measured.

The outer parts of a stellar wind radiate free-free emission at radio wavelengths. At such large distances the wind has already reached its terminal velocity so that the density in the wind falls off with r^{-2}. This allows an analytic solution to the equation of radiative transfer. In this case, the mass-loss rate $\dot{M}$ is related to the radio flux as

$$\frac{\dot{M}}{v_\infty} = \frac{0.095 \mu S_\nu^{3/4} D^{3/2}}{\sqrt{Z^2 \gamma g_\nu \nu}} \tag{E.1}$$

in units of $\mathrm{M}_\odot$ yr^{-1}/(km s^{-1}). In the equation, μ is the mean weight per ion, Z the rms ionic charge, γ the ratio of electron-to-ion density, ν te frequency, g_ν the Gaunt factor, D the distance in kpc, and S_ν the flux in Jy. The Gaunt factor can be approximated by

$$g_\nu = 9.77\Big(1 + 0.13 \log \frac{0.35 T_{\mathrm{eff}}^{3/2}}{Z\nu}\Big). \tag{E.2}$$

The O4 star ζ Pup has an effective temperature of 42400 K and therefore He is fully ionized (He^{++}) in the wind. The helium content expressed as $\epsilon = He/(H + He)$ is 0.17 for this star, and one finds

$$\mu = 1.488 \tag{E.3}$$

$$\gamma = 1.154 \tag{E.4}$$

$$< Z > = 1.211 \tag{E.5}$$

Further for ζ Pup, $D = 0.4$ kpc, $v_\infty = 2200$ km s^{-1}. The radio flux observed at 3.6 cm is $S_\nu = 1.60 \pm 0.$ mJy.

c. Determine the mass loss rate $\dot{M}$ of ζ Pup.

E.2 Radiation force on electrons

The radiative acceleration due to a flux F_ν on material with opacity κ_ν is

$$g_{\rm rad} = \int_0^\infty \frac{\kappa_\nu \pi F_\nu}{c}\, d\nu. \tag{E.6}$$

The net acceleration in the outer layers of a luminous star is therefore

$$g_{\rm eff} = g_{\rm gravity} - g_{\rm rad}. \tag{E.7}$$

In order for an object to remain bound, $g_{\rm eff} > 0$ is required.

In Chapter 2 we looked at the Eddington limit, which is the largest luminosity that an object of mass M can still have and remain bound. We found that

$$\frac{\kappa}{4\pi G c} \leq \frac{M}{L}. \tag{E.8}$$

We can define Γ as the ratio of an object's luminosity to its Eddington luminosity. Using electron scattering as the dominant opacity source ($\kappa = 0.25$), we get

$$\Gamma \equiv \frac{L}{L_{\rm Edd}} \approx 2.7 \times 10^{-5} \frac{L/L_\odot}{M/M_\odot}. \tag{E.9}$$

We are using cgs units, so $G = 6.67 \times 10^{-8}$, L$_\odot$=3.85×10^{33}, and M$_\odot$=2×10^{33} in these units.

d. What is Γ for a typical O-star (M=60 M$_\odot$, L=10^6 L$_\odot$)?

e. The brightest stars in our Galaxy are $\sim 10^6$ L$_\odot$. What is their minimum mass?

f. In the Large Magellanic cloud the metal content is about 5 times lower than in our Galaxy. What would the effect on the maximum luminosity in the LMC be?

E.3 Escape velocity

g. Show that the local escape velocity on a distance r from the stellar center is, in km s^{-1},

$$v_{\rm esc}(r) = 618\sqrt{\frac{M/M_\odot}{r/R_\odot}(1-\Gamma)}, \tag{E.10}$$

with R$_\odot$=6.9 $\times$ 10^{10} cm.

h. Compare the observed wind velocities with the escape velocity at the stellar surfaces R

	ζ Pup	τ Sco	Red supergiant
Spectral type	O4	B0.2V	M
L/L$_\odot$	8×10^5	3×10^4	10^6
R/R$_\odot$	17	6.5	1000
M/M$_\odot$	60	15	15
$v_{\rm wind}$ (km s^{-1})	2200	2000	100

i. What would be the flow timescale in the winds of these stars, when we define

$$t_{\rm flow} = \frac{R_\star}{v_{\rm wind}}? \tag{E.11}$$

E.4 Radiation pressure due to lines

Strong resonance lines can have opacities exceeding the opacity due to electrons by a factor of 1 million or more. One might expect that even a very inefficient transfer of momentum (enclosed in the radiation field) to the stellar wind plasma will result in a significant acceleration of the gas particles. The total amount of momentum contained in the stellar radiation field is L/c.

j. Derive an upper limit to the maximum mass-loss rate $\dot{M}$ for a wind with terminal velocity v_∞ that can be accelerated by radiation pressure exerted on the stellar wind plasma.

For a rapidly expanding atmosphere, Castor (1974) derived that the acceleration due to radiation pressure on spectral lines is given by

$$g_l \approx \frac{\Delta\nu_D F_\nu \kappa_l}{c\tau_l}\Big(1 - e^{-\tau_l}\Big), \tag{E.12}$$

with $\Delta\nu_D$ the (Doppler) width of the line, κ_l the monochromatic line absorption coefficient per unit mass (derived through the line profile function), and F_ν the continuum flux blueward of the rest-wavelength of the line.

The optical depth τ_l is given by

$$\tau_l = \int_r^\infty \kappa_l \rho dr. \tag{E.13}$$

Suppose that the opacity of the line is sharply peaked around a certain frequency ν' with the width $\Delta\nu_{\rm th}$, corresponding to the thermal velocity of the plasma. In a hot star, $\Delta\nu_{\rm th} \approx 10$ km s^{-1}. In a rapidly expanding atmosphere the width of the zone absorbing photons of frequency ν' is

$$\Delta r = \frac{\nu_{\rm th}}{dv/dr}. \tag{E.14}$$

This Δr is called the *Sobolev length*, and under the given conditions is small compared to the scale height $H = P/\rho g$ of the wind. Thus, in this special case, the local gas pressure and density do not change appreciably over Δr.

k. Derive an expression for the optical depth τ_l assuming the Sobolev approximation.

In the following we consider only *strong* lines (Lucy & Solomon 1970). Because $\tau_l \gg 1$, we can write for g_l,

$$g_l = \sum_i \frac{\Delta\nu_i F_{\nu_i}}{c}\frac{\kappa_i}{\tau_i}. \tag{E.15}$$

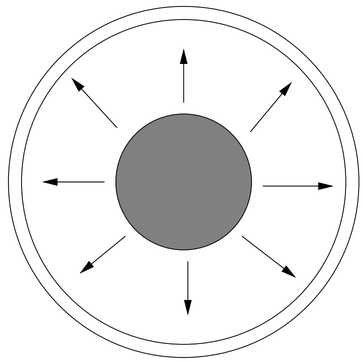

Figure E.5: Geometry of the Sobolev approcimation.

l. Show that this can be written as

$$g_l = \frac{LN_{\rm eff}}{4\pi c^2 \rho r^2}\frac{\Delta v}{\Delta r}, \tag{E.16}$$

using the Sobolev approximation and

$$N_{\rm eff} \equiv \sum_i \frac{4\pi r^2 F_{\nu_i}\nu_i}{L}. \tag{E.17}$$

$N_{\rm eff}$ stands for the number of 'effective' lines driving the wind (~ 100). Assume that $N_{\rm eff}$ is constant through the wind, and consider only the supersonic part of the wind (so that we can neglect gas pressure). The equation of motion in the wind is given by

$$v\frac{dv}{dr} = g_l - \frac{GM(1-\Gamma)}{r^2}. \tag{E.18}$$

In spherical symmetry conservation of mass is

$$\dot{M} = 4\pi r^2 \rho v = \text{constant}. \tag{E.19}$$

m. Show that the mass loss rate $\dot{M}$ relates to the stellar luminosity L as

$$\dot{M} = \frac{LN_{\rm eff}}{c^2}\Big(1-\epsilon\Big), \tag{E.20}$$

with

$$\epsilon \equiv \frac{4\pi\rho c^2 GM(1-\Gamma)}{(dv/dr)LN_{\rm eff}} \approx \text{constant}. \tag{E.21}$$

This, this 'simple' theory predicts that $\dot{M} \propto L$ as is observed.

We can rewrite this as

$$4\pi r^2 \rho v = \tfrac{L N_{\rm eff}}{c^2}\Big(1-\epsilon\Big) \tag{E.22}$$

$$\Rightarrow \frac{\epsilon}{1-\epsilon} v \frac{dv}{dr} = \tfrac{GM(1-\Gamma)}{r^2}. \tag{E.23}$$

n. Derive by integrating this equation from $v(R_\star) = v_0$ to $v(r) = v$ that the terminal velocity of the wind v_∞ relates to the escape velocity $v_{\rm esc}$ at the stellar surface as

$$v_\infty = \sqrt{\frac{1-\epsilon}{\epsilon}} v_{\rm esc}. \tag{E.24}$$

The observed trend (Abbott 1982) is $v_\infty \approx 2.6 v_{\rm esc}$, showing that this 'simple' description works well for these mass losing stars.