Solutions to discussion section 4

1. Interstellar dust attenuation

From $\tau_\lambda = 1.086\,\mathrm{A}_\lambda$ and $\tau(\lambda) = n_d C_{ext} L$ you get the attenuation A_λ in magnitudes, using $C_{ext} = Q_{ext}\sigma_d$, $\sigma_d = \pi r^2$ being the grains cross section, n_d the dust density and L the distance of the star.

$$A_\lambda = \frac{n_d \cdot Q_{ext} \cdot \pi r^2 \cdot L}{1.086} = \frac{2 \cdot 10^{-6}\,\mathrm{m}^{-3} \cdot 0.5 \cdot \pi \cdot 0.1^2\,\mu\mathrm{m}^2 \cdot 1\,\mathrm{kpc}}{1.086} = 0.892$$

2. Heating of interstellar dust

The grain temperature can be obtained by

$$T_d = \left(\frac{\pi J_s}{\sigma}\right)^{1/4} \left(\frac{Q_{abs}}{Q_{emis}}\right)^{1/4}.$$

The given efficiency is $\frac{Q_{emis}}{Q_{abs}} = 0.1\%$, $\sigma = 5.67 \cdot 10^{-5}\,\mathrm{erg}/\mathrm{cm^s sK^4}$ is the Stefan-Boltzmann constant, J is the available energy which is

$$J_s = 10^{10}\,\mathrm{photons}/\mathrm{m^2 snm} \cdot 100\,\mathrm{nm} \cdot 9\,\mathrm{eV} = 9 \cdot 10^{12}\,\mathrm{eV}/\mathrm{m^2 s} = 1.442 \cdot 10^{-3}\,\mathrm{erg}/\mathrm{cm^2 s}.$$

$$\Rightarrow T_d = (80\,\mathrm{K})^{1/4}(10^3)^{1/4} = 16.8\,\mathrm{K}$$

3. Linear molecule

Inserting the values:

$$A_{ul} = 1.165 \cdot 10^{-11} \cdot (0.1)^2 \cdot (115.271 \cdot 10^9)^3 \cdot \frac{1}{3} s^{-1} = 5.9 \cdot 10^{-8} s^{-1}$$

with $\mu = 0.11$ Debye the value is $A_{ul} = 7.2 \cdot 10^{-8} s^{-1}$ like in the table from Tielens.

4. Critical density

CS: $n* = \frac{A_{10}}{\langle \sigma v \rangle} = 1.8 \cdot 10^4 cm^{-3}$
CS: $n* = \frac{A_{20}}{\langle \sigma v \rangle} = 2.2 \cdot 10^5 cm^{-3}$
CO: $n* = \frac{A_{10}}{\langle \sigma v \rangle} = 740 cm^{-3}$

The critical density of the (1-0)-transition of CO is much lower than for the transitions of CS. From the lecture (p.29) we see that the electric dipole moment of CO is very small $\Rightarrow A_{ul}$ is small for CO $\Rightarrow$ n* is low.

5. CO line intensty

a)

$$\begin{aligned} I_{ul} &= N_u A_{ul} h\nu_{ul}/4\pi \\ &= N_u(7.2 \cdot 10^{-8}s^{-1}) \cdot (6.6261 \cdot 10^{-27} erg \cdot s) \cdot (115.271 \cdot 10^9 s^{-1})/(4\pi sr) \\ &= 4.3762 \cdot 10^{-24} N_1 \ erg\ cm^{-2}s^{-1}sr^{-1} \end{aligned}$$

b) Brightness temperature in the Rayleigh limit: $I(\nu) = \frac{2\nu^2 kT_B}{c^2} \Rightarrow T_B = \frac{I_\nu c^2}{2\nu^2 k}$

The integration over frequency corresponds to an integration over velocity due to the Doppler effect:

$$\text{(radio definition):}\ \ v = c(1 - \frac{\nu}{\nu_0}) \Rightarrow \frac{dv}{d\nu} = -\frac{c}{\nu_0}$$

The minus doesn't play a role for the integrated value.

$$\begin{aligned} T_B\Delta v &= \int\int \frac{I_\nu c^2}{2k\nu_0^2} \cdot \frac{c}{\nu_0} d\nu d\Omega \\ &= I_{ul}\left(\frac{c^3}{2k\nu_{ul}^3}\right) erg\ cm^{-2}s^{-1}cm^3Kerg^{-1} \\ &= 10^{-5}\left(\frac{c^3}{2k\nu_{ul}^3}\right)\left(\frac{I_{ul}}{erg\ cm^{-2}s^{-1}sr^{-1}}\right) Kkm/s \end{aligned}$$

c)

$$I_{ul} = T_B\Delta v 10^5 \left(\frac{2k\nu_{ul}^3}{c^3}\right) erg\ s^{-1}sr^{-1}cm^{-2}$$

d)

$$I_{10} = 4.35 \cdot 10^{-24} N_1 erg\ cm^{-2}s^{-1}sr^{-1} \text{see ex. 3a}$$

Insert this into the expression for the brightness temperature (ex. 3b):

$$\begin{aligned} T_B\Delta v &= 2.78 \cdot 10^{-15} N_1 Kkm/s \\ &= 2.78 \cdot 10^{-15}\left(g_1\left(\frac{kT_x}{hcB}\right) N_{CO} exp\left[\frac{E_1}{kT}\right]\right) Kkm/s \\ &= 5.24 \cdot 10^{-14} N_{CO} \end{aligned}$$

$$\Rightarrow N_{CO} = 1.9 \cdot 10^{13} \frac{T_B\Delta v}{Kkm/s} cm^{-2}$$

Values used: $g_1 = 3$, $T_x = 10K$, $E_1 = 5.5K \cdot k$, $B = 1.9225\ cm^{-1}$.