Introduction to manifolds

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In this reader, by <u>differentiable</u> we always mean <u>$\mathcal{C}^\infty$</u>.

We will use the following notation.

Let $U \in \mathbb{R}^n$ be open and $f = (f_1, \ldots, f_m) : U \longrightarrow \mathbb{R}^m$ be a differentiable map. Then the <u>Jacobian</u> (<u>matrix</u>) of f at $x \in U$ is

$$Df(x) := \left(\frac{\partial f_i}{\partial x_j}(x) \right)_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \ldots & \frac{\partial f_1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(x) & \ldots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix} .$$

We also recall the following results which should be well known from calculus.

Theorem 0.0.1 (Implicit Function Theorem)
Consider $\mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$ with coordinates $(x, y) = (x_1, \ldots, x_n, y_1, \ldots, y_m)$. Given is an open subset $U \subset \mathbb{R}^n \times \mathbb{R}^m$, a point $p \in U$, and a differentiable map $f = (f_1, \ldots, f_m) : U \longrightarrow \mathbb{R}^m$ with $f(p) = 0$ and

$$\det \left(\left(\frac{\partial f_i}{\partial y_j}(p) \right)_{i,j=1,\ldots,m} \right) \neq 0 .$$

Then there are open $V \subset \mathbb{R}^n$, $W \subset \mathbb{R}^m$ with $p \in U' := V \times W \subset U$ such that there is a unique differentiable map $g : V \longrightarrow \mathbb{R}^m$ with $U' \cap f^{-1}(0) = \{ \ (x, g(x)) \mid x \in V \ \}$.

Theorem 0.0.2 (Inverse Function Theorem)
Let $U \subset \mathbb{R}^n$ be open, $f = (f_1, \ldots, f_n) : U \longrightarrow \mathbb{R}^n$ a differentiable map, $p \in U$ and

$$\det (Df(p)) \neq 0 .$$

Then there is an open $U' \subset U$, $p \in U'$, and an open $V \subset \mathbb{R}^n$ such that $f|_{U'} : U' \longrightarrow V$ is a diffeomorphism.

1 Submanifolds of Euclidean space

1.1 Submanifolds of $\mathbb{R}^n$

Definition 1.1.1 *A <u>k-dimensional</u> <u>submanifold</u> of $\mathbb{R}^n$ is a subset $X \subset \mathbb{R}^n$ with the following property:*
For every $p \in X$ there is an open $U_p \subset \mathbb{R}^n$ with $p \in U_p$, and a differentiable map

$$f = (f_1, \ldots, f_{n-k}) : U_p \longrightarrow \mathbb{R}^{n-k}$$

such that $f^{-1}(0) = X \cap U_p$ and

$$\mathrm{rk}\,(Df(p)) = n - k \ .$$

Examples 1.1.2 *1. We start by describing the dimensionally extreme cases.*

<u>Claim:</u>

(a) The 0-dimensional submanifolds of $\mathbb{R}^n$ are precisely the discrete subsets.

(b) The n-dimensional submanifolds of $\mathbb{R}^n$ are precisely the open subsets.

<u>Proof:</u> (a) Recall that a subset $X \subset \mathbb{R}^n$ is discrete if and only if for every $p \in X$ there exists an open subset $U \in \mathbb{R}^n$ such that $\{p\} = X \cap U$. In this case, define

$$f : U \longrightarrow \mathbb{R}^n \quad , \quad f(x) := x - p \ ;$$

then f is differentiable with $X \cap U = \{p\} = f^{-1}(0)$ and $\mathrm{rk}(Df(p)) = \mathrm{rk}(\mathrm{id}_{\mathbb{R}^n}) = n = n - 0$. This shows that X is a 0-dimensional submanifold.

Conversely, let $X \subset \mathbb{R}^n$ be a 0-dimensional submanifold and $p \in X$. Then there exists an open $U \subset \mathbb{R}^n$ with $p \in U$ and a differentiable map $f : U \longrightarrow \mathbb{R}^{n-0} = \mathbb{R}^n$ such that $X \cap U = f^{-1}(0)$ and $\mathrm{rk} Df(p) = n$. By the Inverse Function Theorem 0.0.2 we may assume (after making U smaller if necessary) that f is a diffeomorphism and in particular injective. This implies $\{p\} = f^{-1}(0) = X \cap U$. This shows that X is discrete.

(b) If $X \in \mathbb{R}^n$ is open, then for every $p \in X$ take $U = X$ and $f \equiv 0$, the constant function from U to $\{0\} = \mathbb{R}^0 = \mathbb{R}^{n-n}$; the Jacobian of this map has rank 0, and therefore X is an n-dimensional submanifold.

Conversely, let $X \subset \mathbb{R}^n$ be an n-dimensional submanifold and $p \in X$. Then there exists an open $U \subset \mathbb{R}^n$ with $p \in U$ and a differentiable map $f : U \longrightarrow \mathbb{R}^{n-n} = \mathbb{R}^0 = \{0\}$ such that $X \cap U = f^{-1}(0) = U$. This implies that for every $p \in X$ there exists an open $U \subset \mathbb{R}^n$ with $p \in U \subset X$, i.e. that X is an open subset of $\mathbb{R}^n$.

2. Let $S^1 = \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 = 1 \}$ be the unit circle. For every point $p = (p_1, p_2) \in S^1$ we choose $U = \mathbb{R}^2$ and $f : U \longrightarrow \mathbb{R}$, $f(x_1, x_2) = x_1^2 + x_2^2 - 1$. Then f is differentiable, it holds $S^1 = S^1 \cap U = f^{-1}(0)$, the Jacobian of f is $Df(x_1, x_2) = (2x_1, 2x_2)$, and since $(p_1, p_2) \neq (0,0)$ for all $(p_1, p_2) \in S^1$, we have $\mathrm{rk} Df(p_1, p_2) = 1$ for all $(p_1, p_2) \in S^1$. Therefore, S^1 is a 1-dimensional submanifold of $\mathbb{R}^2$.

3. *Suppose that* $X \subset \mathbb{R}^n$ *is (locally) the zero set of a map* $f : U \longrightarrow \mathbb{R}^{n-k}$ *, but that* f *is not everywhere differentiable, or that for some* $p \in X \cap U$ *it holds* $\mathrm{rk}Df(p) < n-k$ *. Does this mean that* X *is* <u>*not*</u> *a submanifold of* $\mathbb{R}^n$*? The answer to this question is no, as we see from the following example.*

   Consider the diagonal $X := \{ (x_1, x_2) \in \mathbb{R}^2 \mid x_1 = x_2 \}$ *; intuitively one expects that this straight line is a one-dimensional submanifold of* $\mathbb{R}^2$*.*
   But X *equals* $f^{-1}(0)$ *for the differentiable function* $f(x_1, x_2) = (x_1 - x_2)^2 = x_1^2 - 2x_1x_2 + x_2^2$ *, whose Jacobian* $Df(x_1, x_2) = (2x_1 - 2x_2, -2x_1 + 2x_2) = (0, 0)$ *for* <u>*all*</u> $(x_1, x_2) \in X$ *, i.e. it holds* $\mathrm{rk}Df(x_1, x_2) = 0 < 2 - 1 = 1$ *for* <u>*all*</u> $(x_1, x_2) \in X$ *. Nevertheless,* X *is indeed a 1-dimensional submanifold, since it is also the zero set of the differentiable function* $x_1 - x_2$ *whose Jacobian* $(1, -1)$ *has rank 1 everywhere.*

   The point we want to make here is that if an X *is given as zero set of a map* f *which does not satisfy all of the conditions of the definition, it can nevertheless be a submanifold; the given* f *might just be the wrong choice.*

4. *We view elements of* $\mathbb{R}^n$ *as* <u>*column*</u> *vectors.*

   A <u>*quadratic*</u> <u>*hypersurface*</u> *in* $\mathbb{R}^n$ *is a subset* $X = f_{A,b,c}^{-1}(0)$ *where*

$$f_{A,b,c} : \mathbb{R}^n \longrightarrow \mathbb{R} \quad , \quad f_{A,b,c}(x) = x^t A x + 2b^t x + c = \sum_{i,j=1}^{n} a_{ij} x_i x_j + 2 \sum_{i=1}^{n} b_i x_i + c \ ,$$

   with $A = (a_{ij})_{i,j=1,\ldots,n}$ *a real symmetric* $(n \times n)$*-matrix,* $b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{R}^n$ *, and* $c \in \mathbb{R}$ *.* X *is called* <u>*smooth*</u> *or* <u>*nondegenerate*</u> *if*

$$\det \begin{pmatrix} A & b \\ b^t & c \end{pmatrix} \neq 0 \ .$$

   <u>*Claim:*</u> *A smooth quadratic hypersurface is an* $(n-1)$*-dimensional differentiable submanifold of* $\mathbb{R}^n$*.*

   <u>*Proof:*</u> *It suffices to show*

$$f_{A,b,c}(x) = 0 \quad \Rightarrow \quad Df_{A,b,c}(x) := \left(\frac{\partial f_{A,b,c}}{\partial x_1} \ldots, \frac{\partial f_{A,b,c}}{\partial x_n} \right)(x) \neq 0 \ .$$

   It is easy to see that $Df_{A,b,c}(x) = (2Ax + 2b)^t$ *.*
   Assume that $f_{A,b,c}(x) = 0$ *and* $Df_{A,b,c}(x) = 0$ *. The second condition means* $Ax + b = 0$ *or* $x^t A = -b^t$ *. In combination with the first condition it follows that* $b^t x + c = 0$ *. This implies*

$$\begin{pmatrix} A & b \\ b^t & c \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + b \\ b^t x + c \end{pmatrix} = 0 \ ,$$

   but this means that $\begin{pmatrix} A & b \\ b^t & c \end{pmatrix}$ *is not invertible; a contradiction.* ■

5. *Let* $M(n) = M(n \times n, \mathbb{R}$ *be the vector space of real* $(n \times n)$*-matrices which is naturally identified with* $\mathbb{R}^{n^2}$.
Let $S(n) = \{ \ A \in M(n) \mid A = A^t \ \}$ *be the vector space of symmetric real* $(n \times n)$*-matrices; this is identified with* $\mathbb{R}^{\frac{n(n+1)}{2}}$ *by taking the coefficients on and above the diagonal as coordinates.*
Let I_n *denote the* $n \times n$ *unit matrix; then* $O(n) = \{ \ A \in M(n) \mid A \cdot A^t = I_n \ \}$ *is the orthogonal group in dimension* n*, i.e. the group of orthogonal* $(n \times n)$*-matrices. Observe that an orthogonal matrix* A *is invertible with inverse* $A^{-1} = A^t$.
Claim: $O(n)$ *is an* $\frac{n(n-1)}{2}$*-dimensional submanifold of* $M(n)$.
Proof: $O(n)$ *is the zero set of the differentiable map*

$$f : M(n) \longrightarrow S(n) \quad , \quad f(A) := A \cdot A^t - I_n \ .$$

For every $A \in M(n)$ *, the Jacobian*

$$Df(A) : M(n) \longrightarrow S(n)$$

is given by

$$\begin{aligned} Df(A)(B) &= \frac{d}{dt}\left(f(A+tB)\right)|_{t=0} = \frac{d}{dt}\left(A \cdot A^t + t(A \cdot B^t + B \cdot A^t) + t^2 B \cdot B^t - I_n\right)|_{t=0} \\ &= \left(A \cdot B^t + B \cdot A^t + 2tB \cdot B^t\right)|_{t=0} = A \cdot B^t + B \cdot A^t \ . \end{aligned}$$

Take now $A \in O(n)$ *and* $C \in S(n)$ *, and define* $B := \frac{1}{2} C \cdot A \in M(n)$ *. Then* $B^t = \frac{1}{2} A^t \cdot C$ *, and we get*

$$Df(A)(B) = \frac{1}{2}\left(A \cdot A^t \cdot C + C \cdot A \cdot A^t\right) = C \ .$$

This means that for every $A \in O(n)$ *the Jacobian* $Df(A)$ *is surjective, i.e. that it has rank* $\dim S(n) = \frac{n(n+1)}{2}$ *, i.e. that* $O(n)$ *is a submanifold of* $M(n)$ *of dimension*

$$n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2} \ .$$

■

Remarks 1.1.3 1. *It is a (nontrivial) fact that not every submanifold of* $\mathbb{R}^n$ *can be given as the zero set of just* one *differentiable map satisfying the conditions of the definition. An example for which this is not possible is the Möbius band realized as a 2-dimensional submanifold in* $\mathbb{R}^3$.

2. $\mathbb{R}^n$*, and hence any subset with the induced topology, is second countable and locally connected; this has the following two consequences.*
   (a) *A discrete subset, i.e. a 0-dimensional submanifold, can contain at most countably many points.*
   (b) *Since local connectivity implies that connected components are open, an open subset, i.e. an* n*-dimensional submanifold, is the disjoint union of at most countably many connected open subsets.*

Theorem 1.1.4 *Consider the following properties of a subset $X \subset \mathbb{R}^n$ equipped with the induced topology.*

1. *X is a k-dimensional submanifold of $\mathbb{R}^n$.*

2. *For every $p \in X$ there exists an open neighborhood U of p in $\mathbb{R}^n$, an open subset $U' \subset \mathbb{R}^n$, and a diffeomorphism $\phi : U \longrightarrow U'$ such that*

$$\phi(X \cap U) = \{ \ x = (x_1, \ldots, x_n) \in U' \mid x_1 = \ldots = x_{n-k} = 0 \ \} \ .$$

3. *For every $p \in X$ there exists an open neighborhood U of p in $\mathbb{R}^n$, an open subset $W \subset \mathbb{R}^k$, and a differentiable map $\eta : W \longrightarrow \mathbb{R}^n$ such that*
   - *$X \cap U = \eta(W)$,*
   - *$\eta : W \longrightarrow X \cap U$ is a homeomorphism,*
   - *$\mathrm{rk}(D\eta(\eta^{-1}(p))) = k$.*

4. *There exists a <u>k-dimensional</u> <u>differentiable</u> <u>atlas</u> for X, i.e. a set $\mathcal{A} = \{ \ (V_i, h_i, W_i) \mid i \in I \ \}$, where I is some index set, with the following properties:*
   - *for every $i \in I$, V_i is open in X, W_i is open in $\mathbb{R}^k$, and $h_i : V_i \longrightarrow W_i$ is a homeomorphism,*
   - *$X = \bigcup\limits_{i \in I} V_i$,*
   - *for all $i, j \in I$, the map*

$$(h_j \circ h_i^{-1})|_{h_i(V_i \cap V_j)} : h_i(V_i \cap V_j) \longrightarrow h_j(V_i \cap V_j)$$

   is differentiable. The elements (V_i, h_i, W_i) are called <u>charts</u> for X.

Then 1., 2. and 3. are equivalent, and they imply 4.

Proof: The cases $k = 0$ and $k = n$ are trivial, so we assume $0 < k < n$.

<u>1. $\Rightarrow$ 4.</u> We take $I := X$. For every $p \in X$ choose $U_p \subset \mathbb{R}^n$ and $f_p = (f_1, \ldots, f_{n-k}) : U_p \longrightarrow \mathbb{R}^{n-k}$ as in Definition 1.1.1. Define $V_p := X \cap U_p$; then it holds $X = \bigcup\limits_{p \in X} V_p$. Because of

$$\mathrm{rk} \left(\left(\frac{\partial f_i}{\partial x_j}(p) \right)_{\substack{i=1,\ldots,n-k \\ j=1,\ldots,n}} \right) (p) = n - k$$

we may, after renumbering the coordinates in $\mathbb{R}^n$ if necessary, assume that

$$\det \left(\left(\frac{\partial f_i}{\partial x_j}(p) \right)_{i,j=1,\ldots,n-k} \right) \neq 0 \ .$$

According to the Implicit Function Theorem 0.0.1, for every (if necessary smaller) U_p there is an open $W_p \subset \mathbb{R}^k \ni (x_{n-k+1}, \ldots, x_n)$ and a unique differentiable map

$$g_p : W_p \longrightarrow \mathbb{R}^{n-k} \ni (x_1, \ldots, x_{n-k})$$

such that

$$f_p(g_p(x_{n-k+1},\ldots,x_n),x_{n-k+1},\ldots,x_n)=0 \text{ for all } (x_{n-k+1},\ldots,x_n)\in W_p \ , \quad (*)$$

and such that

$$\begin{aligned} V_p &= U_p\cap X = U_p\cap f_p^{-1}(0) \\ &= \{\ (x_1,\ldots,x_n)\in U_p \mid (x_{n-k+1},\ldots,x_n)\in W_p \ ,\ (x_1,\ldots,x_{n-k}) = g_p(x_{n-k+1},\ldots,x_n)\ \}\ . \quad (**) \end{aligned}$$

Define $h_p : V_p \longrightarrow W_p$, $h_p(x_1,\ldots,x_n) := (x_{n-k+1},\ldots,x_n)$, i.e. h_p is the restriction to V_p of an orthogonal projection and in particular continuous. Furthermore, h_p is bijective with continuous inverse

$$h_p^{-1}(x_{n-k+1},\ldots,x_n) = (g_p(x_{n-k+1},\ldots,x_n),x_{n-k+1},\ldots,x_n)\ .$$

Hence h_p is a homeomorphism, and h_p^{-1} is differentiable as map from W_p to $\mathbb{R}^n$, so

$$h_q\circ h_p^{-1} = (\text{projection})\circ(\text{diffb. map})$$

is differentiable (where defined) for all $p,q\in X$.

<u>1. $\Rightarrow$ 2.</u> Take $U := U_p$, V_p, W_p and g_p as in the proof of 1. $\Rightarrow$ 4. , and define

$$\phi : U \longrightarrow \mathbb{R}^n \quad , \quad \phi(x_1,\ldots,x_n) := ((x_1,\ldots,x_{n-k}) - g_p(x_{n-k+1},\ldots,x_n), x_{n-k+1},\ldots,x_n)\ .$$

Then ϕ is differentiable, and its Jacobian has the form

$$D\phi = \begin{pmatrix} I_{n-k} & * \\ 0 & I_k \end{pmatrix}\ ,$$

where I_k denotes the $k\times k$ unit matrix. In particular, it holds $\det(D\phi(p))\neq 0$, so (taking U smaller if necessary) by the Inverse Function Theorem 0.0.2 we may assume that ϕ is a diffeomorphism from U to an open U'. From $(*)$ and $(**)$ it follows

$$\phi(X\cap U) = \phi(V_p) = \{\ x=(x_1,\ldots,x_n)\in U' \mid x_1 = \ldots = x_{n-k} = 0\ \}\ .$$

<u>2. $\Rightarrow$ 1.</u> For $p\in X$ take U, ϕ and U' as in 2. Define

$$\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-k} \quad , \quad \pi(x_1,\ldots,x_n) = (x_1,\ldots,x_{n-k})\ ;$$

then π is linear of rank $n-k$, and it is differentiable with Jacobian $D\pi(x)=\pi$ for all $x\in\mathbb{R}^n$. Define now the differentiable map

$$f := \pi\circ\phi : U \longrightarrow \mathbb{R}^{n-k}\ ;$$

then, since ϕ is bijective, it holds

$$X\cap U = \phi^{-1}\left(\{\ x\in U' \mid \pi(x)=0\ \}\right) = \phi^{-1}\left(\pi^{-1}(0)\right) = (\pi\circ\phi)^{-1}(0) = f^{-1}(0)\ .$$

Finally we have

$$Df(p) = D\pi(\phi(p))\circ D\phi(p) = \pi\circ D\phi(p)\ ;$$

this has rank $n-k$ since $D\phi(p)$ is an isomorphism, and π has rank $n-k$.

<u>1. $\Rightarrow$ 3.</u> Take $U := U_p$, V_p, W_p and h_p as in the proof of 1. $\Rightarrow$ 4. , and define $W := W_p$, $\eta := h_p^{-1} : W \longrightarrow V_p = X \cap U \subset \mathbb{R}^n$; as we have seen, η is a homeomorphism from W to $X \cap U$, and differentiable as a map to $\mathbb{R}^n$, so it remains to verify the rank condition.
Since $W \subset \mathbb{R}^k$, it clearly holds $\mathrm{rk}(D\eta(\eta^{-1}(p))) \leq k$. On the other hand, from $h_p \circ \eta = \mathrm{id}_W$ it follows $\mathrm{id}_{\mathbb{R}^k} = Dh_p(p) \circ D\eta(\eta^{-1}(p))$, which implies $\mathrm{rk}(D\eta(\eta^{-1}(p))) \geq k$.

<u>3. $\Rightarrow$ 1.</u> For $p \in X$ take U, W and η as in 3. Write $\eta = (\eta_1, \ldots \eta_n)$. Since $D\eta(\eta^{-1}(p))$ has rank k, we may assume (by renumbering the coordinates in $\mathbb{R}^n$ if necessary) that

$$\det\left(\left(\frac{\partial \eta_i}{\partial x_j}(\eta^{-1}(p))\right)_{i,j=1,\ldots,k}\right) \neq 0 \ .$$

By the Inverse Function Theorem 0.0.2 (and after shrinking U and W if necessary) there exists an open $W' \subset \mathbb{R}^k$ such that

$$g := (\eta_1, \ldots, \eta_k) : W \longrightarrow W'$$

is a diffeomorphism. Let

$$\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^k \quad , \quad (x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k) \ ,$$

the projection; then

$$g = \pi \circ \eta = \pi|_{\eta(W)} \circ \eta = \pi|_{X\cap U} \circ \eta : W \xrightarrow{\eta} X \cap U \xrightarrow{\pi|_{X\cap U}} W' \ ,$$

so

$$\pi|_{X\cap U} = g \circ \eta^{-1} : X \cap U \longrightarrow W'$$

is a homeomorphism with inverse

$$\phi = \eta \circ g^{-1} : W' \longrightarrow X \cap U \ .$$

Observe that ϕ, viewed as a map $(\phi_1, \ldots, \phi_n)$ from W' to $\mathbb{R}^n$, is injective and differentiable.
Now let be $(x_1, \ldots, x_n) \in X \cap U$, then

$$(x_1, \ldots, x_n) = \phi(\pi(x_1, \ldots, x_n)) = \phi(x_1, \ldots, x_k) = (\phi_1(x_1, \ldots, x_k), \ldots, \phi_n(x_1, \ldots, x_k)) \ ,$$

i.e.

$$\forall\ x \in X \cap U \ : \ x_i = \phi_i(x_1, \ldots, x_k) \quad , \quad i = 1, \ldots, n \ . \quad (***)$$

On the other hand, for $(x_1, \ldots, x_k) \in W'$ it holds

$$\begin{aligned}(x_1, \ldots, x_k) &= \pi(\phi(x_1, \ldots, x_k)) = \pi((\phi_1(x_1, \ldots, x_k), \ldots, \phi_n(x_1, \ldots, x_k)) \\ &= (\phi_1(x_1, \ldots, x_k), \ldots, \phi_k(x_1, \ldots, x_k))) \ ,\end{aligned}$$

i.e.

$$\forall\ (x_1, \ldots, x_k) \in W' \ : \ \phi_i(x_1, \ldots, x_k) = x_i \quad , \quad i = 1, \ldots, k \ . \quad (****)$$

Observe that, by replacing U by $U \cap \pi^{-1}(W')$ if necessary, we may assume that

$$(x_1, \ldots, x_n) \in U \quad \Rightarrow \quad (x_1, \ldots, x_k) \in W' \ . \quad (*****)$$

Define now the differentiable map

$$f : U \longrightarrow \mathbb{R}^{n-k} \quad , \quad f(x_1, \ldots, x_n) = (x_{k+1} - \phi_{k+1}(x_1, \ldots, x_k), \ldots, x_n - \phi_n(x_1, \ldots, x_k)) \ ;$$

then from $(***)$ it follows immediately $X \cap U \subset f^{-1}(0)$. Furthermore, since the Jacobian of f has the form

$$Df = (A \ I_{n-k})$$

with some $((n-k) \times k)$-matrix A, it holds $\mathrm{rk} Df = n - k$. Hence it remains to show that

$$x \in U \ \text{ and } \ f(x) = 0 \quad \Rightarrow \quad x \in X \ ;$$

for this it is sufficient to show that such an $x = (x_1, \ldots, x_n)$ is contained in the image of η. Now $(x_1, \ldots, x_k) \in W'$ by $(*****)$ so $\phi(x_1, \ldots, x_k)$ is defined. By $(****)$ it holds

$$\phi(x_1, \ldots, x_k) = (x_1, \ldots, x_k, \phi_{k+1}(x_1, \ldots, x_k), \ldots, \phi_n(x_1, \ldots, x_k)) \ .$$

On the other hand, $f(x) = 0$ means $x_i = \phi_i(x_1, \ldots, x_k)$, $i = k+1, \ldots, n$, so we get

$$x = (x_1, \ldots, x_k, \phi_{k+1}(x_1, \ldots, x_k), \ldots, \phi_n(x_1, \ldots, x_k)) = \phi(x_1, \ldots, x_k) = \eta(g^{-1}(x_1, \ldots, x_k))$$

as wanted. ■

Remarks 1.1.5 *1. Observe that property 4., in contrast to the other ones, makes sense for any topological space since it does not refer to the ambient space $\mathbb{R}^n$ of X. In fact, this property will be the main ingredient of the definition of an abstract manifold in the next chapter.*

2. In general the implication 4. $\Rightarrow$ 1. *is not true; an example is the following. Let be*

$$X = \{ \ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = |x_1| \ \} \ .$$

Now take $I := \{0\}$, $V_0 := X$, $W_0 := \mathbb{R}$, and $h_0 : (x_1, |x_1|) \mapsto x_1$. h_0 is continuous because it is the restriction to X of the linear, and hence continuous, projection $(x_1, x_2) \mapsto x_1$. It is bijective with inverse $x_1 \mapsto (x_1, |x_1|)$, which is continuous because $x_1 \mapsto |x_1|$ is. This means that h_0 is a homeomorphism, and hence that $\{(V_0, h_0, W_0)\}$ satisfies the first two conditions in 4. But the third condition for this setup is trivially satisfied: the only pair (i, j) to check is $(0, 0)$, and for this we have $h_0 \circ h_0^{-1} = \mathrm{id}_{\mathbb{R}}$, which of course is differentiable.

But can X be a submanifold of $\mathbb{R}^2$? And, if yes, of what dimension k? The answer to the second question seems intuitively obvious, namely $k = 1$. Indeed, since X is neither discreet nor open in $\mathbb{R}^2$, this is the only possibility by Example 1.1.2.1.

The proof of the fact that X is no submanifold of dimension 1 is not so easy and will be given in the next paragraph. In any case, this does not follow from the fact that X is the zero set of the function $(x_1, x_2) \mapsto x_2 - |x_1|$ which is not differentiable at $(0, 0)$; it is not obvious that there is no other function with the right properties (compare Examples 1.1.2).

Excercise 1.1.6 *1. Let X be a k-dimensional submanifold of $\mathbb{R}^n$, $p \in X$ and $\phi : U \longrightarrow U'$ as in Theorem 1.1.4.2. We identify $\mathbb{R}^k$ with $\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_{k+1} = \ldots = x_n = 0 \}$; then $W := U' \cap \mathbb{R}^k$ is open in $\mathbb{R}^k$.*
We denote by $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^k$ the projection $\pi(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$.

(a) Show that U, W and $\eta := \phi^{-1}|_W$ satisfy the conditions of Theorem 1.1.4.3.

(b) Show that there exists a k-dimensional differentiable atlas for X (in the sense of Theorem 1.1.4.4) such that each chart is of the form $(X \cap U, \pi \circ \phi|_{X \cap U}, W)$ for some $p \in X$.

(c) Let γ_1,γ_2 be two curves in X through p. Show that $\dot\gamma_1(0) = \dot\gamma_2(0)$ if and only if

$$\frac{d}{dt} (\pi \circ \phi \circ \gamma_1(t)) |_{t=0} = \frac{d}{dt} (\pi \circ \phi \circ \gamma_2(t)) |_{t=0} \ .$$

2. Show that $X \subset \mathbb{R}^n$ is a k-dimensional submanifold if and only if the following holds:
For every $p \in X$ there exists an open $U_p \subset \mathbb{R}^n$ with $p \in U_p$, an open $W_p \subset \mathbb{R}^k$ and a differentiable map $g_p : W_p \longrightarrow \mathbb{R}^{n-k}$ such that, after renumbering the coordinates in $\mathbb{R}^n$ if necessary, it holds

$$X \cap U_p = \{ (y, g_p(y)) \in \mathbb{R}^k \times \mathbb{R}^{n-k} = \mathbb{R}^n \mid y \in W_p \} \ .$$

Show further that then $\eta_p : W_p \longrightarrow \mathbb{R}^n$, $\eta_p(y) := (y, g_p(y))$ satisfies the conditions in Theorem 1.1.4.3, and that $\mathcal{A} := \{ (X \cap U_p, \eta_p^{-1}|_{X \cap U_p}, W_p) \mid p \in X \}$ is a k-dimensional atlas for X in the sense of Theorem 1.1.4.4.

1.2 Tangent spaces

Let $X \subset \mathbb{R}^n$ be a k-dimensional submanifold, and $p \in X$.

Definition 1.2.1 *A curve in X through p is a differentiable map $\gamma : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}^n$, $0 < \epsilon \in \mathbb{R}$, with $\gamma(-\epsilon, \epsilon) \subset X$ and $\gamma(0) = p$.*

Remarks 1.2.2 *1. Since a curve γ in X through p is differentiable as a map to $\mathbb{R}^n$,*

$$\dot\gamma(0) = \frac{d\gamma}{dt}(0) \in \mathbb{R}^n$$

is well defined.

2. There are many curves through every point $p \in X$: if $\eta : W \longrightarrow \mathbb{R}^n$ is as in Theorem 1.1.4.3 with $p = \eta(q)$, and $\gamma' : (-\epsilon, \epsilon) \longrightarrow W$ a curve through q in $\mathbb{R}^k$, then $\gamma := \eta \circ \gamma'$ is a curve in X through p.

Definition 1.2.3 *1. The set*

$$T_pX := \{ \dot\gamma(0) \mid \gamma \text{ a curve in } X \text{ through } p \} \subset \mathbb{R}^n$$

is called the tangent space of X at p.

2. Let $f : U_p \longrightarrow \mathbb{R}^{n-k}$ *be as in Definition 1.1.1; then we define*

$$T_p^f X := \ker Df(p) \ .$$

3. Let $\phi : U \longrightarrow U'$ *be as in Theorem 1.1.4.2; then we define*

$$T_p^\phi X := D\phi(p)^{-1}(\mathbb{R}^k) = D(\phi^{-1})(\phi(p))(\mathbb{R}^k),$$

where we identify $\mathbb{R}^k$ *with* $\{ \ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = \ldots = x_{n-k} = 0 \ \}$ *.*

4. Let $\eta : W \longrightarrow \mathbb{R}^n$ *is as in Theorem 1.1.4.3 with* $p = \eta(q)$ *; then we define*

$$T_p^\eta X := D\eta(q)(\mathbb{R}^k) \ .$$

Theorem 1.2.4 *It holds*

$$T_p X = T_p^f X = T_p^\phi X = T_p^\eta X \ .$$

In particular, T_pX *is a* k*-dimensional linear subspace of* $\mathbb{R}^n$*, and* $T_p^f X$ *resp.* $T_p^\phi X$ *resp.* $T_p^\eta X$ *is independent of the choice of* f *resp.* ϕ *resp.* η*.*

Proof: $T_p^f X$, $T_p^\phi X$ and $T_p^\eta X$ are image respectively kernel of a linear map and hence linear subspaces of $\mathbb{R}^n$, and the rank conditions on the maps involved imply that they all have dimension k. Therefore, it suffices to show that

$$T_p^\eta X \subset T_p^\phi X \subset T_p X \subset T_p^f X \ .$$

By making W sufficiently small, we may assume that $\eta(W) \subset U$ so that $\phi \circ \eta : W \longrightarrow \mathbb{R}^n$ is well defined and differentiable. Since $\eta(W) \subset X \cap U$ and $\phi(X \cap U) \subset \mathbb{R}^k$, we have $(\phi \circ \eta)(W) \subset \mathbb{R}^k$, implying, using the chain rule,

$$\mathbb{R}^k \supset D(\phi \circ \eta)(q)(\mathbb{R}^k) = D\phi(p) \left(D\eta(q)(\mathbb{R}^k) \right) = D\phi(p) \left(T_p^\eta X \right) \ ,$$

i.e.

$$T_p^\eta X \subset D\phi(p)^{-1}(\mathbb{R}^k) = T_p^\phi X \ .$$

Every vector $v \in T_p^\phi X$ is by definition of the form

$$v = D\phi(p)^{-1}(w) = \frac{d}{dt} \left(\phi^{-1}(\phi(p) + t \cdot w) \right) |_{t=0}$$

with $w \in \mathbb{R}^k$.
Since ϕ maps $X \cap U$ bijectively to $\mathbb{R}^k \cap U'$, it holds $\phi(p) \in \mathbb{R}^k$ and hence $\phi(p) + t \cdot w \in \mathbb{R}^k$ for all t. This implies that for ϵ small enough the curve

$$\gamma : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}^n \quad , \quad \gamma(t) = \phi^{-1}(\phi(p) + t \cdot w) \ ,$$

is a curve in X through p, and hence

$$v = \frac{d}{dt} \left(\phi^{-1}(\phi(p) + t \cdot w) \right) |_{t=0} = \dot{\gamma}(0) \in T_p X \ .$$

This shows the second inclusion

$$T_p^\phi X \subset T_p X \ .$$

Finally, let be $\dot\gamma(0) \in T_p X$; by choosing ϵ small enough we may assume $\gamma(-\epsilon, \epsilon) \subset X \cap U_p$. Then $f \circ \gamma \equiv 0$, and hence, by the chain rule,

$$0 = \frac{d}{dt} (f \circ \gamma(t)) \, |_{t=0} = Df(p)(\dot\gamma(0)) \ ,$$

i.e.

$$\dot\gamma(0) \in \ker Df(p) = T_p^f X \ .$$

This proves the last inclusion

$$T_p X \subset T_p^f X \ .$$

■

Example: We consider $\mathbb{R}^{n+1}$ with standard inner product $\langle \ , \ \rangle$, and the submanifold

$$S^n := \{ \ x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle = 1 \ \} \ .$$

For $f(x) := \langle x, x \rangle - 1$, $x \in S^n$ and $y \in \mathbb{R}^{n+1}$ it holds $Df(x)(y) = 2\langle x, y \rangle$, hence

$$T_x^f S^n = x^\perp \subset \mathbb{R}^{n+1} \ .$$

Remark 1.2.5 *Consider* $X = \{ \ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = |x_1| \ \}$; *we will now show, as promised in Remark 1.1.5.2, that this is not a 1-dimensional submanifold of* $\mathbb{R}^2$.

If X *would be a 1-dimensional submanifold, then it would have a 1-dimensional tangent space at the point* $(0,0)$. *In particular, there would be a differentiable curve* $\gamma = (\gamma_1, \gamma_2) : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}^2$ *with* $\gamma(-\epsilon, \epsilon) \subset X$, $\gamma(0) = (0,0)$ *and* $\dot\gamma(0) \neq (0,0)$. $\gamma(-\epsilon, \epsilon) \subset X$ *implies* $\gamma_1^2(t) = \gamma_2^2(t)$ *for all* t*; differentiating this twice we get* $2\dot\gamma_1(t)^2 + 2\gamma_1(t)\ddot\gamma_1(t) = 2\dot\gamma_2(t)^2 + 2\gamma_2(t)\ddot\gamma_2(t)$. *Using* $\gamma_1(0) = \gamma_2(0) = 0$ *this gives*

$$\dot\gamma_1(0)^2 = \dot\gamma_2(0)^2 \ . \quad (*)$$

On the other hand, $\gamma(-\epsilon, \epsilon) \subset X$ *implies* $\gamma_2(t) \geq 0$ *for all* t*, and* $\gamma(0) = (0,0)$ *implies* $\gamma_2(0) = 0$. *This means that* γ_2 *has a minimum at* $t = 0$, *and hence that* $\dot\gamma_2(0) = 0$. *But then* $(*)$ *implies* $\dot\gamma_1(0) = 0$, *too, so* $\dot\gamma(0) = (0,0)$; *a contradiction.*

2 Differentiable manifolds

2.1 Manifolds

Let X be a topological space.

Definition 2.1.1 *1. X is called locally Euclidean if for every point $p \in X$ there exists an open neighborhood U of p in X, an $n \in \mathbb{N}$ and an open subset $V \subset \mathbb{R}^n$, and a homeomorphism $h : U \longrightarrow V$.*

2. An n-dimensional topological atlas for X is a set

$$\mathcal{A} = \{ \ (U_i, h_i, V_i) \mid i \in I \ \} \ ,$$

where I is an index set, and for each $i \in I$
- U_i is an open subset of X and $X = \bigcup_{i \in I} U_i$,
- V_i is an open subset of $\mathbb{R}^n$,
- $h_i : U_i \longrightarrow V_i$ is a homeomorphism.

3. An n-dimensional topological manifold is a pair $(X, \mathcal{A})$, where X is a topological space and $\mathcal{A}$ an n-dimensional topological atlas for X.

It is obvious that an n-dimensional topological manifold is locally Euclidean.

Excercise 2.1.2 *1. Give an example of a locally Euclidean space which is not Hausdorff.*

2. A standard result in topology states that open subsets $V \in \mathbb{R}^n$, $W \in \mathbb{R}^m$ can be homeomorphic only if $m = n$. Use this to show that every connected component of a locally Euclidean space has the structure of a topological manifold of a well defined dimension.

3. Let $(X, \mathcal{A})$ be an n-dimensional topological manifold. Show that for all $i, j \in I$ the gluing map

$$(h_j \circ h_i^{-1})|_{h_i(U_i \cap U_j)} : h_i(U_i \cap U_j) \longrightarrow h_j(U_i \cap U_j)$$

is a homeomorphism.

Example 2.1.3 *Consider $\mathbb{R}^{n+1}$ with standard coordinates $x = (x_1, x_2, \ldots, x_{n+1})$. The n-dimensional unit sphere is defined as*

$$S^n := \{ \ x \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1 \ \} \ .$$

Define for $1 \le i \le n+1$ $U_{i,\pm} := \{ \ x \in S^n \mid \pm x_i > 0 \ \}$; it is clear that these hemispheres cover S^n. Let

$$D^n := \{ \ y \in \mathbb{R}^n \mid \sum_{i=1}^{n} y_i^2 < 1 \ \}$$

be the n-dimensional open unit disk. Then the projections

$$h_{i,\pm} : U_{i,\pm} \longrightarrow D^n \quad , \quad (x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n+1})$$

are continuous with continuous inverse

$$h_{i,\pm}^{-1} = (y_1, \ldots, y_n) \mapsto (y_1 \ldots, y_{i-1}, \ \pm\sqrt{1 - \sum_{j=1}^{n} y_j^2} \ , y_i, \ldots, y_n) \ .$$

This means that S^n is an n-dimensional topological manifold.

Definition 2.1.4 *Let $\mathcal{A} = \{ \ (U_i, h_i, V_i) \mid i \in I \ \}$ be an n-dimensional topological atlas for X. $\mathcal{A}$ is called an n-dimensional differentiable atlas for X if for all $i, j \in I$ the gluing map*

$$(h_j \circ h_i^{-1})|_{h_i(U_i \cap U_j)} : h_i(U_i \cap U_j) \longrightarrow h_j(U_i \cap U_j)$$

is differentiable.

From Exercise 2.1.2 it follows that each gluing map $(h_j \circ h_i^{-1})|_{h_i(U_i \cap U_j)}$ is bijective; the inverse is $(h_i \circ h_j^{-1})|_{h_j(U_i \cap U_j)}$. Since differentiability of $\mathcal{A}$ means that both these maps are differentiable, it holds

Remark 2.1.5 *$\mathcal{A}$ is a differentiable atlas if and only if all gluing maps are diffeomorphisms.*

Examples 2.1.6

1. *By Theorem 1.1.4, every k-dimensional submanifold X of $\mathbb{R}^n$ admits a k-dimensional differentiable atlas.*
2. *If $U \subset \mathbb{R}^n$ is open and $h : X \longrightarrow U$ is a homeomorphism, then $\{(X, h, U)\}$ is an n-dimensional differentiable atlas because $h \circ h^{-1} = \mathrm{id}_U$ is differentiable.*
   A particular example for this is the standard atlas $\mathcal{A}(U) := \{(U, \mathrm{id}_U, U)\}$ for an open subset $U \subset \mathbb{R}^n$; another is the atlas $\{\mathbb{R}, h, \mathbb{R}\}$ where $h(x) = x^3$.
3. *Consider $X = \{ \ (x_1, x_2) \in \mathbb{R}^2 \mid x_2 = |x_1| \ \}$; we have seen in Remark 1.2.5 that X is no submanifold of $\mathbb{R}^2$. But X has the 1-dimensional differentiable atlas $\{X, h, \mathbb{R}\}$ where $h(x_1, x_2) = x_1$, since this projection is continuous with continuous inverse $h^{-1}(x) = (x, |x|)$.*

Example 2.1.7 *The topological atlas $\{ \ (U_{i,\pm}, h_{i\pm}, D^n) \mid 1 \le i \le n+1 \ \}$ for the unit sphere $S^n \subset \mathbb{R}^n$ (see Example 2.1.3) is a differentiable atlas; e.g. for $i < j$ it holds*

$$h_{j,+} \circ h_{i,+}^{-1}(y_1 \ldots, y_n) = (y_1 \ldots, y_{i-1}, \ \sqrt{1 - \sum_{j=1}^{n} y_j^2} \ , y_i, \ldots, y_{j-2}, y_j, \ldots, y_n) \ .$$

Another way to see the differentiability of the atlas is to observe that each gluing map $h_{j,\pm} \circ h_{i,\pm}^{-1}$ is the composition of the differentiable map $h_{i,\pm}^{-1} : D^n \longrightarrow \mathbb{R}^n$ with a projection from $\mathbb{R}^{n+1}$ to $\mathbb{R}^n$, which is differentiable, too.

Two n-dimensional differentiable atlases $\mathcal{A} = \{ (U_i, h_i, V_i) \mid i \in I \}$, $\mathcal{A}' = \{ (U'_j, h'_j, V'_j) \mid j \in J \}$ for a topological space X are called equivalent, notation $\mathcal{A} \sim \mathcal{A}'$, if their union $\mathcal{A} \cup \mathcal{A}'$ is an n-dimensional differentiable atlas, too, i.e. if for all $i \in I$, $j \in J$ the maps

$$h_i \circ (h'_j)^{-1}|_{h'_j(U_i \cap U'_j)} \;:\; h'_j(U_i \cap U'_j) \longrightarrow h_i(U_i \cap U'_j) \;,$$
$$h'_j \circ h_i^{-1}|_{h_i(U_i \cap U'_j)} \;:\; h_i(U_i \cap U'_j) \longrightarrow h'_j(U_i \cap U'_j)$$

are differentiable. It is left as an easy exercise to check that this indeed is an equivalence relation.

An n-dimensional differentiable atlas $\mathcal{A}$ is called maximal or an n-dimensional differentiable structure in X if it holds

$$\mathcal{A} \sim \mathcal{A}' \;\Rightarrow\; \mathcal{A}' \subset \mathcal{A} \;.$$

It is easy to see that the equivalence class of an atlas $\mathcal{A}$ contains the unique maximal atlas

$$\mathcal{M}_{\mathcal{A}} = \bigcup_{\mathcal{A} \sim \mathcal{A}'} \mathcal{A}' \;.$$

Remarks 2.1.8 *1. Let be* $(U_i, h_i, V_i) \in \mathcal{A}$.

  (a) If $U \subset U_i$ *is open, then* $(U, h_i|_U, h_i(U)) \in \mathcal{M}_{\mathcal{A}}$.
  If $V \subset V_i$ *is open, then* $(h_i^{-1}(V), h_i|_{h_i^{-1}(V)}, V) \in \mathcal{M}_{\mathcal{A}}$.

  (b) If $g : V_i \longrightarrow V \subset \mathbb{R}^n$ *is a diffeomorphism, then* $(U_i, g \circ h_i, V) \in \mathcal{M}_{\mathcal{A}}$ *. In particular, for every point* $p \in X$ *there exists a chart* $(U, h, V) \in \mathcal{M}_{\mathcal{A}}$ *with* $V = \{ x \in \mathbb{R}^n \mid \|x\| < 1 \}$ *the open unit ball and* $h(p) = 0$.

2. Consider for $\mathbb{R}$ *the two 1-dimensional differentiable atlases* $\{(\mathbb{R}, \mathrm{id}_{\mathbb{R}}, \mathbb{R}\}$ *and* $\{(\mathbb{R}, h, \mathbb{R}\}$*, where* $h(x) = x^3$ *(see Examples 2.1.6). These are not equivalent since* $(\mathrm{id}_{\mathbb{R}} \circ h^{-1})(x) = x^{\frac{1}{3}}$ *is not differentiable at* $x = 0$.

Excercise 2.1.9 *Consider the standard atlas* $\mathcal{A}(\mathbb{R}^n) = \{(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n}, \mathbb{R}^n)\}$ *. Show that the associated maximal atlas is*

$$\mathcal{M}_{\mathcal{A}(\mathbb{R}^n)} = \{ (U, h, V) \mid U, V \subset \mathbb{R}^n \text{ open },\; h : U \longrightarrow V \text{ diffeomorphism } \} \;.$$

Definition 2.1.10 *An n-dimensional differentiable manifold is a pair* $(X, \mathcal{M})$ *where* X *is a second countable Hausdorff space, and* $\mathcal{M}$ *is a maximal n-dimensional differentiable atlas for* X.

Remarks 2.1.11 *1. If a second countable Hausdorff space admits some n-dimensional differentiable atlas, then it has the structure of an n-dimensional differentiable manifold.*

2. The induced topology in every subset of a second countable Hausdorff space has these properties, too. Therefore:

  (a) Since $\mathbb{R}^n$ *is second countable and Hausdorff, every k-dimensional submanifold of* $\mathbb{R}^n$ *has the structure of an k-dimensional differentiable manifold.*

(b) Let $(X, \mathcal{M})$ be an n-dimensional differentiable manifold and $Y \subset X$ open. Then it is easy to see that

$$\mathcal{M}_Y := \{ \ (Y \cap U, h|_{Y\cap U}, h(Y \cap U)) \mid (U, h, V) \in \mathcal{M} \ \}$$

is a maximal n-dimensional differentiable atlas for Y, so $(Y, \mathcal{M}_Y)$ is an n-dimensional differentiable manifold, too.

Remark 2.1.12 *Example 2.1.6.1 states that a k-dimensional submanifold X of $\mathbb{R}^n$ admits a k-dimensional differentiable atlas. In the proof of the implication 1. $\Rightarrow$ 4. in Theorem 1.1.4 we have constructed an atlas $\mathcal{A}$ for X consisting of charts (U, h, V) with the following properties:*

h is the restriction to an open subset of X of an orthogonal projection,
h^{-1} is differentiable if viewed as a map from $V \subset \mathbb{R}^k$ to $\mathbb{R}^n$.

But any two atlases consisting of charts of this type are equivalent because the composition of a differentiable map with a projection, and vice versa, is differentiable. This means that each of these atlases defines the same k-dimensional differentiable structure in X. We call this the natural differentiable structure in X.

Excercise 2.1.13 *Let X be a k-dimensional submanifold of $\mathbb{R}^n$. Show that an atlas for X as in Exercise 1.1.6.1 defines the natural differentiable structure in X.*

Excercise 2.1.14 *Show that the 0-dimensional differentiable manifolds are precisely the discrete topological spaces with at most countably many points.*

Example 2.1.15 *Let $(X, \mathcal{T})$ be a topological space, Y a set and $\psi : Y \longrightarrow X$ a bijective map. Then there is a unique topology $\mathcal{T}_\psi$ in Y such that $\psi : (Y, \mathcal{T}_\psi) \longrightarrow (X, \mathcal{T})$ is a homeomorphism, namely $\mathcal{T}_\psi = \{ \ \psi^{-1}(U) \mid U \in \mathcal{T} \ \}$.*

Now let V be an n-dimensional real vector space. Every basis $\mathcal{B} = \{b_1, \ldots, b_2\}$ of V determines a bijective linear map (the coordinate isomorphism) $\psi_\mathcal{B} : V \longrightarrow \mathbb{R}^n$ by $\psi_\mathcal{B}(\sum_{i=1}^{n} \lambda_i b_i) = (\lambda_1, \ldots, \lambda_n)$, and hence a topology $\mathcal{T}_\mathcal{B}$ in V such that $\psi_\mathcal{B}$ becomes a homeomorphism.
Let $\mathcal{B}'$ be another basis of V. Since linear maps between Euclidean spaces are continuous, the bijective linear map $\psi_{\mathcal{B}'} \circ \psi_\mathcal{B}^{-1} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a homeomorphism. Hence the composition of homeomorphisms

$$\mathrm{id}_V : (V, \mathcal{T}_\mathcal{B}) \xrightarrow{\psi_\mathcal{B}} \mathbb{R}^n \xrightarrow{\psi_{\mathcal{B}'} \circ \psi_\mathcal{B}^{-1}} \mathbb{R}^n \xrightarrow{\psi_{\mathcal{B}'}^{-1}} (V, \mathcal{T}_{\mathcal{B}'})$$

is a homeomorphism, too; this implies $\mathcal{T}_\mathcal{B} = \mathcal{T}_{\mathcal{B}'}$. In other words, there is a unique topology $\mathcal{T}_V$ in V such that each coordinate isomorphism $\psi_\mathcal{B} : (V, \mathcal{T}_V) \longrightarrow \mathbb{R}^n$ is a homeomorphism; $\mathcal{T}_V$ is called the natural topology in V.

According to Example 2.1.6.2, each $\{(V, \psi_\mathcal{B}, \mathbb{R}^n)\}$ is an n-dimensional differentiable atlas for V with its natural topology. Since each linear isomorphism $\psi_{\mathcal{B}'} \circ \psi_\mathcal{B}^{-1}$ is not only a homeomorphism but even a diffeomorphism, it follows that all these atlases are equivalent, i.e. that they define the same natural differentiable structure in V.

To close this subsection we state without the easy proof

Proposition 2.1.16 *1. Let X and Y be n-dimensional differentiable manifolds with atlases $\mathcal{A}_X$ and $\mathcal{A}_Y$. Then the disjoint union $\mathcal{A}_X \dot{\cup} \mathcal{A}_Y$ is an n-dimensional differentiable atlas for the disjoint union $X \dot{\cup} Y$ equipped with the topology defined by*

$$U \subset X \dot{\cup} Y \text{ is open } \iff U \cap X \subset X \text{ is open and } U \cap Y \subset Y \text{ is open }.$$

The equivalence class of this atlas depends only on the equivalence classes of $\mathcal{A}_X$ and $\mathcal{A}_Y$, and the resulting n-dimensional differentiable manifold is denoted $X+Y$ and called the differentiable sum of X and Y.

2. Let X be an n-dimensional and Y an m-dimensional manifold with atlases $\mathcal{A}_X$ and $\mathcal{A}_Y$. Then an $(n+m)$-dimensional differentiable atlas in the topological product $X \times Y$ is given by

$$\mathcal{A}_{X \times Y} := \{ \ (U \times U', h \times h', V \times V') \mid (U,h,V) \in \mathcal{A}_X \ , \ (U',h',V') \in \mathcal{A}_Y \ \} \ ,$$

where $(h \times h')(p,p') := (h(p), h'(p'))$. The equivalence class of this atlas depends only on the equivalence classes of $\mathcal{A}_X$ and $\mathcal{A}_Y$, and the resulting $(n+m)$-dimensional differentiable manifold is (again) denoted $X \times Y$ and called the differentiable product of X and Y.

2.2 Differentiable maps

Let $(X, \mathcal{M})$ be an n-dimensional differentiable manifold, where $\mathcal{M} = \{ \ (U_i, h_i, V_i) \mid i \in I \ \}$.

Definition 2.2.1 *A function $f : X \longrightarrow \mathbb{R}$ is called differentiable if for every $i \in I$ the function $f \circ h_i^{-1} : V_i \longrightarrow \mathbb{R}$ is differentiable.*

Remarks 2.2.2 *1. If $f : X \longrightarrow \mathbb{R}$ is differentiable, then every $f \circ h_i^{-1} : V_i \longrightarrow \mathbb{R}$ is differentiable and hence continuous. Since continuity is a local question, each h_i is a homeomorphism, and the U_i cover X, it follows that f is continuous.*

2. Let $\mathcal{A} \subset \mathcal{M}$ be any atlas for X. Then for the differentiability of f it is sufficient that $f \circ h_j^{-1}$ is differentiable for all $(U_j, h_j, V_j) \in \mathcal{A}$; this can be seen as follows:
Let be $(U_i, h_i, V_i) \in \mathcal{M}$ and $x \in V_i$; it is sufficient to show that $f \circ h_i^{-1}$ is differentiable in some open neighborhood of x.
Since $\mathcal{A}$ is an atlas, there exists an $(U_j, h_j, V_j) \in \mathcal{A}$ such that $p := h_i^{-1}(x) \in U_j$. Then $U_i \cap U_j$ resp. $h_i(U_i \cap U_j)$ resp. $h_j(U_i \cap U_j)$ is an open neighborhood of p resp. x resp. $h_j(p)$, and it holds

$$(f \circ h_i^{-1})|_{h_i(U_i \cap U_j)} = (f \circ h_j^{-1})|_{h_j(U_i \cap U_j)} \circ (h_j \circ h_i^{-1})|_{h_i(U_i \cap U_j)} \ .$$

This is differentiable since $(h_j \circ h_i^{-1})|_{h_i(U_i \cap U_j)}$ is differentiable because $\mathcal{M}$ is a differentiable structure, and $(f \circ h_j^{-1})|_{h_j(U_i \cap U_j)}$ is differentiable by assumption.

Definition 2.2.3 *Let* $(X, \mathcal{M})$ *be an* n*-dimensional and* $(X', \mathcal{M}')$ *an* m *dimensional differentiable manifold. Let* $f : X \longrightarrow X'$ *be a map.*

1. f *is called* *differentiable* *if it is continuous, and*

$$(h' \circ f \circ h^{-1})|_{h(U \cap f^{-1}(U'))} : h(U \cap f^{-1}(U')) \longrightarrow V'$$

is differentiable for all $(U, h, V) \in \mathcal{M}$, $(U', h', V') \in \mathcal{M}'$.

2. f *is called a* *diffeomorphism* *if it is differentiable and bijective, with differentiable inverse map* $f^{-1} : X' \longrightarrow X$.
$(X, \mathcal{M})$ *and* $(X', \mathcal{M}')$ *are called* *diffeomorphic* *if a diffeomorphism* $f : X \longrightarrow X'$ *exists.*

Remarks 2.2.4 1. *In contrast to the definition of a differentiable function on a manifold, for the definition of a differentiable map between two manifolds we ask* f *to be continuous* a priori *for the following reason. Differentiability of a map between subsets of Euclidean spaces makes sense only if the domain of this map is open. Therefore, if we want to check differentiability of "*f *viewed in charts", i.e. of* $h' \circ f \circ h^{-1}$*, we have to know that its domain, i.e.* $h(U \cap f^{-1}(U'))$*, is open. Since* h *is a homeomorphism, this is true iff* $U \cap f^{-1}(U')$ *is open, which holds if* f *is continuous.*

2. *An argument analogous to that in Remark 2.2.2.2 shows that the differentiability of a map* f *can be checked by verifying the differentiability of* f *viewed in charts from* *some* *atlases for* X *and* X'.

3. *Let* (U, h, V) *be a chart for* X*; then* U *resp.* V *has a differentiable structure as open subset of* X *resp.* $\mathbb{R}^n$*, and* $h : U \longrightarrow V$ *is a diffeomorphism since* $\mathrm{id}_V \circ h \circ h^{-1} = \mathrm{id}_V = h \circ h^{-1} \circ \mathrm{id}_V^{-1}$.
Conversely, if $U \subset X$ *and* $V \subset \mathbb{R}^n$ *are open and* $h : U \longrightarrow V$ *is a diffeomorphism, the* (U, h, V) *belongs to the maximal atlas* $\mathcal{M}$ *of* X*. Indeed,* (U, h, V) *is a topological chart since* h *is a homeomorphism, and for every chart* $(U', h', V') \in \mathcal{M}$ *it holds (where defined)*

$$\begin{aligned} h \circ h'^{-1} &= \mathrm{id}_V \circ h \circ h'^{-1} , \\ h' \circ h^{-1} &= h' \circ h^{-1} \circ \mathrm{id}_V^{-1} . \end{aligned}$$

But the two maps on the right are differentiable because h *is a diffeomorphism.*
In other words it holds

$$\mathcal{M} = \{ (U, h, V) \mid U \subset X \text{ open}, \ V \subset \mathbb{R}^n \text{ open}, \ h : U \longrightarrow V \text{ a diffeomorphism} \} .$$

4. *A diffeomorphism is, in particular, a homeomorphism. Thus, for topological reasons, diffeomorphic manifolds have the same dimension.*

Example 2.2.5 *Consider for* $\mathbb{R}$ *the two atlases* $\{(\mathbb{R}, \mathrm{id}_\mathbb{R}, \mathbb{R})\}$ *and* $\{(\mathbb{R}, h, \mathbb{R})\}$*, where* $h(x) = x^3$ *(see Examples 2.1.6); the differentiable structures* $\mathcal{M}_1 = \mathcal{M}_{\{(\mathbb{R}, \mathrm{id}_\mathbb{R}, \mathbb{R})\}}$ *and* $\mathcal{M}_2 = \mathcal{M}_{\{(\mathbb{R}, h, \mathbb{R})\}}$ *in* $\mathbb{R}$ *are not equal by Remarks 2.1.8. But*

$$f : (\mathbb{R}, \mathcal{M}_1) \longrightarrow (\mathbb{R}, \mathcal{M}_2) \ , \ f(x) = x^{\frac{1}{3}}$$

is a diffeomorphism of manifolds since

$$(h \circ f \circ (\mathrm{id}_{\mathbb{R}})^{-1})(x) = x = (\mathrm{id}_{\mathbb{R}} \circ f^{-1} \circ h^{-1})(x) \ ,$$

i.e.

$$h \circ f \circ (\mathrm{id}_{\mathbb{R}})^{-1} = \mathrm{id}_{\mathbb{R}} = \mathrm{id}_{\mathbb{R}} \circ f^{-1} \circ h^{-1} \ ,$$

i.e. f *and* f^{-1} *are differentiable in the charts* $\mathrm{id}_{\mathbb{R}}$ *and* h.

From now on, when we write "a differentiable manifold X" we assume that some differentiable atlas for X has been fixed.

Lemma 2.2.6 *Let* X_1 , X_2, X_3 *be differentiable manifolds and* $f_1 : X_1 \longrightarrow X_2$, $f_2 : X_2 \longrightarrow X_3$ *differentiable maps. Then* $f := f_2 \circ f_1 : X_1 \longrightarrow X_3$ *is differentiable.*

Proof: Let (U_1, h_1, V_1) resp. (U_3, h_3, V_3) be a chart for X_1 resp. X_3; we have to show $h_3 \circ f \circ h_1^{-1}$ is differentiable in $h_1(U_1 \cap f^{-1}(U_3))$. This is a local question, so let be $x \in h_1(U_1 \cap f^{-1}(U_3))$, $p_1 := h_1^{-1}(x)$, $p_2 := f_1(p_1)$, $p_3 := f_2(p_2)$, and let (U_2, h_2, V_2) be a chart for X_2 with $p_2 \in U_2$. Since f_1 and f_2 are continuous, there are open $W_i \subset U_i$ with $p_i \in W_i$, $i = 1, 2, 3$, such that $f_i(W_i) \subset W_{i+1}$ for $i = 1, 2$. Since $h_1(W_1)$ is an open neighborhood of x it suffices to show that $h_3 \circ f \circ h_1^{-1}$ is differentiable on this. This follows from

$$(h_3 \circ f \circ h_1^{-1})|_{h_1(W_1)} = (h_3 \circ f_2 \circ f_1 \circ h_1^{-1})|_{h_1(W_1)} = \big((h_3 \circ f_2 \circ h_2^{-1})|_{h_2(W_2)}\big) \circ \big((h_2 \circ f_1 \circ h_1^{-1})|_{h_1(W_1)}\big) \ ,$$

and the fact that $(h_3 \circ f_2 \circ h_2^{-1})|_{h_2(W_2)}$ and $(h_2 \circ f_1 \circ h_1^{-1})|_{h_1(W_1)}$ are differentiable because f_1 and f_2 are. ■

Definition 2.2.7 *Let* $f : X \longrightarrow X'$ *be a differentiable map between differentiable manifolds and* $p \in X$. *The* <u>*rank*</u> *of* f *at* p *is*

$$\mathrm{rk}_p f := \mathrm{rk}\left(D(h' \circ f \circ h^{-1})(h(p))\right) \ ,$$

where (U, h, V) *resp.* (U', h', V') *is a chart for* X *resp.* X' *with* $p \in U$ *resp.* $f(p) \in U'$.

Lemma 2.2.8 *The definition of* $\mathrm{rk}_p f$ *is independent of the choice of the charts.*

Proof: Let $(\tilde{U}, \tilde{h}, \tilde{V})$ and $(\tilde{U}', \tilde{h}', \tilde{V}')$ be other charts around p and $f(p)$, then by the chain rule it holds

$$\begin{aligned} D(h' \circ f \circ h^{-1})(h(p)) &= D(h' \circ (\tilde{h}')^{-1} \circ \tilde{h}' \circ f \circ \tilde{h}^{-1} \circ \tilde{h} \circ h^{-1})(h(p)) \\ &= \left[D(h' \circ (\tilde{h}')^{-1})(\tilde{h}'(f(p)))\right] \circ \left[D(\tilde{h}' \circ f \circ \tilde{h}^{-1})(\tilde{h}(p))\right] \circ \left[D(\tilde{h} \circ h^{-1})(h(p))\right] . \end{aligned}$$

Since the gluing maps $h' \circ (\tilde{h}')^{-1}$ and $\tilde{h} \circ h^{-1}$ are diffeomorphisms (Remark 2.1.5), the Jacobians $D(h' \circ (\tilde{h}')^{-1})(\tilde{h}'(f(p)))$ and $D(\tilde{h} \circ h^{-1})(h(p))$ are isomorphisms. This implies

$$\mathrm{rk}\left(D(h' \circ f \circ h^{-1})(h(p))\right) = \mathrm{rk}\left(D(\tilde{h}' \circ f \circ \tilde{h}^{-1})(\tilde{h}(p))\right) \ .$$

■

Theorem 2.2.9 (Rank Theorem) *Let X resp. X' be an n- resp. m-dimensional differentiable manifold, $p \in X$, $f : X \longrightarrow X'$ a differentiable map, and $k \in \mathbb{N}$.*
Suppose that there is an open neighborhood U_0 of p in X such that $\mathrm{rk}_q f = k$ for all $q \in U_0$. Then there are charts (U, h, V) for X with $p \in U$, (U', h', V') for X' with $f(p) \in U'$, such that $f(U) \subset U'$ and

$$\left(h' \circ f \circ h^{-1}\right)(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_k, 0, \ldots, 0) \in \mathbb{R}^m$$

for all $(x_1, x_2, \ldots, x_n) \in V$.

Proof: In this proof, when we write "we may assume that" without comment, then we mean "we may assume, by shrinking charts if necessary, that" (see Remarks 2.1.8).

Choose a chart (U', h'_1, V'_1) around $f(p)$. Since f is continuous, there exists a chart (U, h_1, V_1) around p with $f(U) \subset U'$; we define

$$\hat{f} = (\hat{f}_1, \ldots, \hat{f}_m) := h'_1 \circ f \circ h_1^{-1} : V_1 \longrightarrow V'_1 \ .$$

By assumption we may assume that

$$\mathrm{rk}_q f = k \ \text{ for all } \ q \in U \ ,$$

i.e. that

$$\mathrm{rk}(D\hat{f}(x)) = k \ \text{ for all } \ x \in V_1 \ .$$

This means in particular that the Jacobian matrix $D\hat{f}(h_1(p))$ contains an invertible $k \times k$-submatrix which is invertible. Therefore, after renumbering the coordinates in V_1 and V'_1 if necessary, we may assume that $D\hat{f}(x)$ has the block form

$$D\hat{f}(x) = \left(\frac{\partial \hat{f}_i}{\partial x_j}(x)\right)_{\substack{i=1,\ldots,m \\ j=1,\ldots,n}} = \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix}$$

with $\det A(h_1(p)) \neq 0$. Since $D\hat{f}(x)$, and hence $\det A(x)$, depends differentiably and thus continuously on x, we may assume that $\det A(x) \neq 0$ for all $x \in V_1$. We define

$$g : V_1 \longrightarrow \mathbb{R}^n \quad , \quad g(x_1, \ldots, x_n) := (\hat{f}_1(x_1, \ldots, x_n), \ldots, \hat{f}_k(x_1, \ldots, x_n), x_{k+1}, \ldots, x_n) \ ;$$

then

$$Dg(x) = \begin{pmatrix} A(x) & * \\ 0 & I_{n-k} \end{pmatrix} ,$$

and hence

$$\det Dg(x) = \det A(x) \neq 0 \ \text{ for all } \ x \in V_1 \ .$$

By the Inverse Function Theorem 0.0.2 we may assume that $V := g(V_1)$ is open and $g : V_1 \longrightarrow V$ a diffeomorphism. Then $h := g \circ h_1 : U \longrightarrow V$ is a chart for X around p by Remarks 2.1.8, hence for

$$\check{f} = (\check{f}_1, \ldots, \check{f}_m) := h'_1 \circ f \circ h^{-1} = \hat{f} \circ g^{-1} : V \longrightarrow V'_1$$

it holds

$$\mathrm{rk}(D\check{f}) = \mathrm{rk}_q f \equiv k \ \text{ in } \ V\ . \quad (1)$$

For

$$V \ni (x_1, \dots, x_n) = x = g(y) = g(y_1, \dots, y_n) = (\hat{f}_1(y), \dots, \hat{f}_k(y), y_{k+1}, \dots, y_n)$$

we have

$$(x_1, \dots, x_k) = (\hat{f}_1(y), \dots, \hat{f}_k(y))\ ,$$

and hence

$$\check{f}(x) = \hat{f}(y) = (\hat{f}_1(y), \dots, \hat{f}_m(y)) = (x_1, \dots, x_k, \check{f}_{k+1}(x), \dots, \check{f}_m(x))\ . \quad (2)$$

This means in particular that $D\check{f}$ has the block form

$$D\check{f}(x) = \begin{pmatrix} I_k & 0 \\ * & \check{A}(x) \end{pmatrix}\ .$$

(1) implies $\check{A}(x) \equiv 0$, i.e.

$$\frac{\partial \check{f}_i}{\partial x_j} \equiv 0 \ \text{ for } \ i, j \geq k+1\ . \quad (3)$$

We may assume that V is connected; then (3) implies that the $\check{f}_i$, $k+1 \leq i \leq m$, are independent of the x_j, $k+1 \leq j \leq m$, i.e that

$$\check{f}_i(x_1, \dots, x_n) = \check{f}_i(x_1, \dots, x_k, x^0_{k+1}, \dots, x^0_n)\ ,\ i, j \geq k+1\ , \quad (4)$$

where $(x^0_1, \dots, x^0_n) = h(p)$.

We leave it as an exercise to show (using (2)) that we may assume that for all $z = (z_1, \dots, z_m) \in V'_1$ it holds $(z_1, \dots, z_k, x^0_{k+1}, \dots, x^0_n) \in V$, and define

$$\begin{aligned} g' \quad &: \quad V'_1 \longrightarrow \mathbb{R}^m\ , \\ g'(z) \quad &:= \quad (z_1, \dots, z_k, z_{k+1} - \check{f}_{k+1}(z_1, \dots, z_k, x^0_{k+1}, \dots, x^0_n), \dots, z_m - \check{f}_m(z_1, \dots, z_k, x^0_{k+1}, \dots, x^0_n))\ . \end{aligned}$$

Then the Jacobian matrix Dg' has the block form

$$Dg' = \begin{pmatrix} I_k & 0 \\ * & I_{m-k} \end{pmatrix}\ ,$$

and by the Inverse Function Theorem 0.0.2 we may assume that there is an open $V' \subset \mathbb{R}^m$ such that $g' : V'_1 \longrightarrow V'$ is a diffeomorphism. In particular, $h' := g' \circ h'_1 : U' \longrightarrow V'$ is a chart for Y around $f(p)$, and it holds for $x = (x_1, \dots, x_n) \in V$

$$\begin{aligned} & (h' \circ f \circ h^{-1})(x) = (g' \circ h'_1 \circ f \circ h^{-1})(x) = (g' \circ \check{f})(x) \\ = \ & g'(x_1, \dots, x_k, \check{f}_{k+1}(x), \dots, \check{f}_m(x)) \ \text{ by } \ (2) \\ = \ & g'(x_1, \dots, x_k, \check{f}_{k+1}(x_1, \dots, x_k, x^0_{k+1}, \dots, x^0_n), \dots, \check{f}_m(x_1, \dots, x_k, x^0_{k+1}, \dots, x^0_n)) \ \text{ by } \ (4) \\ = \ & (x_1, \dots, x_k, 0, \dots, 0) \ \text{ by the definition of } \ g'\ . \end{aligned}$$

■

Corollary 2.2.10 *Suppose that in the situation of the Rank Theorem 2.2.9 it holds* $n = m = k$ *. Then* f *is a diffeomorphism locally around* p*, i.e. there exist open neighborhoods* $p \in U$ *,* $f(p) \in U'$ *such that* $f|_U : U \longrightarrow U'$ *is a diffeomorphism.*

Proof: With respect to charts as in the Rank Theorem it holds locally around $h(p)$

$$\left(h' \circ f \circ h^{-1}\right)(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n) \ ,$$

i.e.

$$h' \circ f \circ h^{-1} = \mathrm{id}_{\mathbb{R}^n}|_V \ .$$

Hence $h' \circ f \circ h^{-1}$ is a local diffeomorphism and in particular homeomorphism. Since h' and h^{-1} are homeomorphisms, it follows that f is a local homeomorphism around p, in particular that f^{-1} is continuous around $f(p)$. Furthermore, with respect to these charts it holds

$$h \circ f^{-1} \circ (h')^{-1} = \left(h' \circ f \circ h^{-1}\right)^{-1} = \mathrm{id}_{\mathbb{R}^n}|_{V''}$$

in the neighborhood $V'' = (h' \circ f \circ h^{-1})(V) = h'(f(U))$ of $h'(f(p))$; this shows that f^{-1} is differentiable in the neighborhood $f(U)$ of $f(p)$. ■

2.3 Tangent spaces

Let X be an n-dimensional differentiable manifold and $p \in X$. A differentiable curve in X through p is a differentiable map $\gamma : (-\epsilon, \epsilon) \longrightarrow X$, $\epsilon > 0$, with $\gamma(0) = p$. The set of all these curves is denoted by $\mathcal{K}_p$. We say that two curves $\gamma_1, \gamma_2 \in \mathcal{K}_p$ are equivalent, notation $\gamma_1 \sim \gamma_2$, if the following holds.

Let (U, h, V) be a chart for X with $p \in U$. Then for $\gamma \in \mathcal{K}_p$, and for a suitable $0 < \delta \leq \epsilon$, we get a differentiable curve $h \circ \gamma : (-\delta, \delta) \longrightarrow V$, and we define

$$\dot\gamma(0)_h := \frac{d}{dt}(h \circ \gamma)(0) \in \mathbb{R}^n \ .$$

We say that two curves $\gamma_1, \gamma_2 \in \mathcal{K}_p$ are equivalent, notation $\gamma_1 \sim \gamma_2$, if for a chart (U, h, V) as above it holds

$$\dot\gamma_1(0)_h = \dot\gamma_2(0)_h \ .$$

Lemma 2.3.1 $\sim$ *is a well defined, i.e. independent of the chosen chart, equivalence relation in* $\mathcal{K}_p$*.*

Proof: Assume that $\dot\gamma_1(0)_h = \dot\gamma_2(0)_h$, and that (U', h', V') is another chart with $p \in U'$. Then

$$\begin{aligned}
\dot\gamma_1(0)_{h'} &= \frac{d}{dt}(h' \circ \gamma)(0) = \frac{d}{dt}(h' \circ h^{-1} \circ h \circ \gamma)(0) \\
&= D(h' \circ h^{-1})(h(p)) \left(\frac{d}{dt}(h \circ \gamma)(0)\right) \quad \text{by the chain rule} \\
&= D(h' \circ h^{-1})(h(p))\,(\dot\gamma_1(0)_h) = D(h' \circ h^{-1})(h(p))\,(\dot\gamma_2(0)_h) \quad \text{by assumption} \\
&= \dot\gamma_2(0)_{h'} \quad \text{using the previous arguments reversely.}
\end{aligned}$$

This shows that $\sim$ is well defined. The fact that $=$ is an equivalence relation in $\mathbb{R}^n$ immediately implies that $\sim$ is an equivalence relation in $\mathcal{K}_p$. ■

Definition 2.3.2 *The geometric tangent space of X at p is the quotient*

$$T_p^{geom} X := \mathcal{K}_p/_{\sim} \ ,$$

where $\sim$ is the equivalence relation defined above. The equivalence class of $\gamma \in \mathcal{K}_p$ is denoted $[\gamma] \in T_p^{geom} X$, and called a geometric tangent vector of X at p.

Theorem 2.3.3 *$T_p^{geom} X$ has a natural structure of an n-dimensional vector space.*

Proof: Let (U, h, V) be a chart for X with $p \in U$, and define the map $\Phi_h : \mathbb{R}^n \longrightarrow T_p^{geom} X$ as follows.

Since V is open, for every $v \in \mathbb{R}^n$ there exists an $\epsilon > 0$ such that $h(p) + t \cdot v \in V$ for all $t \in (-\epsilon, \epsilon)$. Then

$$\gamma_v : (-\epsilon, \epsilon) \longrightarrow X \quad , \quad \gamma_v(t) := h^{-1}(h(p) + t \cdot v)$$

is an element of $\mathcal{K}_p$, and we define

$$\Phi_h(v) := [\gamma_v] \in T_p^{geom} X \ .$$

We want to show that Φ_h is bijective; for this first observe that it is easy to see that

$$\dot{\gamma}_v(0)_h = \frac{d}{dt}(h \circ \gamma_v)(0) = v \ . \quad (*)$$

This implies that for $v, w \in \mathbb{R}^n$ it holds

$$\Phi_h(v) = \Phi_h(w) \quad \Leftrightarrow \quad [\gamma_v] = [\gamma_w] \quad \Leftrightarrow \quad \gamma_v \sim \gamma_w \quad \Leftrightarrow \quad \dot{\gamma}_v(0)_h = \dot{\gamma}_w(0)_h \quad \Leftrightarrow \quad v = w \ ,$$

i.e. that Φ_h is injective.

Now take $\gamma \in \mathcal{K}_p$, and define $v_\gamma := \dot{\gamma}(0)_h \in \mathbb{R}^n$. Then from $(*)$ it follows

$$\dot{\gamma}_{(v_\gamma)}(0)_h = v_\gamma = \dot{\gamma}(0)_h \quad \Leftrightarrow \quad \Phi_h(v_\gamma) = [\gamma_{v_\gamma}] = [\gamma] \ ,$$

i.e. that Φ_h is surjective.

Hence, Φ_h is bijective with inverse

$$\Phi_h^{-1}([\gamma]) = v_\gamma = \dot{\gamma}(0)_h \ . \quad (**)$$

If (U', h', V') is another chart for X with $p \in U'$, then we define the bijection $\Phi_{h'} : \mathbb{R}^n \longrightarrow T_p^{geom} M$ in the same way; we are done by Lemma 2.3.4 below if we show that $\Phi_{h'}^{-1} \circ \Phi_h : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear isomorphism.

We have, using $(**)$,

$$\Phi_{h'}^{-1} \circ \Phi_h(v) = \Phi_{h'}^{-1}([\gamma_v]) = \dot{\gamma_v}(0)_{h'} = \frac{d}{dt}(h' \circ \gamma_v)(0) \ .$$

Since by definition

$$(h' \circ \gamma_v)(t) = (h' \circ h^{-1})(h(p) + t \cdot v) \ ,$$

the chain rule implies

$$\Phi_{h'}^{-1} \circ \Phi_h(v) = D(h' \circ h^{-1})(h(p))(v) \ .$$

We are done since $h' \circ h^{-1}$ is a local diffeomorphism, and hence $D(h' \circ h^{-1})(h(p))$ a linear isomorphism, by Remark 2.1.5. ■

Lemma 2.3.4 *Let V be a set, and $\Phi : \mathbb{R}^n \longrightarrow V$ a bijective map.*

1. *A structure of an n-dimensional vector space induced by Φ can be defined in V as follows:*
   For $v, w \in V$ define $v + w := \Phi(\Phi^{-1}(v) + \Phi^{-1}(w))$.
   For $v \in V$, $\lambda \in \mathbb{R}$ define $\lambda \cdot v := \Phi(\lambda \cdot \Phi^{-1}(v))$.
   With respect to this structure, Φ is a linear isomorphism.
2. *Let $\Phi' : \mathbb{R}^n \longrightarrow V$ be another bijective map. The structures of n-dimensional vector space in V induced by Φ and Φ' are the same if and only if*

   $$(\Phi')^{-1} \circ \Phi : \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

   is a linear isomorphism.

Proof: This is an easy exercise in linear algebra and therefore left to the reader. ■

Definition 2.3.5 *For data as in the proof of Theorem 2.3.3, we define*

$$\frac{\partial}{\partial x_i}(p) := \Phi_h(e_i) \ , \ i = 1, \ldots, n \ ,$$

where $(e_1, \ldots, e_n)$ is the unit basis of $\mathbb{R}^n$.

Remark 2.3.6 *Since, according to Lemma 2.3.4.1, Φ_h is a linear isomorphism, $(\frac{\partial}{\partial x_1}(p), \ldots, \frac{\partial}{\partial x_n}(p))$ is a basis of $T_p^{geom}X$. Furthermore, for a linear combination $\sum_{i=1}^{n} \lambda_i \cdot \frac{\partial}{\partial x_i}(p)$ it holds*

$$\begin{aligned}
\sum_{i=1}^{n} \lambda_i \cdot \frac{\partial}{\partial x_i}(p) &= \sum_{i=1}^{n} \Phi_h(\lambda_i \cdot e_i) = \Phi_h(\lambda_1, \ldots, \lambda_n) \\
&= \left[t \mapsto h^{-1} \left(h(p) + t \cdot \sum_{i=1}^{n} \lambda_i \cdot e_i \right) \right] \\
&= \left[t \mapsto h^{-1} \left(h(p) + t \cdot (\lambda_1, \ldots, \lambda_n) \right) \right] \ .
\end{aligned}$$

Excercise 2.3.7 *Let M be an n-dimensional differentiable manifold, $p \in M$, (U,h,V), (U',h',V') two charts for M with $p \in U \cap U'$, $(x_1, x_2, \ldots, x_n)$ resp. $(y_1, y_2, \ldots, y_n)$ coordinates in V resp. V', and $\frac{\partial}{\partial x_i}(p)$, $\frac{\partial}{\partial y_i}(p)$, $i = 1, 2, \ldots, n$ the associated bases of $T_p^{geom}M$ (see Definition 2.3.5).*

1. *Show that for $[\gamma] \in T_p^{geom}M$ it holds*

$$[\gamma] = \sum_{i=1}^{n} \frac{d\gamma_i}{dt}(0) \cdot \frac{\partial}{\partial x_i}(p) ,$$

   where $(\gamma_1, \gamma_2, \ldots, \gamma_n) := h \circ \gamma : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}^n$.

2. *Show that*

$$\frac{\partial}{\partial x_i}(p) = \sum_{j=1}^{n} \frac{\partial g_j}{\partial x_i}(h(p)) \cdot \frac{\partial}{\partial y_j}(p) , \ i = 1, 2, \ldots, n ,$$

   where $g = (g_1, g_2, \ldots, g_n) := h' \circ h^{-1}$ is the gluing map.

We now give an alternative definition of the tangent space which sometimes is more convenient.

A differentiable function near p is a pair (f, U) where $U \subset X$ is open with $p \in U$, and $f : U \longrightarrow \mathbb{R}$ is a differentiable function. We say that two such pairs (f, U), (f', U') are equivalent, notation $(f, U) \sim (f', U')$, if and only if there exists an open $V \subset U \cap U'$ with $p \in V$ such that $f|_V = f'|_V$, i.e. if f and f' coincide in a small neighborhood of p. It is easy two see that $\sim$ is indeed an equivalence relation, and hence we get a quotient

$$\mathcal{E}_p := \{ \ (f, U) \mid (f, U) \ \text{ differentiable function near } p \ \}/_{\sim} ;$$

the equivalence class of a pair (f, U) is denoted by $[f, U]$ and called the germ of a differentiable function at p. Since different functions defining the same germ are the same near p, the value

$$[f, U](p) := f(p)$$

is well defined.

Definition 2.3.8 *Let X be a topological space, $A \subset X$ a subset, V a vector space, and $f : A \longrightarrow V$ a map. Then the support of f is the set*

$$\operatorname{supp}(f) := \overline{\{ \ x \in A \mid f(x) \neq 0 \ \}} ,$$

where $\overline{\{ \ \mid \ \}}$ denotes the topological closure in X.

Excercise 2.3.9 1. *Let X be a differentiable manifold, $U \subset X$ open, and $f : U \longrightarrow \mathbb{R}$ a differentiable function with $\operatorname{supp}(f) \subset U$. Show that the function*

$$\tilde{f} : X \longrightarrow \mathbb{R} \ , \ \ \tilde{f}(p) := \begin{cases} f(p) & \text{if } p \in U , \\ 0 & \text{if } p \notin U , \end{cases}$$

   is differentiable in X.

2. *Show that for every germ* $[f,U] \in \mathcal{E}_p$ *there exists a differentiable* $\tilde{f} : X \longrightarrow \mathbb{R}$ *such that* $[f,U] = [\tilde{f}, X]$.

Hint: For the proof of 2. you may use (besides of course 1.) the following fact from calculus (the existence of a bump function).

Let be $x_0 \in \mathbb{R}^n$ *and* $U \subset \mathbb{R}^n$ *an open neighborhood of* x_0*. Then for every* $\epsilon > 0$ *such that the closed ball* $\bar{B}_\epsilon(p)$ *of radius* ϵ *around* p *is contained in* U*, there exists a differentiable (bump) function* $\phi : \mathbb{R}^n \longrightarrow [0,1]$ *with the following properties:*

1. ϕ *is the constant function 1 on* $\bar{B}_\epsilon(p)$,
2. $\mathrm{supp}(\phi) \subset U$.

One readily verifies that $\mathcal{E}_p$ is a vector space by the following rules:

$$[f,U] + [f',U'] := [f|_{U\cap U'} + f'|_{U\cap U'}, U \cap U'] \ , \ \lambda \cdot [f,U] := [\lambda \cdot f, U] \ .$$

Similarly, a multiplication in $\mathcal{E}_p$ is defined by

$$[f,U] \cdot [f',U'] := [f|_{U\cap U'} \cdot f'|_{U\cap U'}, U \cap U'] \ ;$$

since this multiplication obviously obeys the distributive law with respect to the addition defined above, it follows that $\mathcal{E}_p$ is a real algebra.

An element v of the dual space (see Appendix 5.1)

$$\mathcal{E}_p^* = \mathrm{Hom}(\mathcal{E}_p, \mathbb{R}) = \{ \ v : \mathcal{E}_p \longrightarrow \mathbb{R} \mid v \ \textit{linear} \ \}$$

is called a derivation if for all $[f,U], [f',U'] \in \mathcal{E}_p$ the following product rule holds:

$$v\left([f,U] \cdot [f',U']\right) = v([f,U]) \cdot [f',U'](p) + [f,U](p) \cdot v([f',U']) \ .$$

Lemma 2.3.10 *1. Let* (U,h,V) *be a chart for* X *with* $p \in U$ *and coordinates* $(x_1,\dots,x_n)$ *in* V*. Then for* $i = 1,\dots,n$ *the map*

$$v_i : \mathcal{E}_p \longrightarrow \mathbb{R} \quad , \quad v_i([f,W])) := \frac{\partial (f \circ h^{-1})}{\partial x_i}(h(p)) \ ,$$

is a derivation.

2. View $c \in \mathbb{R}$ *as a constant function in some neighborhood* U *of* p*. If* v *is a derivation on* $\mathcal{E}_p$*, then* $v([c,U]) = 0$.

Proof: 1. If f and f' coincide near p, then $f \circ h^{-1}$ and $f' \circ h^{-1}$ coincide near $h(p)$, and hence it holds

$$\frac{\partial (f \circ h^{-1})}{\partial x_i}(h(p)) = \frac{\partial (f' \circ h^{-1})}{\partial x_i}(h(p)) \ ;$$

this shows that $v_i([f,W]))$ is well defined, i.e. independent of the choice of the representative (f,W) for the germ $[f,W]$. Linearity of v_i follows from

$$\frac{\partial((f+f')\circ h^{-1})}{\partial x_i}(h(p)) = \frac{\partial(f\circ h^{-1}+f'\circ h^{-1})}{\partial x_i}(h(p)) = \frac{\partial(f\circ h^{-1})}{\partial x_i}(h(p)) + \frac{\partial(f'\circ h^{-1})}{\partial x_i}(h(p))$$

and

$$\frac{\partial((\lambda\cdot f)\circ h^{-1})}{\partial x_i}(h(p)) = \frac{\partial(\lambda\cdot(f\circ h^{-1}))}{\partial x_i}(h(p)) = \lambda\cdot\frac{\partial(f\circ h^{-1})}{\partial x_i}(h(p))\ ;$$

this shows that $v_i \in \mathcal{E}_p^*$. That v_i is a derivation follows from the product rule for $\frac{\partial}{\partial x_i}$ as follows:

$$\begin{aligned} v_i\left([f,U]\cdot[f',U']\right) &= \frac{\partial((f\cdot f')\circ h^{-1})}{\partial x_i}(h(p)) = \frac{\partial((f\circ h^{-1})\cdot(f'\circ h^{-1}))}{\partial x_i}(h(p)) \\ &= \left(\frac{\partial(f\circ h^{-1})}{\partial x_i}(h(p))\right)\cdot\left((f'\circ h^{-1})(h(p))\right) \\ &\quad + \left((f\circ h^{-1})(h(p))\right)\cdot\left(\frac{\partial(f'\circ h^{-1})}{\partial x_i}(h(p))\right) \\ &= v_i([f,U])\cdot[f',U'](p) + [f,U](p)\cdot v_i\left([f',U']\right)\ . \end{aligned}$$

2. We have

$$\begin{aligned} v([c,U]) &= v([c\cdot 1,U]) \quad \text{viewing } c \text{ and } 1 \text{ as a constant functions} \\ &= v([c,U]\cdot[1,U]) \\ &= v([c,U])\cdot 1(p) + c(p)\cdot v([1,U]) \quad \text{since } v \text{ is a derivation} \\ &= v([c\cdot 1(p),U]) + v([c(p)\cdot 1,U]) \quad \text{since } v \text{ is linear} \\ &= 2v([c,U])\ ; \end{aligned}$$

this clearly implies $v([c,U])=0$. ∎

Definition 2.3.11 *The* <u>*algebraic*</u> <u>*tangent*</u> <u>*space*</u> *of X at p is*

$$T_p^{alg}X := \{\ v\in\mathcal{E}_p^*\ \mid v \text{ a derivation }\}\ .$$

An element of $T_p^{alg}X$ is called an <u>*algebraic*</u> <u>*tangent*</u> <u>*vector*</u> *of X at p.*

Theorem 2.3.12 *$T_p^{alg}X$ is an n-dimensional linear subspace of $\mathcal{E}_p^*$.*

Proof: Let be $v_1, v_2 \in T_p^{alg}X$. By the definition of addition of linear maps it holds

$$\begin{aligned} (v_1+v_2)\left([f,U]\cdot[f',U']\right) &= v_1\left([f,U]\cdot[f',U']\right) + v_2\left([f,U]\cdot[f',U']\right) \\ &= v_1([f,U])\cdot[f',U'](p) + [f,U](p)\cdot v_1([f',U']) \\ &\quad + v_2([f,U])\cdot[f',U'](p) + [f,U](p)\cdot v_2([f',U']) \\ &= (v_1+v_2)([f,U])\cdot[f',U'](p) + [f,U](p)\cdot(v_1+v_2)([f',U'])\ ; \end{aligned}$$

this shows that $v_1 + v_2 \in T_p^{alg} X$. It is even easier to verify that $\lambda \cdot v \in T_p^{alg} X$ for $\lambda \in \mathbb{R}$ and $v \in T_p^{alg} X$; hence $T_p^{alg} X$ is a linear subspace of $\mathcal{E}_p^*$.

Let (U, h, V) be a chart for X with $p \in U$ and coordinates $(x_1, \dots, x_n)$ in V; we may assume that $h(p) = 0$. Write $h = (h_1, \dots, h_n)$; then for each i it holds $h_i \circ h^{-1} = x_i$, and each pair (h_i, U) is a differentiable function near p.

Let $v_1, \dots, v_n \in T_p^{alg} X$ be the derivatives associated to this chart by Lemma 2.3.10.1; we are done if we can show that these are a basis of $T_p^{alg} X$.

Assume that there are $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} \lambda_i \cdot v_i = 0$, i.e. such that

$$\sum_{i=1}^{n} (\lambda_i \cdot v_i)([f, W]) = \sum_{i=1}^{n} \lambda_i \cdot (v_i([f, W])) = 0$$

for all $[f, W] \in \mathcal{E}_p$. Then in particular for all j it holds

$$0 = \sum_{i=1}^{n} \lambda_i \cdot (v_i([h_j, U])) = \sum_{i=1}^{n} \lambda_i \cdot \frac{\partial (h_j \circ h^{-1})}{\partial x_i}(h(p)) = \sum_{i=1}^{n} \lambda_i \cdot \frac{\partial x_j}{\partial x_i}(h(p)) = \lambda_j \ ,$$

this proves that the v_i are linearly independent.

It remains to show that the v_i generate $T_p^{alg} X$. Take $v \in T_p^{alg} X$ and define $\lambda_i := v([h_i, U])$; we are done if we can show that $v = \sum_{i=1}^{n} \lambda_i \cdot v_i$, i.e. that $v([f, W]) = \sum_{i=1}^{n} \lambda_i \cdot v_i([f, W])$ for all $[f, W] \in \mathcal{E}_p$.

For such an $[f, W]$, choose an open ball B around $h(p) = 0$ inside $V \cap h(W)$, and define the differentiable function $g := f \circ h^{-1} : B \longrightarrow \mathbb{R}$. Then $f|_{h^{-1}(B)} = g \circ h|_{h^{-1}(B)}$, i.e. $[f, W] = [g \circ h, h^{-1}(B)]$.

According to Lemma 2.3.13 below there are differentiable functions ψ_i in B such that $\psi_i(0) = \frac{\partial g}{\partial x_i}(0)$ for all i, and $g(x) = g(0) + \sum_{i=1}^{n} \psi_i(x) \cdot x_i$. In $h^{-1}(B)$ this means

$$f = g \circ h = f(p) + \sum_{i=1}^{n} (\psi_i \circ h) \cdot (x_i \circ h) = f(p) + \sum_{i=1}^{n} (\psi_i \circ h) \cdot h_i \ .$$

Since v is linear and vanishes on constants (Lemma 2.3.10.2) we get [1]

$$\begin{aligned} v(f) &= \sum_{i=1}^{n} v((\psi_i \circ h) \cdot h_i) \\ &= \sum_{i=1}^{n} v(\psi_i \circ h) \cdot h_i(p) + \sum_{i=1}^{n} (\psi_i \circ h)(p) \cdot v(h_i) \quad \text{since } v \text{ is a derivation} \\ &= \sum_{i=1}^{n} \psi_i(0) \cdot v(h_i) \quad \text{since} \quad h(p) = 0 \ , \text{ so in particular } h_i(p) = 0 \end{aligned}$$

[1] Here and later on we save on notations by writing only functions in place of (equivalence classes of) pairs. It is a good exercise to write down the details at least once.

$$
\begin{aligned}
&= \sum_{i=1}^{n} \lambda_i \cdot \frac{\partial g}{\partial x_i}(0) \quad \text{by definition of the } \lambda_i \text{ and choice of the } \psi_i \\
&= \sum_{i=1}^{n} \lambda_i \cdot \frac{\partial (f \circ h^{-1})}{\partial x_i}(0) \quad \text{by definition of } g \\
&= \sum_{i=1}^{n} \lambda_i \cdot v_i(f) \quad \text{by definition of the } v_i \ ;
\end{aligned}
$$

this is what we wanted. ■

Lemma 2.3.13 *Let* $B \subset \mathbb{R}^n$ *be an open ball around* 0*, and* $g : B \longrightarrow \mathbb{R}$ *a differentiable function. Then the functions*

$$
\psi_i : B \longrightarrow \mathbb{R} \quad , \quad \psi_i(x) := \int_0^1 \frac{\partial g}{\partial x_i}(t \cdot x) dt \quad , \quad i = 1, \ldots, n \ ,
$$

are differentiable with $\psi_i(0) = \frac{\partial g}{\partial x_i}(0)$ *for all* i*, and* $g(x) = g(0) + \sum_{i=1}^{n} \psi_i(x) \cdot x_i$.

Proof: We know from standard calculus that the ψ_i are differentiable, and it holds

$$
\psi_i(0) = \int_0^1 \frac{\partial g}{\partial x_i}(0) dt = \frac{\partial g}{\partial x_i}(0) \ .
$$

For fixed $x \in B$ define the differentiable function

$$
h : [0,1] \longrightarrow \mathbb{R} \quad , \quad h(t) := g(t \cdot x) \ .
$$

The chain rule implies

$$
\frac{dh}{dt}(t) = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(t \cdot x) \cdot x_i \ ,
$$

and using the Fundamental Theorem of Calculus we get

$$
\begin{aligned}
g(x) - g(0) &= h(1) - h(0) = \int_0^1 \frac{dh}{dt}(t) dt = \int_0^1 \left(\sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(t \cdot x) \cdot x_i \right) dt \\
&= \sum_{i=1}^{n} \left(\int_0^1 \frac{\partial g}{\partial x_i}(t \cdot x) dt \right) \cdot x_i = \sum_{i=1}^{n} \psi_i(x) \cdot x_i \ .
\end{aligned}
$$

■

Theorem 2.3.14 *1. Let γ be a curve in X through p. Then*

$$v_\gamma : \mathcal{E}_p \longrightarrow \mathbb{R} \quad , \quad v_\gamma([f,U]) := \frac{d}{dt}(f \circ \gamma)(0)$$

is an element of $T_p^{alg}X$.

2. The map

$$\Psi_p : T_p^{geom}X \longrightarrow T_p^{alg}X \quad , \quad \Psi_p([\gamma]) := v_\gamma$$

is a well defined linear isomorphism.

Proof: 1. This is shown in the same way as Lemma 2.3.10.1 using $f \circ \gamma$ in place of $f \circ h^{-1}$ and the product rule for $\frac{d}{dt}$ instead of $\frac{\partial}{\partial x_i}$; details are therefore left to the reader.

2. Let (U,h,V) be a chart for X with $p \in U$, coordinates $(x_1, \ldots, x_2)$ in V, and $h(p) = 0$.

Assume that $[\gamma] = [\tilde{\gamma}] \in T_p^{geom}X$, i.e. that $\gamma \sim \tilde{\gamma}$ in $\mathcal{K}_p$. Then it holds for every $[f,u] \in \mathcal{E}_p$

$$\begin{aligned}
\Psi_p([\gamma])([f,U]) &= v_\gamma([f,U]) = \frac{d}{dt}(f \circ \gamma)(0) = \frac{d}{dt}\left((f \circ h^{-1}) \circ (h \circ \gamma)\right)(0) \\
&= D(f \circ h^{-1})(0) \cdot \frac{d}{dt}(h \circ \gamma)(0) \quad \text{by the chain rule and since } (h \circ \gamma)(0) = h(p) = 0 \\
&= D(f \circ h^{-1})(0) \cdot \frac{d}{dt}(h \circ \tilde{\gamma})(0) \quad \text{by the definition of } \sim \text{ in } \mathcal{K}_p \\
&= \Psi_p([\tilde{\gamma}])([f,U]) \quad \text{by reversing the argument above ;}
\end{aligned}$$

this shows that Ψ_p is well defined.

Let (see Theorem 2.3.3 and Remark 2.3.6) $\Phi_h : \mathbb{R}^n \longrightarrow T_p^{geom}X$ be the linear isomorphism associated to (U,h,V); it suffices to show that $\Psi_p \circ \Phi_h : \mathbb{R}^n \longrightarrow T_p^{alg}X$ is a linear isomorphism. Since $(e_1, \ldots, e_n)$ resp. $(v_1, \ldots, v_n)$ is a basis of $\mathbb{R}^n$ resp. $T_p^{alg}X$ (see the proof of Theorem 2.3.12), this is certainly the case if

$$\Psi_p \circ \Phi_h(\lambda_1, \ldots, \lambda_n) = \sum_{i=1}^{n} \lambda_i \cdot v_i$$

for all $(\lambda_1, \ldots \lambda_n) \in \mathbb{R}^n$, i.e.

$$(\Psi_p \circ \Phi_h(\lambda_1, \ldots, \lambda_n))(f) = \sum_{i=1}^{n} \lambda_i \cdot v_i(f)$$

for all (germs of; see the footnote for the proof of Theorem 2.3.12) differentiable functions f near p.

Since $h(p) = 0$, $\Phi_h(\lambda_1, \ldots, \lambda_n)$ is the class of the curve $\gamma : t \mapsto h^{-1}(t \cdot (\lambda_1, \ldots, \lambda_n))$ by Remark 2.3.6; this implies

$$\begin{aligned}
(\Psi_p \circ \Phi_h(\lambda_1, \ldots, \lambda_n))(f) &= \Psi_p([\gamma])(f) = v_\gamma(f) = \frac{d}{dt}\Big[t \mapsto (f \circ h^{-1})(t \cdot (\lambda_1, \ldots, \lambda_n))\Big](0) \\
&= \sum_{i=1}^{n} \frac{\partial(f \circ h^{-1})}{\partial x_i}(0) \cdot \lambda_i \quad \text{by the chain rule} \\
&= \sum_{i=1}^{n} \lambda_i \cdot v_i(f) \quad \text{by definition of the } v_i \text{ .}
\end{aligned}$$

$\blacksquare$

Remark 2.3.15 *1. The isomorphism Ψ_p is a natural one, independent of any choice.*

2. Since $\frac{\partial}{\partial x_i}(p) = \Phi_h(e_i)$ by definition, it holds $v_i = \Psi_p \circ \Phi_h(e_i) = \Psi_p\left(\frac{\partial}{\partial x_i}(p)\right)$.

2.4 The tangent map

Let X resp. Y be an n- resp. m-dimensional differentiable manifold, $f : X \longrightarrow Y$ a differentiable map, and $p \in X$.

Definition 2.4.1 *The geometric resp. algebraic tangent map of f at p is defined by*

$$Df^{geom}(p) : T_p^{geom}X \longrightarrow T_{f(p)}^{geom}Y \quad , \quad T_p^{geom}X \ni [\gamma] \mapsto [f \circ \gamma] \in T_{f(p)}^{geom}Y$$

resp.

$$Df^{alg}(p) : T_p^{alg}X \longrightarrow T_{f(p)}^{alg}Y \quad , \quad T_p^{alg}X \ni v \mapsto \left[\mathcal{E}_{f(p)} \ni [g, W] \mapsto v([g \circ f, f^{-1}(W)])\right] \in T_{f(p)}^{alg}Y \ .$$

Theorem 2.4.2 *1. $Df^{geom}(p)$ and $Df^{alg}(p)$ are well defined linear maps.*

2. Let $\Psi_p : T_p^{geom}X \longrightarrow T_p^{alg}X$ and $\Psi_{f(p)} : T_{f(p)}^{geom}Y \longrightarrow T_{f(p)}^{alg}Y$ the natural isomorphisms from Theorem 2.3.14; then $Df^{alg}(p) \circ \Psi_p = \Psi_{f(p)} \circ Df^{geom}(p)$, i.e. the following diagram commutes:

$$\begin{array}{ccc}
T_p^{geom}X & \xrightarrow{Df^{geom}(p)} & T_{f(p)}^{geom}Y \\
\Big\downarrow{\scriptstyle \Psi_p} & & \Big\downarrow{\scriptstyle \Psi_{f(p)}} \\
T_p^{alg}X & \xrightarrow[Df^{alg}(p)]{} & T_{f(p)}^{alg}Y
\end{array}$$

3. Let (U, h, V) resp. (U', h', V') be a chart for X resp. Y with $p \in U$ resp. $f(p) \in U'$. Let $(x_1, \ldots, x_n)$ resp. $(y_1, \ldots, y_m)$ be the coordinates in V resp. V', and let

$$\frac{\partial}{\partial x_1}(p), \ldots, \frac{\partial}{\partial x_n}(p) \in T_p^{geom}X \quad , \quad v_1, \ldots, v_n \in T_p^{alg}X$$

resp.

$$\frac{\partial}{\partial y_1}(f(p)), \ldots, \frac{\partial}{\partial y_m}(f(p)) \in T_{f(p)}^{geom}Y \quad , \quad v_1', \ldots, v_m' \in T_{f(p)}^{alg}Y$$

be the bases associated to these charts (see Remark 2.3.6, Lemma 2.3.10 and the proof of Theorem 2.3.12). Then the matrix of $Df^{geom}(p)$ as well as of $Df^{alg}(p)$ with respect to these bases is the Jacobian matrix $D(h' \circ f \circ h^{-1})(h(p))$.

4. If Z is another differentiable manifold and $f' : Y \longrightarrow Z$ a differentiable map, then the <u>*chain rules*</u>

$$\begin{aligned} D(f' \circ f)^{geom}(p) &= (Df')^{geom}(f(p)) \circ Df^{geom}(p) \ , \\ D(f' \circ f)^{alg}(p) &= (Df')^{alg}(f(p)) \circ Df^{alg}(p) \ , \end{aligned}$$

hold.

Proof: 1. We first show that $Df^{geom}(p)$ and $Df^{alg}(p)$ are well defined. Regarding $Df^{geom}(p)$, we have to show that $\gamma \sim \sigma$ implies $f \circ \gamma \sim f \circ \sigma$. For this we take charts (U, h, V) and (U', h', V') as in 3. Then

$$\gamma \sim \sigma \iff \frac{d}{dt}(h \circ \gamma)(0) = \frac{d}{dt}(h \circ \sigma)(0)$$

implies, using the chain rule,

$$\begin{aligned} \frac{d}{dt}(h' \circ f \circ \gamma)(0) &= \frac{d}{dt}(h' \circ f \circ h^{-1} \circ h \circ \gamma)(0) = D(h' \circ f \circ h^{-1})(h(p)) \cdot \frac{d}{dt}(h \circ \gamma)(0) \\ &= D(h' \circ f \circ h^{-1})(h(p)) \cdot \frac{d}{dt}(h \circ \sigma)(0) = \frac{d}{dt}(h' \circ f \circ h^{-1} \circ h \circ \sigma)(0) \\ &= \frac{d}{dt}(h' \circ f \circ \sigma)(0) \ , \end{aligned}$$

i.e. $f \circ \gamma \sim f \circ \sigma$ as wanted. Regarding $Df^{alg}(p)$ first notice that if g and g' are functions which coincide in a neighborhood of $f(p)$, then the functions $g \circ f$ and $g' \circ f$ coincide in a neighborhood of p, and hence it holds (in sloppy notation) $v(g \circ f) = v(g' \circ f)$. That indeed $Df^{alg}(p)(v) \in T^{alg}_{f(p)}Y$ follows easily from $(\lambda g + \mu g') \circ f = \lambda(g \circ f) + \mu(g' \circ f)$, $(g \cdot g') \circ f = (g \circ f) \cdot (g' \circ f)$, and $v \in T^{alg}_p X$. Linearity of $Df^{alg}(p)$ is obvious, and using this the linearity of $Df^{geom}(p)$ will follow from 2.

2. For all $[\gamma] \in T^{geom}_p X$ and functions g around $f(p)$ it holds

$$\begin{aligned} \left(\Psi_{f(p)}\left(Df^{geom}(p)([\gamma])\right)\right)(g) &= \left(\Psi_{f(p)}\left([f \circ \gamma]\right)\right)(g) \quad \text{by definition of } Df^{geom}(p) \\ &= \frac{d}{dt}(g \circ f \circ \gamma)(0) \quad \text{by definition of } \Psi_{f(p)} \ , \end{aligned}$$

and on the other hand

$$\begin{aligned} \Big(Df^{alg}(p)\left(\Psi_p([\gamma])\right)\Big)(g) &= \left(\Psi_p([\gamma])\right)(g \circ f) \quad \text{by definition of } Df^{alg}(p) \\ &= \frac{d}{dt}(g \circ f \circ \gamma)(0) \quad \text{by definition of } \Psi_p \ . \end{aligned}$$

3. Because of 2. it suffices to show the statement for $Df^{geom}(p)$ or $Df^{alg}(p)$, since Ψ_p (resp. $\Psi_{f(p)}$) maps $\frac{\partial}{\partial x_i}(p)$ (resp. $\frac{\partial}{\partial y_j}(p)$) to v_i (resp. v'_j) by Remark 2.3.15. For every function g around $f(p)$ it holds

$$\Big(Df^{alg}(p)(v_i)\Big)(g) = v_i(g \circ f) \quad \text{by definition of } Df^{alg}(p)$$

$$
\begin{aligned}
&= \frac{\partial(g\circ f\circ h^{-1})}{\partial x_i}(h(p)) \ \text{ by definition of } v_i \\
&= \frac{\partial(g\circ h'^{-1}\circ h'\circ f\circ h^{-1})}{\partial x_i}(h(p)) \\
&= \sum_{j=1}^{m}\frac{\partial(g\circ h'^{-1})}{\partial y_j}(h'(p))\cdot\frac{\partial(h'\circ f\circ h^{-1})_j}{\partial x_i}(h(p)) \ \text{ by the chain rule} \\
&= \sum_{j=1}^{m}\frac{\partial(h'\circ f\circ h^{-1})_j}{\partial x_i}(h(p))\cdot v'_j(g) \ \text{ by definition of } v'_j \\
&= \left(\sum_{j=1}^{m}\frac{\partial(h'\circ f\circ h^{-1})_j}{\partial x_i}(h(p))\cdot v'_j\right)(g)\ .
\end{aligned}
$$

This is equivalent to

$$
Df^{alg}(p)(v_i) = \sum_{j=1}^{m}\frac{\partial(h'\circ f\circ h^{-1})_j}{\partial x_i}(h(p))\cdot v'_j\ ,
$$

i.e. the claim.

4. For every $[\gamma]\in T_p^{geom}X$ it holds

$$
\begin{aligned}
\big(D(f'\circ f)^{geom}(p)\big)([\gamma]) &= [f'\circ f\circ\gamma] = \big((Df')^{geom}(f(p))\big)([f\circ\gamma]) \\
&= \big((Df')^{geom}(f(p))\big)\Big(Df^{geom}(p)([\gamma])\Big) \\
&= \Big((Df')^{geom}(f(p))\circ Df^{geom}(p)\Big)([\gamma]))\ ;
\end{aligned}
$$

this proves the geometric case. The algebraic one follows from this and 2. ■

Excercise 2.4.3 *Prove the statement for $Df^{geom}(p)$ in Theorem 2.4.2.3 directly, i.e. without using Df^{alg} and 2.4.2.2. More precisely, use only definitions and facts from the geometric context, and standard analysis.*
Similarly, prove the statement of Theorem 2.4.2.4 in the algebraic case directly, without referring to the geometric case.

Remark 2.4.4 *In what follows we will identify $T_p^{geom}X$ and $T_p^{alg}X$ by the natural isomorphism Ψ_p and just write T_pX. Furthermore, since this identification is compatible with tangent maps by Theorem 2.4.2, we will just write $Df(p)$ instead of $Df^{geom}(p)$ or $Df^{alg}(p)$.*

An immediate consequence of Definition 2.2.7 and Theorem 2.4.2.3 is

Corollary 2.4.5 *It holds* $\mathrm{rk}_p f = \mathrm{rk}(Df_p)$.

Definition 2.4.6 *Let X, Y and $f : X \longrightarrow Y$ be as above.*

1. *f is called an* <u>immersion</u> *if* $\mathrm{rk}_p f = n$, *i.e. if* $Df(p)$ *is injective, for all* $p \in X$.

2. *f is called an* <u>submersion</u> *if* $\mathrm{rk}_p f = m$, *i.e. if* $Df(p)$ *is surjective, for all* $p \in X$.

3. *f is called an* <u>embedding</u> *if it is an immersion and* $f : X \longrightarrow f(X)$ *is a homeomorphism when* $f(X) \subset Y$ *is equipped with the induced topology.*

Excercise 2.4.7 *Let be* $\mathbb{R}_+ := \{ x \in \mathbb{R} \mid x > 0 \}$. *Consider the differentiable sum (see Proposition 2.1.16)* $\mathbb{R} + \mathbb{R}_+$, *and the map*

$$f : \mathbb{R} + \mathbb{R}_+ \longrightarrow \mathbb{R}^2 \quad , \quad f(x) := \begin{cases} (x, 0) & \text{if } x \in \mathbb{R} \ , \\ (0, x) & \text{if } x \in \mathbb{R}_+ \ . \end{cases}$$

Show that f is an injective immersion, but not an embedding.

2.5 Submanifolds

Let X be an n-dimensional differentiable manifold.

Definition 2.5.1 *A subset* $Y \subset X$ *is a* <u>k-dimensional</u> <u>submanifold</u> *of X if for every* $p \in Y$ *there exists a chart* (U, h, V) *for X with* $p \in U$ *such that*

$$h(Y \cap U) = \{ x = (x_1, \ldots, x_n) \in V \mid x_{k+1} = \ldots = x_n = 0 \} \ .$$

Remark 2.5.2 *A diffeomorphism* $\mathbb{R}^n \supset U \xrightarrow{\phi} U' \subset \mathbb{R}^n$ *is a chart for the standard differentiable structure in* $\mathbb{R}^n$ *by Exercise 2.1.9. Therefore, Theorem 1.1.4 implies that a subset* $Y \subset \mathbb{R}^n$ *is a k-dimensional submanifold in the sense of Definition 1.1.1 if and only if it is a k-dimensional submanifold in the sense of Definition 2.5.1.*

Excercise 2.5.3 *(compare Example 1.1.2) Show that the 0-dimensional (resp. n-dimensional) submanifolds of X are precisely the discrete (resp. open) subsets of X.*

Example 2.5.4 *Consider the n-dimensional unit sphere* $S^n \subset \mathbb{R}^{n+1}$,

$$S^n = \{ x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1 \} \ ,$$

with the differentiable atlas $\{ (U_{i,\pm}, h_{i\pm}, D^n) \mid 1 \leq i \leq n+1 \}$ *as in Example 2.1.7 (see also Example 2.1.3). By viewing* $\mathbb{R}^n$ *as the subset* $\mathbb{R}^n = \{ x \in \mathbb{R}^{n+1} \mid x_{n+1} = 0 \}$, *the $(n-1)$-dimensional unit sphere* $S^{n-1} \subset \mathbb{R}^n$ *as the subset*

$$S^{n-1} = \{ x \in S^n \mid x_{n+1} = 0 \} \subset S^n \ .$$

For every $p \in S^{n-1}$ *there exists a* $U_{i,\pm}$, $1 \leq i \leq n$, *with* $p \in U_{i,\pm}$, *and it holds*

$$h_{i\pm}(S^{n-1} \cap U_{i,\pm}) = \{ (x_1, \ldots, x_n) \in D^n \mid x_n = 0 \} \ .$$

This shows that S^{n-1} is an $(n-1)$-dimensional submanifold of S^n.

The argument can be easily generalized to show that, for every $k \le n$, S^k is a k-dimensional submanifold of S^n.

Excercise 2.5.5 *Show directly, i.e. without using the results below, that the inclusion map*

$$S^k \hookrightarrow S^n \quad , \quad S^k \ni (x_1, \ldots, x_{k+1}) \mapsto (x_1, \ldots, x_{k+1}, 0, \ldots, 0) \in S^n \ ,$$

is an embedding.

Proposition 2.5.6 *A k-dimensional submanifold Y of X, equipped with the topology induced from X, has a k-dimensional differentiable structure such that the inclusion map $i_Y : Y \hookrightarrow X$ is an embedding.*

Proof: Let

$$\mathrm{pr} : \mathbb{R}^n \longrightarrow \mathbb{R}^k \quad , \quad \mathrm{pr}(x_1, \ldots, x_n) := (x_1, \ldots, x_k) \ ,$$

be the linear projection, and

$$i : \mathbb{R}^k \hookrightarrow \mathbb{R}^n \quad , \quad i(x_1, \ldots, x_k) := (x_1, \ldots, x_k, 0, \ldots, 0)$$

the linear inclusion. pr and i are differentiable (so in particular continuous), and obviously it holds $\mathrm{pr} \circ i = \mathrm{id}_{\mathbb{R}^k}$. If we identify $\mathbb{R}^k$ with $i(\mathbb{R}^k) \subset \mathbb{R}^n$ via the injective map i, then it is easily seen that the standard topology in $\mathbb{R}^k$ coincides with the topology induced from the standard topology in $\mathbb{R}^n$; this is equivalent to saying that $i : \mathbb{R}^k \longrightarrow i(\mathbb{R}^k)$ is a homeomorphism with inverse $\mathrm{pr}|_{i(\mathbb{R}^k)} : i(\mathbb{R}^k) \longrightarrow \mathbb{R}^k$.

Let (U, h, V) be a chart for X as in Definition 2.5.1, and define

$$U' := Y \cap U \ , \ V' := \mathrm{pr}(h(Y \cap U)) \subset \mathbb{R}^k \ , \ h' := \mathrm{pr} \circ h|_{U'} : U' \longrightarrow V' \ .$$

By assumption it holds that U' is open in Y and that $h(U') = V \cap i(\mathbb{R}^k)$; since V is open in $\mathbb{R}^n$ this means that $h(U')$ is open in $i(\mathbb{R}^k)$, and hence that $V' = \mathrm{pr}(h(U'))$ is open in $\mathbb{R}^k$. Since h is bijective, the map $h|_{U'} : U' \longrightarrow h(U')$ is bijective, too, and since $\mathrm{pr}|_{i(\mathbb{R}^k)} : i(\mathbb{R}^k) \longrightarrow \mathbb{R}^k$ is bijective it follows that h' is bijective. Furthermore, since the restriction of a continuous map to a subset with the induced topology remains continuous, it follows that $h|_{U'}$ and hence $h' = \mathrm{pr} \circ h|_{U'}$ is continuous. Since its inverse $h^{-1} \circ i|_{V'}$ is continuous, too, h' is a homeomorphism, and hence a k-dimensional topological chart for Y. Since a chart (U, h, V) as above exists around every point in Y, we get an k-dimensional atlas for Y. Now observe that any two of the charts (U', h', V'), $(\tilde{U}', \tilde{h}', \tilde{V}')$ for Y obtained in this way glue differentiably; this is true because the gluing map is of the form

$$\tilde{h}' \circ (h')^{-1} = \mathrm{pr} \circ (\tilde{h} \circ h^{-1}) \circ i$$

and $\tilde{h} \circ h^{-1}$ is differentiable. Thus we have produced a k-dimensional differentiable atlas in Y.

That Y has the topology induced from X is equivalent to saying that the inclusion $i_Y : Y \hookrightarrow X$ is a homeomorphism onto its image $Y \subset X$; hence it remains to show that i_Y is an immersion, i.e. that it has constant rank k. For this, choose around $p \in Y$ a chart (U, h, V) for X around p as above with associated chart (U', h', V') for Y; then

$$\begin{aligned} \left(h \circ i_Y \circ (h')^{-1}\right)(x_1, \ldots, x_k) &= \left(h \circ (h')^{-1}\right)(x_1, \ldots, x_k) = \left(h \circ h^{-1} \circ i|_{V'}\right)(x_1, \ldots, x_k) \\ &= i|_{V'}(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, \ldots, 0) \ . \end{aligned}$$

It is obvious that the Jacobian of this map has rank k at every point. ■

Theorem 2.5.7 *For a subset* $Y \subset X$ *the following are equivalent.*

1. *Y is a k-dimensional submanifold of X.*
2. *For every $p \in Y$ there exists an open neighborhood U of p in X, an $(n-k)$-dimensional manifold Z, a point $q \in Z$, and a differentiable map $f : U \longrightarrow Z$ such that $Y \cap U = f^{-1}(q)$ and $\mathrm{rk}_p f = n - k$.*
3. *Y, with the topology induced from X, has a k-dimensional differentiable structure such that the inclusion map $i_Y : Y \hookrightarrow X$ is an embedding.*

Proof: 1. $\Longrightarrow$ 2.: We take a chart (U, h, V) for X around p as in Definition 2.5.1, and define $Z := \mathbb{R}^{n-k}$, $q := 0 \in \mathbb{R}^{n-k}$, and $f := \mathrm{pr} \circ h : U \longrightarrow Z$, where $\mathrm{pr} : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-k}$ is the projection $\mathrm{pr}(x_1, \ldots, x_n) := (x_{k+1}, \ldots, x_n)$. Then f is differentiable with $f^{-1}(q) = Y \cap U$. $\mathrm{id}_{\mathbb{R}^{n-k}}$ is a chart for $\mathbb{R}^{n-k}$, and it holds

$$\left(\mathrm{id}_{\mathbb{R}^{n-k}} \circ f \circ h^{-1}\right)(x) = \mathrm{pr}(x) \ ,$$

hence

$$D\left(\mathrm{id}_{\mathbb{R}^{n-k}} \circ f \circ h^{-1}\right)(x) = \begin{pmatrix} 0_{(n-k)\times k} & I_{n-k} \end{pmatrix}$$

where $0_{(n-k)\times k}$ is the $(n-k) \times k$ zero matrix and I_{n-k} is the $(n-k) \times (n-k)$ unit matrix. This means in particular that

$$\mathrm{rk}_p(f) = \mathrm{rk}\left(D\left(\mathrm{id}_{\mathbb{R}^{n-k}} \circ f \circ h^{-1}\right)(h(p))\right) = n - k \ .$$

2. $\Longrightarrow$ 1.: For $p \in Y$ take U, Z, f and q as in 2. Let (U', h', V') be a chart for Z with $q \in U'$ and $h'(q) = 0$. After shrinking U around p if necessary, we may assume that $f(U) \subset U'$ (because f is continuous), and that U is the domain of a chart (U, h, V) for X. Define

$$\tilde{f} := h' \circ f \circ h^{-1} : V \longrightarrow V' \ ;$$

then $\tilde{f}$ is differentiable, and since h and h' are bijective it holds

$$\tilde{f}^{-1}(0) = h(f^{-1}((h')^{-1}(0)) = h(f^{-1}(q)) = h(Y \cap U) \ .$$

Furthermore, we have

$$\mathrm{rk}(D\tilde{f}(h(p))) = \mathrm{rk}_p f = n - k \ ,$$

so that $h(Y \cap U)$ is a k-dimensional submanifold of $\mathbb{R}^n$ in the sense of Definition 1.1.1. By Theorem 1.1.4 (and after shrinking U again if necessary) we may assume that there exists an open $V_1 \subset \mathbb{R}^n$ and a diffeomorphism $g : V \longrightarrow V_1$ such that

$$g(h(Y \cap U)) = (g \circ h)(Y \cap U) = \{ (x_1, \dots, x_n) \in V_1 \mid (x_{k+1}, \dots, x_n) = 0 \} . \quad (*)$$

Since g is a diffeomorphism, $(U, g \circ h, V_1)$ is a chart for X, too, and $(*)$ says that it meets the requirements of Definition 2.5.1.

<u>1. $\Longrightarrow$ 3.:</u> This follows immediately from Proposition 2.5.6.

<u>3. $\Longrightarrow$ 1.:</u> Since i_Y is an immersion and hence has constant rank k, according to the Rank Theorem 2.2.9 around every point $p \in Y$ there is a chart (U', h', V') resp. (U, h, V) for Y resp. X with $p \in U'$ and $p = i_Y(p) \in U$, such that $U' = i_Y(U') \subset U$ and

$$(h \circ i_Y \circ (h')^{-1})(x_1, \dots, x_k) = (x_1, \dots, x_k, 0, \dots, 0) \quad (**)$$

for all $(x_1, \dots, x_k) \in V'$.

That i_Y is a homeomorphism onto its image is equivalent to saying that the topology of the manifold Y coincides with the topology induced from X. Therefore, there exists an open $U_1 \subset X$ such that $U' = Y \cap U_1$, so, by replacing U by $U \cap U_1$ if necessary, we may assume that $U' = Y \cap U$. Since $Y \cap U = U' = (h')^{-1}(V')$, $(**)$ implies

$$h(Y \cap U) \subset \{ (x_1, \dots, x_n) \in V \mid x_{k+1} = \dots = x_n = 0 \} .$$

On the other hand, from $(**)$ it follows

$$h(Y \cap U) = (h \circ i_Y \circ (h')^{-1})(V') = V' \times \{0\} \subset \mathbb{R}^k \times \{0\} \subset \mathbb{R}^n .$$

Since V' is open in $\mathbb{R}^k$, there exists an open $V_1 \subset \mathbb{R}^n$ such that $V' \times \{0\} = (\mathbb{R}^k \times \{0\}) \cap V_1$ (compare the proof of Proposition 2.5.6). Thus, by replacing V by $V \cap V_1$ and U by $h^{-1}(V \cap V_1)$ if necessary, we get

$$h(Y \cap U) = V' \times \{0\} = (\mathbb{R}^k \times \{0\}) \cap V = \{ (x_1, \dots, x_n) \in V \mid x_{k+1} = \dots = x_n = 0 \} .$$

■

Excercise 2.5.8 *Let X and Y be differentiable manifold, and $f : Y \longrightarrow X$ an embedding. Adapt and extend the arguments of the proof of the previous Theorem to show that $f(Y)$ is a submanifold of X (and hence a differentiable manifold), such that $f : Y \longrightarrow f(Y)$ is a diffeomorphism.*

3 Differential forms

For the facts about duality and exterior powers used in the following we refer to the Appendices 5.1 and 5.2.

3.1 The exterior algebra of a manifold

Let X be an n-dimensional differentiable manifold. Consider for $k \in \mathbb{N}_0$ the disjoint union

$$\Lambda^k T^*X := \coprod_{p \in X} \Lambda^k T_p^*X \ ,$$

where T_p^*X is the dual of the tangent space T_pX.

Recall that a chart (U, h, V) for X with coordinates $(x_1, \ldots, x_n)$ in V produces for each $p \in X$ a basis $\frac{\partial}{\partial x_1}(p), \ldots, \frac{\partial}{\partial x_n}(p)$ of T_pX; we denote by $dx_1(p), \ldots, dx_n(p)$ the dual basis of T_p^*X. Then for $1 \le k \le n$ the $dx_{i_1}(p) \wedge \ldots \wedge dx_{i_k}(p)$, $1 \le i_1 < \ldots < i_k \le n$, are a basis of $\Lambda^k T_p^*X$.

Remark 3.1.1 *If (U', h', V') is another chart around $p \in U$ with coordinates $(y_1, \ldots, y_n)$ in V', then we know that*

$$\frac{\partial}{\partial x_i}(p) = \sum_{j=1}^{n} \frac{\partial\left((h' \circ h^{-1})_j\right)}{\partial x_i}(h(p)) \cdot \frac{\partial}{\partial y_j}(p) \ , \ 1 \le i \le n \ .$$

Hence it holds

$$dy_j(p) = \sum_{i=1}^{n} \frac{\partial\left((h' \circ h^{-1})_j\right)}{\partial x_i}(h(p)) \cdot dx_i(p) \ , \ 1 \le j \le n$$

and

$$dy_1(p) \wedge \ldots \wedge dy_n(p) = \det\left(D(h' \circ h^{-1})(h(p))\right) \cdot dx_1(p) \wedge \ldots \wedge dx_n(p)$$

Definition 3.1.2 *A <u>differential</u> <u>form</u> of <u>degree</u> k or <u>k-form</u> on X is a map*

$$\omega : X \longrightarrow \Lambda^k T^*X$$

such that $\omega(p) \in \Lambda^k T_p^*X$ for all $p \in X$.

A k-form ω is called <u>differentiable</u> if for every chart (U, h, V) as above it holds

$$\omega|_U = \sum_{1 \le i_1 < \ldots < i_k \le n} a_{i_1 \ldots i_k} \cdot dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

with differentiable functions $a_{i_1 \ldots i_k}$.
We denote by $\Omega^k X$ the space of differentiable k-forms on X.

Remarks 3.1.3 *1. $\Omega^k X$ is a vector space. The addition and scalar multiplication are defined by*

$$(\omega + \eta)(p) := \omega(p) + \eta(p) \ , (a \cdot \omega)(p) := a \cdot \omega(p) \ \text{ for all } \ p \in X \ .$$

$\omega + \eta$ and $a \cdot \omega$ are again differentiable because if with respect to a chart (U, h, V) it holds e.g.

$$\omega|_U = \sum_{1 \leq i_1 < \ldots < i_k \leq n} a_{i_1 \ldots i_k} \cdot dx_{i_1} \wedge \ldots \wedge dx_{i_k} \ , \ \eta|_U = \sum_{1 \leq i_1 < \ldots < i_k \leq n} b_{i_1 \ldots i_k} \cdot dx_{i_1} \wedge \ldots \wedge dx_{i_k} \ ,$$

then

$$(\omega + \eta)|_U = \sum_{1 \leq i_1 < \ldots < i_k \leq n} (a_{i_1 \ldots i_k} + b_{i_1 \ldots i_k}) \cdot dx_{i_1} \wedge \ldots \wedge dx_{i_k} \ .$$

Similarly, for $f \in \mathcal{C}^\infty(X, \mathbb{R})$, $\omega \in \Omega^k X$ it holds $f \cdot \omega \in \Omega^k X$, where $(f \cdot \omega)(p) := f(p) \cdot \omega(p)$. This means that $\Omega^k X$ is a $\mathcal{C}^\infty(X, \mathbb{R})$-module.

2. *Since $\Lambda^0 T_p^* X = \mathbb{R}$ by definition, a (differentiable) 0-form is nothing but a (differentiable) function, i.e. $\Omega^0 X = \mathcal{C}^\infty(X, \mathbb{R})$.*
3. *For $k > n$ it holds $\Omega^k X = \{0\}$.*
4. *To check the differentiability of a k-form it suffices to check it for the charts in some atlas for X.*

We have maps

$$\wedge : \Omega^k X \times \Omega^l X \longrightarrow \Omega^{k+l} X \ , \ (\omega, \eta) \mapsto \omega \wedge \eta \ , \ (\omega \wedge \eta)(p) := \omega(p) \wedge \eta(p) \ \text{ for all } \ p \in X \ ;$$

in case $k = 0$ the wedge $\wedge$ is the usual (pointwise) multiplication.
In particular, for $\omega \in \Omega^k X$, $\eta \in \Omega^l X$ it holds $\omega \wedge \eta = (-1)^{k \cdot l} \eta \wedge \omega$.
We get a map

$$\wedge : \bigoplus_{k=0}^n \Omega^k X \times \bigoplus_{l=0}^n \Omega^l X \longrightarrow \bigoplus_{m=0}^n \Omega^k X \ , \ (\sum_{k=0}^n \omega_k, \sum_{l=0}^n \eta_l) \mapsto \sum_{m=0}^n \left(\sum_{k+l=m} \omega_k \wedge \eta_l \right) \ ;$$

It is easy to see that $\left(\bigoplus_{k=0}^n \Omega^k X, +, \wedge \right)$ is a non-commutative ring with unit $1 \in \Omega^0 X$, and that this ring structure is compatible with the vector space structure such that $\bigoplus_{k=0}^n \Omega^k X$ becomes a real algebra, the <u>exterior</u> <u>algebra</u> of X.

Let $f : X \longrightarrow \mathbb{R}$ be a differentiable function. For $p \in X$ we define $df(p) : T_p X \longrightarrow \mathbb{R}$ by

$$df(p)(v) := v(f) \ ,$$

where we view $v \in T_p X$ as a derivation and f as a function around p. It is obvious that $df(p)$ is linear and hence an element of $T_p^* X$.
Observe that if we identify a geometric tangent vector $[\gamma]$ with the derivation v_γ, then

$$df(p)([\gamma]) = df(p)(v_\gamma) = v_\gamma(f) = \frac{d(f \circ \gamma)}{dt}(0) \ .$$

Proposition 3.1.4 *With respect to a chart (U,h,V) around p with coordinates $(x_1,\ldots,x_n)$ in V it holds*

$$df(p)=\sum_{i=1}^{n}\frac{\partial(f\circ h^{-1})}{\partial x_i}(h(p))\cdot dx_i(p)\ .$$

In particular, the 1-form $df : p\mapsto df(p)$ is differentiable; df is called the (exterior) differential of f.

Proof: Let v_i be the algebraic tangent vector corresponding to $\frac{\partial}{\partial x_i}(p)$; then it holds

$$df(p)(v_i)=v_i(f)=\frac{\partial(f\circ h^{-1})}{\partial x_i}(h(p))\ ,$$

so the claim follows from Lemma 5.1.6. ■

Examples 3.1.5 *1. If we define* $x_i(p):=i-$th coordinate of $h(p)$, *then x_i is a differentiable function in U. The value of the exterior differential dx_i of this function at p coincides with the basisvector $dx_i(p)$ introduced before; this follows from Proposition 3.1.4 since $x_i\circ h^{-1}$ is the function $(x_1,\ldots,x_n)\mapsto x_i$.*

2. ii) We consider $\mathbb{R}$ with standard atlas $\{(\mathbb{R},\mathrm{id}_{\mathbb{R}},\mathbb{R})$ and standard coordinate x, so $\frac{\partial}{\partial x}(p)$ resp. $dx(p)$ is a basis vector of $T_p\mathbb{R}$ resp. $T_p^*\mathbb{R}$. Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a differentiable function, and let us for the moment write $\frac{\partial}{\partial x}$ for the usual derivative on functions, i.e. $f'=\frac{\partial f}{\partial x}$. It is a natural question to ask if there is a relation between the three kinds of derivatives of f, namely f' (usual derivative), df (exterior differential) and Df (tangent map).
It follows from Proposition 3.1.4 that

$$df=\frac{\partial(f\circ\mathrm{id}_{\mathbb{R}})}{\partial x}\cdot dx=f'\cdot dx\ .$$

This implies

$$df(p)\left(\frac{\partial}{\partial x}(p)\right)=f'(p)\ .$$

On the other hand, it holds

$$Df(p)\left(\frac{\partial}{\partial x}(p)\right)=\frac{\partial(\mathrm{id}_{\mathbb{R}}\circ f\circ\mathrm{id}_{\mathbb{R}}^{-1})}{\partial x}(\mathrm{id}_{\mathbb{R}}(p))\cdot\frac{\partial}{\partial x}(f(p))=f'(p)\cdot\frac{\partial}{\partial x}(f(p))\ .$$

Now observe that $\Phi_p:T_p\mathbb{R}\longrightarrow\mathbb{R}$, $\Phi(a\cdot\frac{\partial}{\partial x}(p)):=a$, is a natural linear isomorphism for all $p\in\mathbb{R}$. It follows that

$$df(p)=\Phi_{f(p)}\circ Df(p)\ .$$

Now let Y be an m-dimensional manifold, and $f:X\longrightarrow Y$ a differentiable map. Then for each $p\in X$ we have the linear tangent map

$$Df(p):T_pX\longrightarrow T_{f(p)}Y$$

and its dual

$$f^*(p) := Df(p)^* : T^*_{f(p)}Y \longrightarrow T^*_pX$$

given by

$$f^*(p)(u^*)(v) = u^*(Df(p)(v)) \text{ for all } v \in T_pX \ , \ u^* \in T^*_{f(p)}Y \ .$$

Lemma 3.1.6 *Let be* $p \in X$ *and* (U, h, V) *resp.* (U', h', V') *charts for* X *resp.* Y *around* p *resp.* $f(p)$ *with coordinates* $(x_1, \ldots, x_n)$ *resp.* $(y_1, \ldots, y_m)$ *in* V *resp.* V'*. Then it holds*

$$f^*(p)(dy_j(f(p)) = \sum_{i=1}^{n} \frac{\partial(h' \circ f \circ h^{-1})_j}{\partial x_i}(h(p)) \cdot dx_i(p)$$

Proof: This follows from

$$Df(p)\left(\frac{\partial}{\partial x_i}(p)\right) = \sum_{j=1}^{m} \frac{\partial(h' \circ f \circ h^{-1})_j}{\partial x_i}(h(p)) \cdot \frac{\partial}{\partial y_j}(f(p))$$

and the general theory of duality. ■

We generally define for $1 \leq k \leq n$

$$f^*(p) := \Lambda^k Df(p)^* : \Lambda^k T^*_{f(p)}Y \longrightarrow \Lambda^k T^*_pX \ ;$$

applying this pointwise we obtain linear pullback maps mapping k-forms on Y to k-forms on X. Furthermore, we define

$$f^* : \mathcal{C}^\infty(Y, \mathbb{R}) = \Omega^0 Y \longrightarrow \Omega^0 X = \mathcal{C}^\infty(X, \mathbb{R}) \ , \ f^*(g) := g \circ f \ .$$

Then locally it holds

$$f^*(a \cdot dy_{i_1} \wedge \ldots \wedge dy_{i_k}) = (a \circ f) \cdot f^*(dy_{i_1}) \wedge \ldots \wedge f^*(dy_{i_k}) = f^*(a) \cdot f^*(dy_{i_1}) \wedge \ldots \wedge f^*(dy_{i_k}) \ .$$

In particular, from Lemma 3.1.6 it follows that the pullback of a differentiable k-form on Y is differentiable on X, i.e. that we get linear pullback maps

$$f^* : \Omega^k Y \longrightarrow \Omega^k X$$

satisfying $f^*(\omega \wedge \eta) = f^*(\omega) \wedge f^*(\eta)$.

Lemma 3.1.7 *Let* $g : Y \longrightarrow \mathbb{R}$ *a differentiable function on* Y*. Then it holds*

$$d(g \circ f) = d(f^*(g)) = f^*(dg) \ .$$

Proof: For any (algebraic) tangent vector $v \in T_pX$ it holds

$$[f^*(p)(dg(f(p)))](v) = dg(f(p))[Df(p)(v)] = [Df(p)(v)](g) = v(g \circ f) = [d(g \circ f)(p)](v) \ .$$

■

Example: Consider $\mathbb{R}^2$ with standard coordinates (x, y) associated to the chart $h = \mathrm{id}_{\mathbb{R}^2}$. For $\mathbb{R}^2 \setminus \{0\}$, we also have charts k via with polar coordinates (r, φ), i.e. of the form

$$k^{-1}(r, \varphi) = (r \cdot \cos\varphi, r \cdot \sin\varphi) \ .$$

Viewing x and y as the component functions of the gluing map $h \circ k^{-1}$ we have

$$x = x(r, \varphi) = r \cdot \cos\varphi \quad , \quad y = y(r, \varphi) = r \cdot \sin\varphi \ .$$

This implies

$$dx = \frac{\partial x}{\partial r} \cdot dr + \frac{\partial x}{\partial \varphi} \cdot d\varphi = \cos\varphi \cdot dr - r \cdot \sin\varphi \cdot d\varphi \ ,$$

and

$$dy = \frac{\partial y}{\partial r} \cdot dr + \frac{\partial y}{\partial \varphi} \cdot d\varphi = \sin\varphi \cdot dr + r \cdot \cos\varphi \cdot d\varphi \ .$$

Observe that this implies

$$dx \wedge dy = r \cdot dr \wedge d\varphi \ .$$

Now consider for $S^1 \subset \mathbb{R}^2$ a chart κ of the form $\kappa^{-1}(\varphi) = (\cos\varphi, \sin\varphi)$. Let $i : S^1 \hookrightarrow \mathbb{R}^2$ be the inclusion map; then

$$(k \circ i \circ \kappa^{-1})(\varphi) = (1, \varphi) = ((k \circ i \circ \kappa^{-1})_1(\varphi), (k \circ i \circ \kappa^{-1})_2(\varphi)) \ .$$

This implies

$$i^*dr = \frac{\partial (k \circ i \circ \kappa^{-1})_1}{\partial \varphi} d\varphi = 0 \quad , \quad i^*d\varphi = \frac{\partial (k \circ i \circ \kappa^{-1})_2}{\partial \varphi} d\varphi = d\varphi \ .$$

Since in a point $(x, y) \in S^1$ it holds $r = 1$, we get

$$i^*dx = i^*(\cos\varphi \cdot dr - \sin\varphi \cdot d\varphi) = \cos\varphi \cdot i^*dr - \sin\varphi \cdot i^*d\varphi = -\sin\varphi \cdot d\varphi \ ,$$

and similarly

$$i^*dy = \cos\varphi \cdot d\varphi \ .$$

Roughly speaking, these relations are obtained because *"on S^1 it holds $r \equiv 1$, implying $dr \equiv 0$, and $\varphi = \varphi$ "*.

3.2 The exterior differential

Let X be an n-dimensional differentiable manifold. We state without proof

Theorem 3.2.1 *There are unique linear maps (exterior differentials)*

$$d : \Omega^k X \longrightarrow \Omega^{k+1} X \quad , \quad k \in \mathbb{N}_0 \ ,$$

with the following properties:

1. *For* $k = 0$, *this is the map* $f \mapsto df$ *as defined in the previous section.*
2. *For all* $\omega \in \Omega^k X$ *and all* $U \subset X$ *open it holds* $(d\omega)|_U = d(\omega|_U)$.
3. *Locally in a chart* (U, h, V) *with coordinates* $(x_1, \ldots, x_n)$ *in* V *it holds*

$$d(f \cdot dx_{i_1} \wedge \ldots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} = \left(\sum_{i=1}^{n} \frac{\partial (f \circ h^{-1})}{\partial x_i} \circ h \right) \cdot dx_i \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \ .$$

Proposition 3.2.2 *1. For* $\omega \in \Omega^k X$, $\eta \in \Omega^l X$ *it holds*

$$d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^k \omega \wedge d(\eta) \ .$$

2. For $\omega \in \Omega^k X$ *it holds*

$$d^2 \omega = (d \circ d) \omega = 0 \ .$$

3. If $f : X \longrightarrow Y$ *is a differentiable map and* $\omega \in \Omega^k Y$, *then it holds*

$$f^*(d\omega) = d(f^* \omega) \ .$$

Proof: 1. For two functions f and g on X, $p \in X$ and an algebraic tangent vector $v \in T_p X$ it holds

$$\begin{aligned} d(f \cdot g)(p)(v) &= v(f \cdot g) = v(f) \cdot g(p) + f(p) \cdot v(g) = [df(p)(v)] \cdot g(p) + f(p) \cdot dg(p)(v) \\ &= [(df \cdot g + f \cdot dg)(p)](v) \ , \end{aligned}$$

which means

$$d(f \cdot g) = df \cdot g + f \cdot dg \ ;$$

This is the claim for $k = l = 0$. In the general case, since d is linear and the $\wedge$-product is bilinear, and because of 2. in Theorem 3.2.1, it suffices to consider the case $\omega = f \cdot dx_{i_1} \wedge \ldots \wedge dx_{i_k}$, $\eta = g \cdot dx_{j_1} \wedge \ldots \wedge dx_{j_l}$. Then $\omega \wedge \eta = f \cdot g \cdot dx_{i_1} \wedge \ldots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_l}$ and hence

$$\begin{aligned} d(\omega \wedge \eta) &= d(f \cdot g) \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_l} \quad \text{by 3. in Theorem3.2.1} \\ &= (df \cdot g + f \cdot dg) \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_l} \quad \text{as seen above} \\ &= (df \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}) \wedge (g \cdot dx_{j_1} \wedge \ldots \wedge dx_{j_l}) \\ &\quad + ((-1)^k f \cdot dx_{i_1} \wedge \ldots \wedge dx_{i_k}) \wedge (dg \wedge dx_{j_1} \wedge \ldots \wedge dx_{j_l}) \\ &= d(\omega) \wedge \eta + (-1)^k \omega \wedge d(\eta) \ . \end{aligned}$$

2. Again it suffices to prove the claim locally in a chart. For a function f we have, using the local formula for d of a function and 3. in Theorem 3.2.1

$$\begin{aligned} d^2 f &= d\left(\sum_{i=1}^n \frac{\partial(f\circ h^{-1})}{\partial x_i}\circ h\cdot dx_i\right) = \sum_{i=1}^n d\left(\frac{\partial(f\circ h^{-1})}{\partial x_i}\circ h\right)\wedge dx_i \\ &= \sum_{i=1}^n\left(\sum_{j=1}^n \frac{\partial}{\partial x_j}\left(\frac{\partial(f\circ h^{-1})}{\partial x_i}\circ h\circ h^{-1}\right)\circ h\cdot dx_j\right)\wedge dx_i \\ &= \sum_{i=1}^n\left(\sum_{j=1}^n \frac{\partial^2(f\circ h^{-1})}{\partial x_i\partial x_j}\circ h\cdot dx_j\right)\wedge dx_i \\ &= \sum_{i<j}\left(\frac{\partial^2(f\circ h^{-1})}{\partial x_i\partial x_j}-\frac{\partial^2(f\circ h^{-1})}{\partial x_j\partial x_i}\right)\circ h\cdot dx_j\wedge dx_i = 0 \end{aligned}$$

since $\frac{\partial^2}{\partial x_i\partial x_j}=\frac{\partial^2}{\partial x_j\partial x_i}$ on $\mathcal{C}^\infty$-functions. Now recall that dx_i is in fact the d of a function x_i, so $d^2x_i=0$ by what we have just seen. Using 1. (and induction) we deduce

$$\begin{aligned} d^2(f\cdot dx_{i_1}\wedge\ldots\wedge dx_{i_k}) &= d(df\wedge dx_{i_1}\wedge\ldots\wedge dx_{i_k}) \\ &= d^2f\wedge dx_{i_1}\wedge\ldots\wedge dx_{i_k}+\sum_{l=1}^k(-1)^l df\wedge dx_{i_1}\wedge\ldots\wedge d^2x_{i_l}\wedge\ldots\wedge dx_{i_k} \\ &= 0\ . \end{aligned}$$

3. By Lemma 3.1.7, the claim holds for $k=0$. Since f^* is defined pointwise in X it holds

$$f^*(\omega|_U) = (f^*\omega)|_{f^{-1}(U)}\ ,$$

and since f^* is linear it again suffices to consider the case $\omega = a\cdot dy_{i_1}\wedge\ldots\wedge dy_{i_k}$. For this we have

$$\begin{aligned} f^*(d\omega) &= f^*(da\wedge dy_{i_1}\wedge\ldots\wedge dy_{i_k}) = f^*(da)\wedge f^*(dy_{i_1})\wedge\ldots\wedge f^*(dy_{i_k}) \\ &= d(f^*(a))\wedge d(f^*(y_{i_1}))\wedge\ldots\wedge d(f^*(y_{i_k}))\quad (\text{case } k=0\ ) \\ &= d(f^*(a))\wedge d(f^*(y_{i_1}))\wedge\ldots\wedge d(f^*(y_{i_k})) \\ &\quad +\sum_{l=1}^k(-1)^{l+1}f^*(a)\cdot d(f^*(y_{i_1}))\wedge\ldots\wedge d(d(f^*(y_{i_l})))\wedge\ldots\wedge d(f^*(y_{i_k}))\quad (d\circ d=0) \\ &= d\Big(f^*(a)\cdot d(f^*(y_{i_1}))\wedge\ldots\wedge d(f^*(y_{i_k}))\Big) = d\Big(f^*(a)\cdot f^*(dy_{i_1})\wedge\ldots\wedge f^*(dy_{i_k})\Big) \\ &= d(f^*\omega)\ . \end{aligned}$$

■

Definition 3.2.3 *$\omega\in\Omega^kX$ is called* __closed__ *if $d\omega=0$, and* __exact__ *if there exists an $\eta\in\Omega^{k-1}X$ with $\omega=d\eta$.*

Every exact form is closed by Proposition 3.2.2 2, so one can ask if the converse holds, i.e., is every closed form exact? The following theorem (which we state without proof) asserts that there is a purely *topological* condition on X which is sufficient for this to be true, but the subsequent exercise shows that the answer in general is no.

Theorem 3.2.4 (Lemma of Poincaré) *Assume that X is contractible, i.e. that there is a point $p_0 \in X$ and a continuous map $F : X \times [0,1] \longrightarrow X$ such that $F(p,0) = p$ and $F(p,1) = p_0$ for all $p \in X$. Then every closed differential form on X is exact.*

We consider the unit circle $S^1 = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$. It can be shown that there is a 1-dimensional topological atlas for S^1 consisting of two charts (U_1, h_1, V_1), (U_2, h_2, V_2), where

$$U_1 = S^1 \setminus \{(1,0)\} \ , \ V_1 = (0, 2\pi) \ , \ h_1^{-1}(\phi_1) = (\cos\phi_1, \sin\phi_1) \ ,$$

$$U_2 = S^1 \setminus \{(-1,0)\} \ , \ V_2 = (\pi, 3\pi) \ , \ h_2^{-1}(\phi_2) = (\cos\phi_2, \sin\phi_2) \ ,$$

where ϕ_i is the coordinate in V_i, $i = 1, 2$.

Excercise 3.2.5 *1. Show that the given atlas for S^1 is differentiable, so it produces differentiable 1-forms $d\phi_i$ in U_i, $i = 1, 2$.*

2. Show that there exists a unique $\omega \in \Omega^1 S^1$ such that $\omega|_{U_1} = d\phi_i$, $i = 1, 2$.

3. Show that this ω is closed but not exact.

4 Integration on manifolds

4.1 Orientations on a manifold

Let X be a connected n-dimensional differentiable manifold, $n \geq 1$, and

$$\Omega_0^n X := \{ \ \omega \in \Omega^n X \mid \omega(p) \neq 0 \ \text{ for all } \ p \in X \ \} \ .$$

Lemma 4.1.1 *For $\omega, \eta \in \Omega_0^n X$ there exists a unique nowhere vanishing differentiable function $f_{\omega,\eta}$ on X with $\omega = f_{\omega,\eta} \cdot \eta$.*

Proof: Since for every $p \in X$ it holds $\dim \Lambda^n T_p^* X = 1$ and $\omega(p) \neq 0 \neq \eta(p)$, there exists a unique $f_{\omega,\eta}(p) \in \mathbb{R}$ such that $\omega(p) = f_{\omega,\eta}(p) \cdot \eta(p)$. It remains to show that the thus defined function $f_{\omega,\eta}$ on X is differentiable; this can be done locally.

Let (U, h, V) be a chart for X with coordinates $(x_1, \ldots, x_n)$ in V; then

$$\omega|_U = f_\omega \cdot dx_1 \wedge \ldots \wedge dx_n \ , \ \eta|_U = f_\eta \cdot dx_1 \wedge \ldots \wedge dx_n \ ,$$

with nowhere vanishing differentiable functions f_ω and f_η in U. This implies that $\omega|_U = \frac{f_\omega}{f_\eta} \cdot \eta|_U$, and hence that $f_{\omega,\eta}|_U = \frac{f_\omega}{f_\eta}$ is differentiable. ■

Since X is connected it holds either $f_{\omega,\eta} > 0$ or $f_{\omega,\eta} < 0$. It is easy to see that

$$\omega \sim \eta \quad :\Longleftrightarrow \quad f_{\omega,\eta} > 0$$

defines an equivalence relation in $\Omega_0^n X$, and that either $\Omega_0^n X = \emptyset$ or $\Omega_0^n X/_\sim$ has precisely two elements.

Definition 4.1.2 *X is called <u>orientable</u> if $\Omega_0^n X \neq \emptyset$. An <u>orientation</u> of X is then an element of $\Omega_0^n X/_\sim$, and an <u>oriented</u> manifold is a manifold together with a fixed orientation on it.*

Since for $\omega \in \Omega_0^n X$ it holds $0 \neq \omega(p) \in \Lambda^n T_p^* X$, ω defines simultaneously orientations in all tangent spaces $T_p X$ by Corollary 5.3.3. Observe that these orientations coincide for two forms in the same class in $\Omega_0^n X$ since they differ by a strictly positive function.

To prove another useful characterization of orientability of manifolds we need the following concept.

Let X be a topological space and $X = \bigcup_{i \in I} U_i$ an open cover. A <u>partition</u> <u>of</u> <u>unity</u> <u>subordinate</u> to this cover is a collection $\{ \ \tau_i \mid i \in I \ \}$ of continuous functions $\tau_i : X \longrightarrow [0,1]$ with the following properties.

1. $\text{supp}(\tau_i) = \overline{\{ \ x \in X \mid \tau_i(x) \neq 0 \ \}} \subset U_i$ for all $i \in I$.
2. For every $x \in X$ there exists an open neighborhood U of p in X for which there are at most finitely many $i \in I$ with $\text{supp}(\tau_i) \cap U \neq \emptyset$.

3. For every $x \in X$ it holds $\sum\limits_{i \in I} \tau_i(x) = 1$.

Observe that the sum in 3. is well defined because by 2. for every $x \in X$ there are only finitely many $i \in I$ with $\tau_i(x) \neq 0$.
In the case of a differentiable manifold X, a partition of unity is called differentiable if all τ_i are differentiable functions.

We state without proof the following result whose proof uses the second countability of X in an essential way.

Proposition 4.1.3 *If X is a differentiable manifold and $X = \bigcup\limits_{i \in I} U_i$ an open cover, then there exists a differentiable partition of unity subordinate to this cover.*

Theorem 4.1.4 *For an n-dimensional differentiable manifold X with $n \geq 1$ the following are equivalent:*

1. *X is orientable.*
2. *There exists a differentiable atlas $\mathcal{A} = \{ (U_i, h_i, V_i) \mid i \in I \}$ for X such that*

$$\forall\, i,j \in I\ \forall\, p \in U_i \cap U_j \ :\ \det\Big(D(h_j \circ h_i^{-1})(h_i(p))\Big) > 0\ . \qquad (*)$$

Proof: We will use the following notations: if $\{ (U_i, h_i, V_i) \mid i \in I \}$ is a differentiable atlas for X, then for $i \in I$ we denote by $(x_1^i, x_2^i, \ldots, x_n^i)$ the coordinates is V_i and write

$$\omega^i := dx_1^i \wedge dx_2^i \wedge \ldots \wedge dx_n^i \in \Omega^n U_i\ .$$

Observe that $\omega^i(p) \neq 0$ for all $p \in U_i$.

1. ⇒ 2.: Let $[\omega] \in {}^{\Omega_0^n X}\!/\!\sim$ be an orientation of X, and $\mathcal{A} = \{ (U_i, h_i, V_i) \mid i \in I \}$ a differentiable atlas for X such that each V_i is an open ball around $0 \in \mathbb{R}^n$; then in particular each U_i is connected.

For each $i \in I$ we have $\omega|_{U_i} = f_i \cdot \omega^i$ with a differentiable function f_i in U_i. Since $\omega(p) \neq 0 \neq \omega^i(p)$ it holds $f_i(p) \neq 0$ for all $p \in U_i$. Because f_i is continuous and U_i is connected, it follows that f_i is either strictly positive or strictly negative. In the second case, we replace h_i by $g \circ h_i$, where $g : V_i \longrightarrow V_i$ is the diffeomorphism $g(x_1^i, x_2^i, \ldots, x_n^i) = (-x_1^i, x_2^i, \ldots, x_n^i)$; this means that we replace dx_1^i by $-dx_1^i$ but keep $dx_2^i, \ldots, dx_n^i$, i.e. that we replace ω^i by $-\omega^i$ and hence f_i by $-f_i$.

In other words, we may assume that for all $i \in I$ and all $p \in U_i$ it holds $f_i(p) > 0$. We know (see Remark 3.1.1) that for all $i, j \in I$ and all $p \in U_i \cap U_j$ it holds

$$\frac{1}{f_j(p)} \cdot \omega(p) = \omega^j(p) = \det\Big(D(h_j \circ h_i^{-1})(h_i(p))\Big) \cdot \omega^i(p) = \det\Big(D(h_j \circ h_i^{-1})(h_i(p))\Big) \cdot \frac{1}{f_i(p)} \cdot \omega(p)\ .$$

Since $\omega(p)$ is a basis vector of the 1-dimensional vector space $\Lambda^n T_p^* X$, it follows

$$\det\Big(D(h_j \circ h_i^{-1})(h_i(p))\Big) = \frac{f_i(p)}{f_j(p)} > 0\ .$$

2. $\Rightarrow$ 1.: Let $\mathcal{A} = \{ (U_i, h_i, V_i) \mid i \in I \}$ be an atlas as in 2., and $\{ \tau_i \mid i \in I \}$ a differentiable partition of unity subordinate to the open cover $\bigcup_{i \in I} U_i$ of X. Then for each $i \in I$ we define $\eta^i \in \Omega^n X$ by

$$\eta^i(p) := \begin{cases} \tau_i(p) \cdot \omega^i(p) & \text{if } p \in U_i \ , \\ 0 & \text{if } p \notin U_i \ ; \end{cases}$$

observe that η^i is indeed well defined and differentiable since $\operatorname{supp}(\tau_i) \subset U_i$. Because each point in X has an open neighborhood in which at most finitely many of the τ_i, and hence of the η^i, are not zero,

$$\omega := \sum_{i \in I} \eta^i \in \Omega^n X$$

is well defined, too. For $p \in X$ define $I_p := \{ i \in I \mid p \in U_i \}$; then $\eta^i(p) = 0$ for all $i \notin I_p$, implying $\omega(p) = \sum_{i \in I_p} \eta^i(p)$. Since also $\tau_i(p) = 0$ for all $i \notin I_p$, it holds $\sum_{i \in I_p} \tau_i(p) = 1$; in particular, there exists $i_0 \in I_p$ with $\tau_{i_0}(p) > 0$. We have

$$\begin{aligned} \omega(p) &= \tau_{i_0}(p) \cdot \omega^{i_0}(p) + \sum_{i_0 \neq i \in I_p} \tau_i(p) \cdot \omega^i(p) \\ &= \tau_{i_0}(p) \cdot \omega^{i_0}(p) + \sum_{i_0 \neq i \in I_p} \tau_i(p) \cdot \det\Big(D(h_i \circ h_{i_0}^{-1})(h_{i_0}(p))\Big) \cdot \omega^{i_0}(p) \\ &= \left(\tau_{i_0}(p) + \sum_{i_0 \neq i \in I_p} \tau_i(p) \cdot \det\Big(D(h_i \circ h_{i_0}^{-1})(h_{i_0}(p))\Big) \right) \cdot \omega^{i_0}(p) \ . \end{aligned}$$

Since $\tau_i(p) \geq 0$ for all i, $\tau_{i_0}(p) > 0$ and $\det\Big(D(h_i \circ h_{i_0}^{-1})(h_{i_0}(p))\Big) > 0$ for all $i \in I_p$ by assumption, we see that $\omega(p) \neq 0$ since it is a strictly positive multiple of $\omega^{i_0}(p) \neq 0$. ∎

A differentiable atlas $\mathcal{A} = \{ (U_i, h_i, V_i) \mid i \in I \}$ of X is called oriented if it satisfies the condition $(*)$ in the theorem above. Observe that this condition is equivalent to

$$\forall\ i, j \in I\ \forall\ p \in U_i \cap U_j\ \exists\ a > 0\ :\ \omega^i(p) = a \cdot \omega^j(p) \ , \qquad (**)$$

where the ω^i are as in the proof.

By Theorem 4.1.4, the existence of an oriented atlas is equivalent to the existence of a nowhere vanishing n-form. To understand this equivalence more precisely, let $\mathcal{D}$ be the maximal differentiable atlas of X and define

$$\mathbb{A} := \{ \mathcal{A} \subset \mathcal{D} \mid \mathcal{A} \text{ is an oriented atlas } \} \ .$$

It is easy to see that

$$\mathcal{A} \simeq \mathcal{A}' \quad :\Longleftrightarrow \quad \mathcal{A} \cup \mathcal{A}' \in \mathbb{A}$$

defines an equivalence relation $\simeq$ in $\mathbb{A}$.

Let now be $\omega \in \Omega_0^n X$ and $\mathcal{A} \in \mathbb{A}$ an oriented atlas as above. We say that ω and $\mathcal{A}$ are compatible if

$$\forall\ i \in I\ \forall\ p \in U_i\ \exists\ a > 0\ :\ \omega^i(p) = a \cdot \omega(p) \ . \qquad (***)$$

Then (from in particular $(**)$ and $(***)$) it follows:

1. In the proof of the theorem, we constructed for every $\omega \in \Omega_0^n X$ a compatible $\mathcal{A}_\omega \in \mathbb{A}$, and for every $\mathcal{A} \in \mathbb{A}$ a compatible $\omega_\mathcal{A} \in \Omega_0^n X$.
2. If $\omega \in \Omega_0^n X$ is compatible with $\mathcal{A} \in \mathbb{A}$, then $\eta \in \Omega_0^n X$ is equivalent to ω if and only if it is compatible with $\mathcal{A}$, too.
3. If $\mathcal{A} \in \mathbb{A}$ is compatible with $\omega \in \Omega_0^n X$, then $\mathcal{A}' \in \mathbb{A}$ is equivalent to $\mathcal{A}$ if and only if it is compatible with ω, too.

Using the symbol $\approx$ for compatibility, we get

$$\omega \sim \eta \iff \eta \approx \mathcal{A}_\omega \iff \mathcal{A}_\omega \simeq \mathcal{A}_\eta \ ,$$

and

$$\mathcal{A} \simeq \mathcal{A}' \iff \omega_{\mathcal{A}'} \approx \mathcal{A} \iff \omega_\mathcal{A} \sim \omega_{\mathcal{A}'} \ .$$

We conclude

Corollary 4.1.5 *The map*

$$\Omega_0^n X/_\sim \longrightarrow \mathbb{A}/_\simeq \ , \quad [\omega] \mapsto [\mathcal{A}_\omega] \ ,$$

is well defined and bijective, with inverse $[\mathcal{A}] \mapsto [\omega_\mathcal{A}]$ *. In particular, either* $\mathbb{A}$ *is empty or* $\mathbb{A}/_\simeq$ *has precisely two elements.*

Remarks 4.1.6 *1. Let* $(X, [\omega])$ *be an oriented manifold. A chart* (U, h, V) *for* X *with coordinates* $(x_1, \ldots, x_n)$ *in* V *is called* *compatible* *with* $[\omega]$ *if it is contained in an atlas which is compatible with* ω*, i.e. if the form* $dx_1 \wedge \ldots \wedge dx_n$ *equals* $f \cdot \omega|_U$ *with a positive function* f*.*
If (U, h, V) *is any connected chart for* X*, then either it is compatible with* $[\omega]$*, or the chart obtained by composing* h *with the diffeomorphism* $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{k-1}, -x_k, x_{k+1}, \ldots, x_n)$ *for some* $1 \le k \le n$ *.*

2. The *standard orientation* *of the manifold* $\mathbb{R}^n$ *is the class of the* n*-form* $dx_1 \wedge dx_2 \wedge \ldots \wedge dx_n$ *associated to the standard coordinates.*

Remark 4.1.7 *Observe that the contents of this section so far really makes sense only for* $n \ge 1$ *. On the other hand, by Exercise 2.1.14 a 0-dimensional manifold is an at most countable collection of discrete points* $X = \{ p_i \mid i \in I \}$ *, with* $T_{p_i}X = \{0\}$ *for all* $i \in I$ *, so in agreement with Definition 5.3.1 4. we define an orientation in such* X *to be a map* $\mathfrak{o} : X \longrightarrow \{1, -1\}$ *. In particular, if* X *is connected, i.e. a single point, there are precisely two orientations in* X*.*

4.2 The integral

Let X be an n-dimensional differentiable manifold.

Lemma 4.2.1 *Let* $K \subset X$ *be a compact subset.*
Then there exist $N \in \mathbb{N}$, *charts* (U_i', h_i', V_i') *and* (U_i, h_i, V_i) *with* $\bar{U}_i \subset U_i'$, $h_i = h_i'|_{U_i}$, *and differentiable functions* $\tau_i : U := \bigcup_{i=1}^{N} U_i \longrightarrow [0,1]$ *with* $U_i = \tau_i^{-1}((0,1])$, $1 \leq i \leq N$, *such that* $K \subset U$ *and* $\sum_{i=1}^{N} \tau_i(p) = 1$ *for all* $p \in U$.

Proof: Choose an atlas $\{ (U_i', h_i', V_i') \mid i \in \mathbb{N} \}$ for X, and a partition of unity $\{ \sigma^i \mid i \in \mathbb{N} \}$ subordinate to the open cover $X = \bigcup_{i \in \mathbb{N}} U_i'$.
For all $i \in \mathbb{N}$ define $U_i := \sigma_i^{-1}((0,1])$; then U_i is open in X with $\bar{U}_i = \operatorname{supp}(\sigma^i) \subset U_i'$ and it holds $X = \bigcup_{i \in \mathbb{N}} U_i$. Since K is compact, for some $N \in \mathbb{N}$ it holds $K \subset \bigcup_{i=1}^{N} U_i$.
For $1 \leq i \leq N$ define

$$h_i := h_i'|_{U_i} \ , \ V_i := h_i(U_i) \subset V_i' \ , \ U := \bigcup_{i=1}^{N} U_i \ , \ \tau^i := \sigma^i|_U \ .$$

The (U_i, h_i, V_i) are charts, and it holds $U_i = (\tau^i)^{-1}((0,1])$ and $\tau(p) := \sum_{i=1}^{N} \tau^i(p) > 0$ for all $p \in U$; therefore, the functions $\tau_i := \frac{\tau^i}{\tau} : U \longrightarrow [0,1]$ have the desired properties. ∎

Remark: If X is oriented, we can start in the proof above with an oriented atlas; as a consequence, the (U_i', h_i', V_i')'s and (U_i, h_i, V_i)'s will be oriented, too.

Let X be an oriented n-dimensional differentiable manifold, and $\alpha \in \Omega^n X$ a differentiable n-form with compact support

$$K := \operatorname{supp}(\alpha) := \overline{\{ p \in X \mid \alpha(p) \neq 0 \}} \ .$$

Choose for K oriented charts (U_i', h_i', V_i'), (U_i, h_i, V_i) and functions $\tau_i : U \longrightarrow [0,1]$ as in Lemma 4.2.1, and let $(x_1^i, \ldots, x_n^i)$ be the coordinates in V_i', $1 \leq i \leq N$.

Remark 4.2.2 *Consider the 1-forms* dx_j^i *in* U_i, $j = 1, \ldots, n$, *induced by the chart* (U_i, h_i, V_i). *Then*

$$\left(h_i^{-1}\right)^* (dx_j^i) = dx_j^i \ ,$$

where on the right hand side we view dx_j^i *as the 1-form in* V_i *induced by the standard atlas* id_{V_i}.

Define differentiable functions $\alpha^i : U_i \longrightarrow \mathbb{R}$ by

$$\alpha|_{U_i} =: \alpha^i \cdot dx_1^i \wedge \ldots \wedge dx_n^i \ ;$$

then in V_i it holds (see the Remark above)

$$\left(h_i^{-1}\right)^* (\alpha|_{U_i}) = (\alpha^i \circ h_i^{-1}) \cdot dx_1^i \wedge \ldots \wedge dx_n^i \ ,$$

and hence, defining functions α_i in V_i,

$$\left(h_i^{-1}\right)^* (\tau_i \cdot \alpha|_{U_i}) = ((\tau_i \cdot \alpha^i) \circ h_i^{-1}) \cdot dx_1^i \wedge \ldots \wedge dx_n^i =: \alpha_i \cdot dx_1^i \wedge \ldots \wedge dx_n^i \ .$$

Since K is compact, $K \cap \bar{U}_i$ and

$$\operatorname{supp}(\alpha_i) \cap \bar{V}_i = h_i'(K \cap \bar{U}_i)$$

are compact, too; in particular, α_i is bounded in V_i. The open set $\{\ x \in V_i \mid \alpha_i(x) \neq 0\ \}$ is contained in the compact set $\operatorname{supp}(\alpha_i) \cap \bar{V}_i$, and hence bounded in $\mathbb{R}^n$.

From standard analysis we recall the following

Fact: Let $A \subset \mathbb{R}^n$ be an open subset, and $f : A \longrightarrow \mathbb{R}$ a continuous and bounded function with $\{\ x \in A \mid f(x) \neq 0\ \} \subset \mathbb{R}^n$ bounded. Then f is integrable over A, and $\int\limits_A f(x)dx$ is finite.

Therefore, the following definition makes sense.

Definition 4.2.3 *The* <u>*integral*</u> <u>*of*</u> α <u>*over*</u> X *is defined as*

$$\int\limits_X \alpha := \sum_{i=1}^N \int\limits_{V_i} (h_i^{-1})^*(\tau_i \cdot \alpha|_{U_i}) = \sum_{i=1}^N \int\limits_{V_i} \alpha_i(x) dx_1^i \wedge \ldots \wedge dx_n^i := \sum_{i=1}^N \int\limits_{V_i} \alpha_i(x) dx^i \ .$$

Theorem 4.2.4 $\int\limits_X \alpha$ *is well defined, independent of the chosen data.*

Proof: For $1 \leq j \leq \tilde{N}$, let $(\tilde{U}_j', \tilde{h}_j', \tilde{V}_j')$, $(\tilde{U}_j, \tilde{h}_j, \tilde{V}_j)$, $\tilde{U}$, $\tilde{\tau}_j$ be analogous data (in particular with oriented charts!) with coordinates $(y_1^j, \ldots, y_n^j)$ in $\tilde{V}_j'$, and define $\tilde{\alpha}_j$ as above.

Since $K \subset U \cap \tilde{U}$, and α vanishes outside K, we may assume that $U = \tilde{U}$. Then it holds

$$U_i = \bigcup_{j=1}^{\tilde{N}} U_i \cap \tilde{U}_j \quad , \quad V_i = \bigcup_{j=1}^{\tilde{N}} h_i(U_i \cap \tilde{U}_j) \ .$$

Because of $\sum\limits_{j=1}^{\tilde{N}} \tilde{\tau}_j \equiv 1$ and the linearity of $(h_i^{-1})^*$ it holds

$$(h_i^{-1})^*(\tau_i \cdot \alpha|_{U_i}) = \sum_{j=1}^{\tilde{N}} (h_i^{-1})^*(\tau_i \cdot \tilde{\tau}_j \cdot \alpha|_{U_i}) = \sum_{j=1}^{\tilde{N}} (h_i^{-1})^*(\tau_i \cdot \tilde{\tau}_j \cdot \alpha|_{U_i \cap \tilde{U}_j}) \ ;$$

the last equality holds because $\tau_i \cdot \tilde{\tau}_j$ vanishes outside $U_i \cap \tilde{U}_j$.

The first set of data thus gives

$$\int_X \alpha = \sum_{i=1}^{N}\sum_{j=1}^{\tilde N} \int_{h_i(U_i\cap\tilde U_j)} (h_i^{-1})^*(\tau_i\cdot\tilde\tau_j\cdot\alpha|_{U_i\cap\tilde U_j}) \ ,$$

and analogously the second gives

$$\int_X \alpha = \sum_{i=1}^{N}\sum_{j=1}^{\tilde N} \int_{\tilde h_j(U_i\cap\tilde U_j)} (\tilde h_j^{-1})^*(\tau_i\cdot\tilde\tau_j\cdot\alpha|_{U_i\cap\tilde U_j}) \ ,$$

The following Lemma implies

$$\int_{h_i(U_i\cap\tilde U_j)} (h_i^{-1})^*(\tau_i\cdot\tilde\tau_j\cdot\alpha|_{U_i\cap\tilde U_j}) = \int_{\tilde h_j(U_i\cap\tilde U_j)} (\tilde h_j^{-1})^*(\tau_i\cdot\tilde\tau_j\cdot\alpha|_{U_i\cap\tilde U_j})$$

for all pairs (i,j), and hence the Theorem. ■

Lemma 4.2.5 *Let (U,h,V) and (U,k,W) be two oriented charts for X with coordinates $(x_1,\ldots,x_n)$ resp. $(y_1,\ldots,y_n)$ in V resp. W, and $\omega\in\Omega^nU$. If*

$$(k^{-1})^*(\omega) = f\cdot dy_1\wedge\ldots\wedge dy_n \in \Omega^1W \ ,$$

then

$$(h^{-1})^*(\omega) = (f\circ\phi)\cdot|\det(D\phi)|\cdot dx_1\wedge\ldots\wedge dx_n \in \Omega^1V \qquad (*)$$

where $\phi := k\circ h^{-1} : V \longrightarrow W$.
Since ϕ is a diffeomorphism, the Transformation Formula from calculus (Theorem 4.2.6 below) implies

$$\int_V (h^{-1})^*(\omega) = \int_V (f\circ\phi)(x)\cdot|\det(D\phi(x))|dx = \int_W f(y)dy = \int_W (k^{-1})^*(\omega) \ .$$

Proof: Write in U

$$\omega = g\cdot dy_1\wedge\ldots\wedge dy_n \ ;$$

then according to Remark 3.1.1 it holds in U

$$\omega = g\cdot(\det(D\phi)\circ h)\cdot dx_1\wedge\ldots\wedge dx_n \ .$$

Using Remark 4.2.2 we get in W

$$(k^{-1})^*(\omega) = (g\circ k^{-1})\cdot dy_1\wedge\ldots\wedge dy_n \ ,$$

i.e.

$$f = g\circ k^{-1} \ ,$$

and in V

$$(h^{-1})^*(\omega) = (g \cdot (\det(D\phi) \circ h)) \circ h^{-1} \cdot dx_1 \wedge \ldots \wedge dx_n = (g \circ h^{-1}) \cdot \det(D\phi)) \cdot dx_1 \wedge \ldots \wedge dx_n \ .$$

It holds $g \circ h^{-1} = f \circ \phi$, and since the charts are oriented it holds $\det(D\phi) = |\det(D\phi)|$; this proves $(*)$ and hence the Lemma. ■

Theorem 4.2.6 Transformation Formula: *Let $A, B \subset \mathbb{R}^n$ be open, $\phi : A \longrightarrow B$ a diffeomorphism, and $f : B \longrightarrow \mathbb{R}$ an integrable function.*
Then $(f \circ \phi) \cdot |\det(D\phi)| : A \longrightarrow \mathbb{R}$ is integrable, too, and it holds

$$\int_A |\det(D\phi)| \cdot (f \circ \phi) = \int_{\phi^{-1}(B)} |\det(D\phi)| \cdot (f \circ \phi) = \int_B f \ .$$

Remarks 4.2.7 1. *Define $\Omega^n_c X := \{ \ \alpha \in \Omega^n X \ | \ \mathrm{supp}(\alpha) \text{ compact } \}$. This is a linear subspace of $\Omega^n X$, and the map*

$$\int_X : \Omega^n_c X \longrightarrow \mathbb{R} \quad , \quad \alpha \mapsto \int_X \alpha \ ,$$

is linear.

2. *Let be X, $\alpha \in \Omega^n_c X$, and $A \subset X$ open. Then, even if the support of $\alpha|_A$ might not be compact, the integral*

$$\int_A \alpha = \int_A \alpha|_A = \sum_{i=1}^{N} \int_{h_i(A \cap U_i)} a_i(x) dx$$

is always well defined.

3. *Consider $\mathbb{R}^n$ with its standard differentiable structure and orientation.*
Let be $\alpha = f \cdot dx_1 \wedge \ldots \wedge dx_n \ \in \Omega^n \mathbb{R}^n$, and $A \subset \mathbb{R}^n$ open. If $\mathrm{supp}(\alpha)$ is compact, then

$$\int_A \alpha = \int_A f(x) dx_1 dx_2 \ldots dx_n \ ,$$

where the right hand side is the usual (Lebesgue or Riemann) integral. Observe that this integral is also well defined for arbitrary $\alpha \in \Omega^n \mathbb{R}^n$ if $\overline{A}$ is compact.
Furthermore, using the notations from above it holds

$$(h_i^{-1})^*(\tau_i \cdot \alpha|_{U_i}) = a_i \cdot dx^i_1 \wedge \ldots \wedge dx^i_n \ ,$$

since $(h_i^{-1})^*(dx^i_k) = dx^i_k$, so

$$\int_{V_i} a_i(x) dx = \int_{V_i} (h_i^{-1})^*(\tau_i \cdot \alpha|_{U_i}) \ .$$

The definition of the integral given above is not very useful for calculations, e.g. because in general it is very hard to explicitly construct a partition of unity. But in many cases a fact of the following type is helpful.

Proposition 4.2.8 *Let X be an n-dimensional oriented manifold, and $\alpha \in \Omega^n X$ a differentiable n-form with compact support* $\mathrm{supp}(\alpha)$.
Let $Y \subset X$ be a submanifold with $\dim Y < \dim X$, and (U_i, h_i, V_i) , $1 \leq i \leq N$, oriented charts for $X \setminus Y$ such that $U_i \cap U_j = \emptyset$ for $1 \leq i \neq j \leq N$, and

$$\mathrm{supp}(\alpha) \setminus Y \subset \bigcup_{i=1}^{N} U_i \ .$$

Define functions α_i in V_i as above. Then it holds

$$\int_X \alpha = \int_{X \setminus Y} \alpha = \sum_{i=1}^{N} \int_{V_i} \alpha_i(x) dx \ .$$

Examples 4.2.9 *1. Consider $\mathbb{R}^2$ with standard coordinates (x, y), and, for differentiable functions f and g, the 1-form $\alpha = f(x, y)dx + g(x, y)dy \in \Omega^1\mathbb{R}^2$. Let $S^1 \subset \mathbb{R}^2$ be the unit circle with inclusion map $\iota : S^1 \longrightarrow \mathbb{R}^2$. $\iota^*(\alpha)$ has compact support since S^1 is compact, hence*

$$\int_{S^1} (f dx + g dy) := \int_{S^1} \iota^*(\alpha)$$

is well defined.

It is easily verified that an oriented atlas for S^1 is given by the two charts

$$h^{-1} : t \mapsto (\cos(t), \sin(t)) \ , \ t \in (0, 2\pi) \ , \ k^{-1} : t \mapsto (\cos(t), \sin(t)) \ , \ t \in (\pi, 3\pi) \ ,$$

and a correct interpretation of Proposition 4.2.8 yields

$$\int_{S^1} (f dx + g dy) = \int_0^{2\pi} (h^{-1})^* \left(\iota^*(\alpha)\right) = \int_0^{2\pi} (h^{-1})^*(\alpha) \ ,$$

using $(h^{-1})^* \circ \iota^* = (\iota \circ h^{-1})^*$ and $\iota \circ h^{-1} = h^{-1} : (0, 2\pi) \longrightarrow \mathbb{R}^2$.
Now observe that

$$\left[(h^{-1})^*(dx)\right](t) = \left[d(x \circ h^{-1})\right](t) = \left[d(\cos(t))\right](t) = \frac{d\cos}{dt}(t)dt = -\sin(t)dt \ ,$$

and similarly

$$\left[(h^{-1})^*(dy)\right](t) = \cos(t)dt \ .$$

This implies

$$\begin{aligned}\left[(h^{-1})^*(\alpha)\right](t) &= \left[(h^{-1})^*(fdx+gdy)\right](t)\\ &= \left[(f\circ h^{-1})\cdot(h^{-1})^*(dx)+(g\circ h^{-1})\cdot(h^{-1})^*(dy)\right](t)\\ &= \Big(-\sin(t)f(\cos(t),\sin(t))+\cos(t)g(\cos(t),\sin(t))\Big)dt\ .\end{aligned}$$

As an explicit example, take $f\equiv 1$, $g\equiv 0$, *i.e.* $\alpha = dx$; *then we get*

$$\int_{S^1} dx = -\int_0^{2\pi}\sin(t)dt = 0\ .$$

2. *Consider the open square* $A := \{\ (x,y)\in\mathbb{R}^2 \mid -1<x<1\ ,\ -1<y<1\ \}$. *Although the rectangle* ∂A *is not smooth in its four corner points,* $\int_{\partial A}\alpha$ *is well defined (of course only after choosing an orientation) for every* $\alpha\in\Omega^1\mathbb{R}^2$; *the integral can be calculated by integrating along the four open sides.*

4.3 Subsets with smooth boundary

Let X be an n-dimensional differentiable manifold and $A\subset X$ an open subset with topological boundary ∂A. [2] We always equip ∂A with the subspace topology induced from X.

Notation: For an open subset $V\subset\mathbb{R}^n$ we write

$$\begin{aligned} V_+ &:= \{\ (x_1,\ldots,x_n)\in V \mid x_1>0\ \}\ ,\\ V_- &:= \{\ (x_1,\ldots,x_n)\in V \mid x_1<0\ \}\ ,\\ V_0 &:= \{\ (x_1,\ldots,x_n)\in V \mid x_1=0\ \}\ .\end{aligned}$$

Definition 4.3.1 *We say that* A *has a* <u>*smooth*</u> <u>*boundary*</u> *if for every* $p\in\partial A$ *there exists a chart* (U,h,V) *for* X *such that* $h(A\cap U)=V_-$.

Theorem 4.3.2 *Suppose that the open subset* $A\subset X$ *has a smooth boundary, and that* (U,h,V) *is a chart as in the definition above.*

1. *We define* $U_A := \partial A\cap U$; *then it holds*

$$h(U_A) = V_0\ .$$

2. *If we identify* $\mathbb{R}^{n-1}$ *with* $\{0\}\times\mathbb{R}^{n-1}\subset\mathbb{R}^n$, *then*

$$h_A := h|_{U_A} : U_A \longrightarrow V_0$$

is an $(n-1)$*-dimensional topological chart for* ∂A*; these charts* (U_A,h_A,V_0) *form a differentiable* $(n-1)$*-dimensional atlas for* ∂A *which is an* $(n-1)$*-dimensional submanifold of* X*.*

---

[2] Recall that $\partial A = \overline{A}\setminus\mathring{A}$ is the set of points $p\in X$ such that for every neighborhood V of p in X it holds $V\cap A\neq\emptyset\neq V\cap(X\setminus A)$.

3. An orientation in X induces an orientation in ∂A.

Proof: 1. This is an exercise in topology (using that h is a homeomorphism) and therefore left to the reader.

2. This follows from 1. and the theory of submanifolds.

3. We consider first the case $n \geq 2$; then by using only connected charts and by composing with $(x_1, x_2, \ldots, x_n) \mapsto (x_1, -x_2, \ldots, x_n)$ if necessary we may assume that all the charts for ∂A as in 1. or 2. are coming from charts for X which are compatible with the given orientation in X; we want to show that this atlas for ∂A is oriented.
Let (U, h, V), (U', h', V') be two of these charts, and define

$$\psi := (\psi_1, \psi_2, \ldots, \psi_n) = h' \circ h^{-1}|_{h(U \cap U')} \quad , \quad \psi^A := (\psi_2^A, \ldots, \psi_n^A) = h'_A \circ h_A^{-1}|_{h_A(U_A \cap U'_A)} \ .$$

We know that $\det(D\psi(x)) > 0$ for all $x \in h(U \cap U')$, and have to show that $\det(D\psi^A(x)) > 0$ for all $x \in h_A(U_A \cap U'_A)$.
By construction for $(x_2, \ldots, x_n) \in h_A(U_A \cap U'_A)$ it holds

$$\psi(0, x_2, \ldots, x_n) = (0, \psi_2^A(x_2, \ldots, x_n), \ldots, \psi_n^A(x_2, \ldots, x_n)) \ .$$

This implies $\psi_1(0, x_2, \ldots, x_n) \equiv 0$, and hence $\frac{\partial \psi_1}{\partial x_k}(0, x_2, \ldots, x_n) = 0$, $2 \leq k \leq n$. Furthermore we get

$$D\psi(0, x_2, \ldots, x_n) = \begin{pmatrix} \frac{\partial \psi_1}{\partial x_1}(0, x_2, \ldots, x_n) & 0 & \cdots & 0 \\ * & & & \\ \vdots & & D(\psi^A)(x_2, \ldots, x_n) & \\ * & & & \end{pmatrix} ,$$

hence it suffices to show that $\frac{\partial \psi_1}{\partial x_1}(0, x_2, \ldots, x_n) \geq 0$. For this observe that for $t > 0$ it holds

$$(t, x_2, \ldots, x_n) \notin V_- \ \Rightarrow \ h^{-1}(t, x_2, \ldots, x_n) \notin A \ \Rightarrow \ \psi(t, x_2, \ldots, x_n) \notin V'_- \ \Rightarrow \ \psi_1(t, x_2, \ldots, x_n) \geq 0 \ ,$$

and thus

$$\begin{aligned} \frac{\partial \psi_1}{\partial x_1}(0, x_2, \ldots, x_n) &= \lim_{t \to 0, t > 0} \frac{\psi_1(t, x_2, \ldots, x_n) - \psi_1(0, x_2, \ldots, x_n)}{t} \\ &= \lim_{t \to 0, t > 0} \frac{\psi_1(t, x_2, \ldots, x_n)}{t} \geq 0 \end{aligned}$$

as wanted.

In the case $n = 1$, ∂A is a 0-dimensional submanifold, i.e. a set of discrete points. We give a point $p \in \partial A$ the orientation $+1$ if a connected chart as in the definition is compatible with the given orientation in X, and the orientation -1 otherwise. ∎

Remark 4.3.3 *In terms of oriented bases of tangent spaces the induced orientation in a smooth boundary is described as follows:*
Let be $p \in \partial A$ *and take an oriented chart as before with coordinates* $(x_1, \ldots, x_n)$*. Then, intuitively,* $\frac{\partial}{\partial x_1}(p)$ *is pointing outward from* A*. A basis* $\{b_2, \ldots, b_n\}$ *of* $T_p\partial A$ *is then oriented if and only if* $\{\frac{\partial}{\partial x_1}(p), b_2, \ldots, b_n\}$ *is an oriented basis of* T_pX*.*

Example 4.3.4 *Consider* $\mathbb{R}^2$ *with its standard differentiable structure.*
The circle $S^1 = \{\ x \in \mathbb{R}^2 \mid \|x\| = 1\ \}$ *is the boundary of the open disc* $D^2 = \{\ x \in \mathbb{R}^2 \mid \|x\| < 1\ \}$ *, and (e.g. using polar coordinates) one verifies that* $S^1 = \partial D^2$ *is a smooth boundary. The inverse of a chart for* S^1 *compatible with the orientation induced by the standard orientation in* $\mathbb{R}^2$ *is then e.g. one of type* $\varphi \mapsto (\cos\varphi, \sin\varphi)$ *, but not one of type* $\varphi \mapsto (\sin\varphi, \cos\varphi)$ *.*
If we view S^1 *as the boundary of* $\{\ x \in \mathbb{R}^2 \mid \|x\| > 1\ \}$ *, then the contrary is true.*

4.4 The Theorem of Stokes

We begin with a useful lemma. Consider $\mathbb{R}^n$ with standard coordinates $x = (x_1, \ldots, x_n)$ and define

$$\begin{aligned}
&U := \{\ x \in \mathbb{R}^n \mid -1 < x_i < 1\ ,\ i = 1, \ldots, n\ \}\ , \\
&U_- := \{\ x \in U \mid x_1 < 0\ \}\ , \\
&\partial U_- := \{\ x \in U \mid x_1 = 0\ \}\ , \\
&V := \{\ (x_2, \ldots, x_n) \in \mathbb{R}^{n-1} \mid (0, x_2, \ldots, x_n) \in \partial U_-\ \}\ , \\
&h : \partial U_- \longrightarrow V\ ,\ (0, x_2, \ldots, x_n) \mapsto (x_2, \ldots, x_n)\ .
\end{aligned}$$

Observe that U_- is an open subset of U with smooth boundary ∂U_-, and that h is a chart for the submanifold $\partial U_- \subset U$.

Let be $\alpha \in \Omega^{n-1}\mathbb{R}^n$ with $\text{supp}(\alpha) \subset U$; then $\text{supp}(\alpha)$ is compact.

Lemma 4.4.1 *It holds*

1. $\int\limits_U d\alpha = 0$ *;*
2. $\int\limits_{U_-} d\alpha = \int\limits_{\partial U_-} \iota^*(\alpha)$ *, where* $\iota : \partial U_- \longrightarrow U$ *is the inclusion map.*

Proof: We will use without comment the following fact: If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a continuous function and $A \subset \mathbb{R}^n$ open, then

$$\int\limits_A f(x)dx = \int\limits_{\overline{A}} f(x)dx\ .$$

Let a_i, $1 \le i \le n$, be the differentiable functions with

$$\alpha = \sum_{i=1}^{n} a_i \cdot dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n\ ;$$

then $\operatorname{supp}(\alpha) \subset U$ is equivalent to $\operatorname{supp}(a_i) \subset U$, $1 \le i \le n$. In particular, we have

$$a_i(x) = 0 \ \text{ if } \ |x_j| = 1 \ \text{ for some } \ 1 \le j \le n\,. \quad (*)$$

Furthermore it holds

$$d\alpha = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial a_i}{\partial x_j} \cdot dx_j \wedge dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n = \sum_{i=1}^{n} (-1)^{i-1} \cdot \frac{\partial a_i}{\partial x_i} \cdot dx_1 \wedge \ldots \wedge dx_n\,.$$

1. We have

$$\int_U d\alpha = \sum_{i=1}^{n} (-1)^{i-1} \cdot \left(\int_U \frac{\partial a_i}{\partial x_i} \cdot dx_1 \wedge \ldots \wedge dx_n \right) \;;$$

it suffices to show that each of the n integrals vanish. For this, first note that by the definition of the integral of differential forms on $\mathbb{R}^n$ and by the Theorem of Fubini it holds

$$\begin{aligned} \int_U \frac{\partial a_i}{\partial x_i} \cdot dx_1 \wedge \ldots \wedge dx_n &= \int_U \frac{\partial a_i}{\partial x_i}(x) dx_1 dx_2 \ldots dx_n = \int_{-1}^{1}\int_{-1}^{1} \ldots \int_{-1}^{1} \frac{\partial a_i}{\partial x_i}(x) dx_1 dx_2 \ldots dx_n \\ &= \int_{-1}^{1} \ldots \int_{-1}^{1} \left(\int_{-1}^{1} \frac{\partial a_i}{\partial x_i}(x) dx_i \right) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n\,. \end{aligned}$$

Using the Fundamental Theorem of Calculus and $(*)$ we get

$$\int_{-1}^{1} \frac{\partial a_i}{\partial x_i}(x) dx_i = a_i(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) - a_i(x_1, \ldots, x_{i-1}, -1, x_{i+1}, \ldots, x_n) = 0$$

and thus the claim.

2. As in 1. we have

$$\int_{U_-} d\alpha = \sum_{i=1}^{n} (-1)^{i-1} \cdot \left(\int_{U_-} \frac{\partial a_i}{\partial x_i} \cdot dx_1 \wedge \ldots \wedge dx_n \right) = \sum_{i=1}^{n} (-1)^{i-1} \cdot \left(\int_{-1}^{1} \ldots \int_{-1}^{1}\int_{-1}^{0} \frac{\partial a_i}{\partial x_i}(x) dx_1 dx_2 \ldots dx_n \right)\,.$$

For $i \neq 1$ we get

$$\int_{-1}^{1} \ldots \int_{-1}^{1}\int_{-1}^{0} \frac{\partial a_i}{\partial x_i}(x) dx_1 dx_2 \ldots dx_n = \int_{-1}^{1} \ldots \int_{-1}^{1}\int_{-1}^{0} \left(\int_{-1}^{1} \frac{\partial a_i}{\partial x_i}(x) dx_i \right) dx_1 \ldots dx_{i-1} dx_{i+1} \ldots dx_n = 0$$

by the same argument as in 1. Hence it remains to show that the summand with $i = 1$ equals $\int_{\partial U_-} \iota^*(\alpha)$. For this we first calculate as before

$$\int_{-1}^{1} \ldots \int_{-1}^{1} \left(\int_{-1}^{0} \frac{\partial a_1}{\partial x_1}(x) dx_1 \right) dx_2 \ldots dx_n = \int_{-1}^{1} \ldots \int_{-1}^{1} \left(a_1(0, x_2, \ldots, x_n) - a_1(-1, x_2, \ldots, x_n) \right) dx_2 \ldots dx_n$$

$$= \int_{-1}^{1} \dots \int_{-1}^{1} a_1(0, x_2, \dots, x_n) dx_2 \dots dx_n \ ,$$

since $a_1(-1, x_2, \dots, x_n) = 0$. On the other hand, we have

$$\iota^*(x_i) = x_i \circ \iota = \begin{cases} 0 & \text{if } i = 1, \\ x_i & \text{if } i \neq 1, \end{cases}$$

and hence

$$\iota^*(dx_i) = d(\iota^*(x_i)) = \begin{cases} 0 & \text{if } i = 1, \\ dx_i & \text{if } i \neq 1. \end{cases}$$

Furthermore, it holds

$$a_i \circ \iota(x_2, \dots, x_n) = a_i(0, x_2, \dots, x_n) \ ,$$

so

$$\begin{aligned} \iota^*(\alpha)(x_2, \dots, x_n) &= \sum_{i=1}^{n} a_i(0, x_2, \dots, x_n) \cdot \iota^*(dx_1) \wedge \dots \wedge \iota^*(dx_{i-1}) \wedge \iota^*(dx_{i+1}) \wedge \dots \wedge \iota^*(dx_n) \\ &= a_1(0, x_2, \dots, x_n) \cdot dx_2 \wedge \dots \wedge dx_n \ . \end{aligned}$$

Thus it holds indeed

$$\int_{\partial U_-} \iota^*(\alpha) = \int_{-1}^{1} \dots \int_{-1}^{1} a_1(0, x_2, \dots, x_n) dx_2 \dots dx_n \ .$$

■

Let M be an n-dimensional differentiable manifold, and $A \subset M$ open with smooth boundary ∂A. Denote by $\iota : \partial A \hookrightarrow M$ the inclusion map. We fix an orientation in M and the induced orientation in ∂A.

Let be $\alpha \in \Omega^{n-1} M$ with compact support $\operatorname{supp}(\alpha)$. Every point $p \notin \operatorname{supp}(\alpha)$ has an open neighborhood U with $\alpha|_U = 0$, and thus $d\alpha|_U = 0$. This implies $\operatorname{supp}(d\alpha) \subset \operatorname{supp}(\alpha)$; in particular, $\operatorname{supp}(d\alpha)$ is compact. For $p \in \partial A \setminus \operatorname{supp}(\alpha)$ it holds $\alpha(p) = 0$ and hence $\iota^*(\alpha)(p) = \iota^*(\alpha(p)) = 0$. This implies $\operatorname{supp}(\iota^*(\alpha)) \subset \partial A \cap \operatorname{supp}(\alpha)$; in particular, $\operatorname{supp}(\iota^*(\alpha))$ is compact, too. Therefore, both integrals in the following are well defined.

Theorem 4.4.2 (Theorem of Stokes) *For every* $\alpha \in \Omega^{n-1} M$ *with compact support it holds*

$$\int_A d\alpha = \int_{\partial A} \iota^*(\alpha) \ .$$

Proof: Let $U, U_-, \partial U_-, h, V$ be as above. Since A is open, for every point $p \in A$ there is an oriented chart (U_p, h_p, U) for M with $p \in U_p \subset A$, and since ∂A is smooth, for every $q \in \partial A$ there is a chart (V_q, k_q, U) for M with $k_q(A \cap V_q) = U_-$; observe that then $(A \cap V_q, k_q|_{A \cap V_q}, U_-)$ is an oriented chart for A, and $(\partial A \cap V_q, h \circ k_q|_{\partial A \cap V_q}, V)$ is an oriented chart for ∂A.

Define $K := \overline{A} \cap \mathrm{supp}(\alpha) = (A \cup \partial A) \cap \mathrm{supp}(\alpha)$; since $\mathrm{supp}(\alpha)$ is compact and $\overline{A}$ is closed, K is compact. Since

$$K \subset A \cup \partial A \subset \left(\bigcup_{p \in A} U_p \right) \cup \left(\bigcup_{q \in \partial A} V_q \right) ,$$

there are $k, l \in \mathbb{N}$ and $p_1, \dots, p_k \in A$, $q_1, \dots, q_l \in \partial A$ such that

$$K \subset \left(\bigcup_{i=1}^{k} U_{p_i} \right) \cup \left(\bigcup_{j=1}^{l} V_{q_j} \right) =: M' .$$

It holds $\mathrm{supp}(d\alpha) \subset \mathrm{supp}(\alpha)$. This means that only points in K contribute to the integrals in the theorem, so we may assume that $M = M'$.

Write $U_i := U_{p_i}$, $h_i := h_{p_i}$, $V_j := V_{q_j}$, $k_j := k_{q_j}$, $W_j := V_j \cap \partial A$, $\kappa_j := h \circ k_j|_{W_j}$.
Let $\{ \tau_i, \sigma_j \mid 1 \le i \le k \ , \ 1 \le j \le l \}$ be a partition of unity subordinate to the open cover

$$M = \left(\bigcup_{i=1}^{k} U_i \right) \cup \left(\bigcup_{j=1}^{l} V_j \right) ,$$

and define $\alpha_i := \tau_i \cdot \alpha$, $\alpha_j := \sigma_j \cdot \alpha$. Then $\mathrm{supp}(\alpha_i) \subset U_i$, $\mathrm{supp}(\alpha_j) \subset V_j$, and

$$\alpha = \sum_{i=1}^{k} \alpha_i + \sum_{j=1}^{l} \alpha_j \quad , \quad d\alpha = \sum_{i=1}^{k} d\alpha_i + \sum_{j=1}^{l} d\alpha_j .$$

If we define

$$\beta_i := (h_i^{-1})^* \alpha_i \ , \ \gamma_j := (k_j^{-1})^* \alpha_j \ ,$$

then since d commutes with pullback it holds

$$d\beta_i = (h_i^{-1})^* d\alpha_i \ , \ d\gamma_j = (k_j^{-1})^* d\alpha_j \ ,$$

and by the definition of the integral we have

$$\int_A d\alpha = \sum_{i=1}^{k} \int_U (h_i^{-1})^* d\alpha_i + \sum_{j=1}^{l} \int_{U_-} (k_j^{-1}|_{U_-})^* d\alpha_j = \sum_{i=1}^{k} \int_U d\beta_i + \sum_{j=1}^{l} \int_{U_-} d\gamma_j|_{U_-} .$$

Since $\mathrm{supp}(\alpha_i) \subset U_i$, $\mathrm{supp}(\alpha_j) \subset V_j$, it holds $\mathrm{supp}(\beta_i) \subset U$, $\mathrm{supp}(\gamma_j) \subset U$. In particular, $\int_U d\beta_i = 0$ by the first part of the lemma, and the second part of the lemma implies

$$\int_A d\alpha = \sum_{j=1}^{l} \int_{U_-} d\gamma_j|_{U_-} = \sum_{j=1}^{l} \int_{\partial U_-} \iota^* \gamma_j \ ,$$

where $\bar{\iota} : \partial U_- \hookrightarrow U$ is the inclusion map. Next observe that

$$\int_{\partial U_-} \bar{\iota}^* \gamma_j = \int_V (h^{-1})^* (\bar{\iota}^* \gamma_j)$$

and

$$(h^{-1})^*(\bar{\iota}^*\gamma_j) = (\bar{\iota} \circ h^{-1})^* \gamma_j = (\bar{\iota} \circ h^{-1})^*((k_j^{-1})^* \alpha_j) = (k_j^{-1} \circ \bar{\iota} \circ h^{-1})^* \alpha_j = (k_j^{-1} \circ h^{-1})^* \alpha_j = (\kappa_j^{-1})^* \alpha_j \ ;$$

This implies

$$\int_A d\alpha = \sum_{j=1}^{l} \int_V (\kappa_j^{-1})^* \alpha_j \ .$$

Since $\mathrm{supp}(\alpha_i) \subset U_i \subset A$, it holds $\alpha_i|_{\partial A} = 0$ and hence $\iota^* \alpha_i = 0$, implying

$$\int_{\partial A} \iota^* \alpha = \int_{\partial A} \left(\sum_{i=1}^{k} \iota^* \alpha_i + \sum_{j=1}^{l} \iota^* \alpha_j \right) = \sum_{j=1}^{l} \int_{\partial A} \iota^* \alpha_j \ .$$

On the other hand, $\mathrm{supp}(\alpha_j) \subset V_j$ implies $\mathrm{supp}(\iota^* \alpha_j) \subset W_j$ and hence

$$\int_{\partial A} \iota^* \alpha_j = \int_{W_j} \iota^* \alpha_j = \int_V (\kappa_j^{-1})^* (\iota^* \alpha_j) = \int_V (\iota \circ \kappa_j^{-1})^* \alpha_j = \int_V (\kappa_j^{-1})^* \alpha_j \ ,$$

which completes the proof. ■

A corollary of the Theorem of Stokes which is used often is

Corollary 4.4.3 *Let M be an n-dimensional oriented compact differentiable manifold, and* $\alpha \in \Omega^{n-1} M$. *Then* $\int_M d\alpha = 0$.

Proof: Taking $A = M$ it holds $\partial A = \emptyset$. Since $\mathrm{supp}(\alpha)$ is closed in M and M is compact, $\mathrm{supp}(\alpha)$ is compact, too, so the Theorem of Stokes applies. The claim follows since obviously the integral over $\emptyset$ is zero. ■

Example 4.4.4 *Consider on $\mathbb{R}^n$ the n-form $\omega_0 = dx_1 \wedge \ldots \wedge dx_n$, called the (standard) volume form. For an open or closed subset $A \subset \mathbb{R}^n$ its volume* $\mathrm{vol}(A)$ *is defined by*

$$\mathrm{vol}(A) = \int_A dx = \int_A dx_1 \ldots dx_n = \int_A \omega_0 \ ;$$

this integral exists by standard calculus, and it is finite if e.g. A is bounded. The volume of A is called length (resp. area) of A if $n = 1$ (resp. $n = 2$).

Of course we know that the area of the open unit disc $D^2 = \{ x \in \mathbb{R}^2 \mid \|x\| < 1 \}$ *equals* π*, i.e. that it holds*

$$\int\limits_{D^2} dx_1 \wedge dx_2 = \pi \ .$$

Observe that $dx_1 \wedge dx_2 = d\alpha$ *, where* $\alpha := x_1 dx_2$ *, so the Theorem of Stokes asserts that*

$$\pi = \int\limits_{S^1} i^*(\alpha)$$

because $\partial D^2 = S^1 \subset \mathbb{R}^2$ *with the orientation induced by the standard orientation in* $\mathbb{R}^2$ *(see Example 4.3.4). This is in fact true, because according to Example 4.2.9.1 we have*

$$\int\limits_{S^1} i^*(\alpha) = \int\limits_0^{2\pi} \cos^2(t) dt = \pi \ ;$$

the last equality being a simple exercise in calculus.

4.5 The Integral Theorem of Greene

Let $\alpha = (\alpha_1, \alpha_2) : [a,b] \longrightarrow \mathbb{R}^2$ be a differentiable curve, and $f : [a,b] \longrightarrow \mathbb{R}$ a differentiable function.

Definition 4.5.1 *1.* α *is called* <u>*regular*</u> *if* $\dot\alpha(t) \neq 0$ *for all* $t \in [a,b]$.

2. A <u>*reparametrization*</u> *of* α *is a curve* $\alpha \circ g : [c,d] \longrightarrow \mathbb{R}^2$ *, where* $g : [c,d] \longrightarrow [a,b]$ *is differentiable and surjective with* $\dot g(s) > 0$ *for all* $s \in [c,d]$.

3. The <u>*integral*</u> *of* f *along* α *is*

$$\int\limits_\alpha f := \int\limits_a^b f(t) \cdot \|\dot\alpha(t)\| dt \ .$$

Lemma 4.5.2 *For a reparametrisation* $\beta = \alpha \circ g$ *of* α *it holds* $\int\limits_\alpha f = \int\limits_\beta f \circ g$.

Proof: From $t = g(s)$ it follows $dt = \dot g ds$. Furthermore, we have

$$\dot\beta = \frac{d}{dt}(\alpha \circ g) = (\dot\alpha \circ g) \cdot \dot g \ ,$$

hence $\|\dot\beta\| = \|\dot\alpha \circ g\| \cdot \dot g$ since $\dot g > 0$. Integration by substitution yields

$$\int\limits_\alpha f = \int\limits_a^b f(t) \cdot \|\dot\alpha(t)\| dt = \int\limits_c^d f(g(s)) \cdot \|\dot\alpha((g(s))\| \cdot \dot g(s) ds = \int\limits_c^d (f \circ g)(s)) \cdot \|\dot\beta(s)\| ds = \int\limits_\beta f \circ g \ .$$

■

Let v be a vector field on $\mathbb{R}^2$, i.e. a differentiable map $v = (v_1, v_2) : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$.

Definition 4.5.3 *1. The* *<u>divergence</u>* *of v is the function*

$$\operatorname{div}(v) := \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} : \mathbb{R}^2 \longrightarrow \mathbb{R} \ .$$

2. The <u>oriented</u> <u>normal</u> <u>component</u> of v along a regular curve $\alpha : [a, b] \longrightarrow \mathbb{R}^2$ is the function

$$n_\alpha^v : [a, b] \longrightarrow \mathbb{R}$$

defined by

$$n_\alpha^v(t) := \frac{< v, (\dot\alpha_2, -\dot\alpha_1) >}{\|\dot\alpha\|}(t) = \frac{v_1(\alpha(t)) \cdot \dot\alpha_2(t) - v_2(\alpha(t)) \cdot \dot\alpha_1(t)}{\|\dot\alpha(t)\|} \ .$$

Let $U \subset \mathbb{R}^2$ be open with $\bar{U}$ compact, and with smooth boundary ∂U. Assume further that $\alpha : [a, b] \longrightarrow \partial U$ is a bijective regular curve in $\mathbb{R}^2$ such that $(\alpha|_{(a,b)})^{-1}$ is an oriented chart for $\partial U \setminus \{\alpha(a)\}$.

Theorem 4.5.4 (Integral Theorem of Greene) *For every differentiable vector field $v = (v_1, v_2)$ on $\mathbb{R}^2$ it holds*

$$\int_U \operatorname{div}(v) dx dy = \int_\alpha n_\alpha^v \ .$$

Proof: Define $\omega := v_1 dy - v_2 dx \in \Omega^1(\mathbb{R}^2)$; then it holds $d\omega = \operatorname{div}(v) \cdot dx \wedge dy$, and hence

$$\int_U d\omega = \int_U \operatorname{div}(v) dx dy \ .$$

Using $\alpha^*(\omega) = ((v_1 \circ \alpha)(t) \cdot \dot\alpha_2(t) - (v_2 \circ \alpha)(t) \cdot \dot\alpha_1(t))\, dt$ and the Theorem of Stokes, we get

$$\int_U d\omega = \int_{\partial U} \omega|_{\partial U} = \int_a^b \alpha^*(\omega) = \int_a^b n_\alpha^v(t) \cdot \|\dot\alpha(t)\| dt = \int_\alpha n_\alpha^v \ .$$

■

4.6 The Fixed Point Theorem of Brouwer

For $n \geq 1$ let $D^n := \{ \ x \in \mathbb{R}^n \mid \|x\| < 1 \ \}$ be the open n-dimensional unit ball; then $D^n \subset \mathbb{R}^n$ is open with smooth boundary $\partial D^n = S^{n-1} = \{ \ x \in \mathbb{R}^n \mid \|x\| < 1 \ \}$ (the $(n-1)$-dimensional unit sphere) and closure $\bar{D}^n = \{ \ x \in \mathbb{R}^n \mid \|x\| \leq 1 \ \}$ (the closed n-dimensional unit ball).

The aim of this section is to prove, using a suitable version of the Theorem of Stokes, the following theorem.

Theorem 4.6.1 (Fixed Point Theorem of Brouwer) *Let $f : \bar{D}^n \longrightarrow \mathbb{R}^n$ be a differentiable map such that $f(\bar{D}^n) \subset \bar{D}^n$. Then f has a fixed point, i.e. there exists a point $p \in \bar{D}^n$ with $f(p) = p$.*

For this we have to make precise what e.g. "differentiable"means for a map $f : \bar{D}^n \longrightarrow \mathbb{R}^n$, so we begin with some general stuff.

Let X be an n-dimensional differentiable manifold and $A \subset X$ open with smooth boundary ∂A; then $\bar{A} = A \cup \partial A$ is the closure of A.

Definition 4.6.2

1. *Let Y be another differentiable manifold. A map $f : \bar{A} \longrightarrow Y$ is called differentable if for every point $p \in \bar{A}$ there exists an open $U \subset X$ with $p \in U$ and a differentiable map $f_U : U \longrightarrow Y$ such that $f_U|_{U \cap \bar{A}} = f|_{U \cap \bar{A}}$.*
2. *A differentiable k-form on $\bar{A}$ is a map $\alpha : \bar{A} \longrightarrow \coprod_{p \in \bar{A}} \Lambda^k T_p^* X$ with $\alpha(p) \in \Lambda^k T_p^* X$ for all $p \in \bar{A}$, such that for every point $p \in \bar{A}$ there exists an open $U \subset X$ with $p \in U$ and an $\alpha_U \in \Omega^k U$ with $\alpha_U|_{U \cap \bar{A}} = \alpha|_{U \cap \bar{A}}$.*

It is obvious that for a differentiable map $f : \bar{A} \longrightarrow Y$ (resp. a differentiable k-form α on $\bar{A}$) the restriction $f|_A : A \longrightarrow Y$ (resp. $\alpha|_A$) is differentiable in the usual sense, i.e. with respect to the manifold structure in A as an open subset of the manifold X.

Excercise 4.6.3

1. *Let $f : \bar{A} \longrightarrow Y$ be a differentiable map and $\alpha \in \Omega^k Y$. Show that $f^*(\alpha)$ is a differentiable k-form on $\bar{A}$.*
2. *Let α be a differentiable k-form on $\bar{A}$. Show that $i^*(\alpha) \in \Omega^k(\partial A)$, where $i : \partial A \hookrightarrow X$ is the inclusion map.*
3. *Show that if $\partial A \neq \emptyset$ there exists an $\alpha \in \Omega^{n-1} \partial A$ with $\int_{\partial A} \alpha \neq 0$.*
4. *Show that the Theorem of Stokes holds for a differentiable $(n-1)$-form on $\bar{A}$ with compact support.*

Lemma 4.6.4 *Assume that $\dim Y = n - 1$, $\alpha \in \Omega^{n-1} Y$, and $f : \bar{A} \longrightarrow Y$ is differentiable. If $f^*(\alpha)$ has compact support, then it holds $\int_{\partial A} (f \circ i)^*(\alpha) = 0$.*

Proof: It holds

$$\begin{aligned} \int_{\partial A} (f \circ i)^*(\alpha) &= \int_{\partial A} i^*(f^*(\alpha)) = \int_A d(f^*(\alpha)) \text{ by Exercise 4.6.3.4} \\ &= \int_A f^*(d\alpha) \text{ by Proposition 3.2.2.3} \\ &= 0 \ ; \end{aligned}$$

the last equality holds because $d\alpha$ vanishes as an n form on the $(n-1)$-dimensional manifold Y and f^* is linear. ■

Proposition 4.6.5 *If* $\bar{A} \neq \emptyset$ *is compact, there is no differentiable map* $f : \bar{A} \longrightarrow X$ *such that*

$$f(\bar{A}) \subset \partial A \quad , \quad f|_{\partial A} = \mathrm{id}_{\partial A} \ . \quad (*)$$

Proof: Assume to the contrary that an f exists satisfying condition $(*)$. Since $\bar{A} \neq \emptyset$ and $f(\bar{A}) \subset \partial A$ it holds $\partial A \neq \emptyset$, so by Exercise 4.6.3.3 there is an $\alpha \in \Omega^{n-1}\partial A$ with $\int_{\partial A} \alpha \neq 0$. Observe that compactness of $\bar{A}$ implies the compactness of $\mathrm{supp}((f \circ i)^*(\alpha))$. Then it follows

$$\begin{aligned} 0 &\neq \int_{\partial A} \alpha = \int_{\partial A} (\mathrm{id}_{\partial A})^*(\alpha) = \int_{\partial A} (f \circ i)^*(\alpha) \ \text{ since the second equation in } (*) \text{ means } f \circ i = \mathrm{id}_{\partial A} \\ &= 0 \ \text{ by Lemma 4.6.4 ;} \end{aligned}$$

a contradiction. ■

Now we are ready to give the

Proof of Theorem 4.6.1: Assume that there exists an f as in the Theorem without fixed point. Let be $p \in \bar{D}^n$; since $p \neq f(p) \in \bar{D}^n$, the line through p and $f(p)$ intersects $S^{n-1} = \partial \bar{D}^n$ in precisely two points. In particular, there is a unique $t_p \leq 0$ such that $q_p := p + t_p \cdot (f(p) - p) \in S^{n-1}$. It is easy to see that the map $g : \bar{D}^n \longrightarrow \mathbb{R}^n$, $g(p) := q_p$, is differentiable with $g(\bar{D}^n) \subset S^{n-1}$ and $g|_{S^{n-1}} = \mathrm{id}_{S^{n-1}}$, thus contradicting Proposition 4.6.5. ■

5 Appendix: (Multi)Linear algebra

5.1 Duality

Let V be a real vector space.

Definition 5.1.1 *The dual vector space is*

$$V^* := \mathrm{Hom}(V,\mathbb{R}) = \{\ v^* : V \longrightarrow \mathbb{R} \mid v^* \text{ lineair } \}\ .$$

Since V^* is again a vector space, we have its dual space $(V^*)^* = \mathrm{Hom}(V^*,\mathbb{R})$, called the bidual of V. Observe that we have a natural linear map $\delta : V \longrightarrow (V^*)^*$ defined by $\delta(v)(v^*) := v^*(v)$ for all $v \in V$, $v^* \in V^*$.

Remark 5.1.2 *It can be shown (using the Axiom of Choice) that δ is always injective. Later we will show that δ is an isomorphism if V is finite dimensional.*

Let U be another vector space over $\mathbb{R}$, and $f : V \longrightarrow U$ a linear map. The dual map $f^* : U^* \longrightarrow V^*$ is defined by

$$f^*(u^*) := u^* \circ f : V \longrightarrow \mathbb{R}$$

for all $u^* \in U^*$, i.e. $f^*(u^*)(v) := u^*(f(v))$ for all $v \in V$ and $u^* \in U^*$. Notice that f^* is linear because

$$(f^*(\lambda u_1^* + \mu u_2^*))(v) = (\lambda u_1^* + \mu u_2^*)(f(v)) = \lambda(u_1^*(f(v))) + \mu(u_2^*(f(v)))$$

coincides with

$$(\lambda f^*(u_1^*) + \mu f^*(u_2^*))(v) = \lambda(f^*(u_1^*)(v)) + \mu(f^*(u_2^*)(v)) = \lambda(u_1^*(f(v))) + \mu(u_2^*(f(v)))$$

for all $\lambda, \mu \in \mathbb{R}$, $u_1^*, u_2^* \in U^*$ and $v \in V$.

Proposition 5.1.3

1. *The map*

$$\mathrm{Hom}(V,U) \longrightarrow \mathrm{Hom}(U^*,V^*) \quad , \quad f \mapsto f^*\ ,$$

   is linear.
   If f is surjective, then f^* is injective.

2. *If W is a third vector space and $g : U \longrightarrow W$ a linear map, then*

$$(g \circ f)^* = f^* \circ g^* : W^* \longrightarrow V^*\ .$$

Proof: 1. For all $f_1, f_2 \in \mathrm{Hom}(V,U)$, $\lambda, \mu \in \mathbb{R}$, $u^* \in U^*$ and $v \in V$ it holds

$$\begin{aligned}\Big((\lambda f_1 + \mu f_2)^*(u^*)\Big)(v) &= u^*\Big((\lambda f_1 + \mu f_2)(v)\Big) = u^*\Big(\lambda f_1(v) + \mu f_2(v)\Big) \\ &= \lambda u^*(f_1(v)) + \mu u^*(f_2(v)) = \lambda f_1^*(u^*)(v) + \mu f_2^*(u^*)(v) \\ &= \Big(\lambda f_1^*(u^*) + \mu f_2^*(u^*)\Big)(v) = \Big((\lambda f_1^* + \mu f_2^*)(u^*)\Big)(v)\ .\end{aligned}$$

This means

$$(\lambda f_1 + \mu f_2)^*(u^*) = (\lambda f_1^* + \mu f_2^*)(u^*)$$

for all $u^* \in U^*$, i.e. $\lambda f_1 + \mu f_2 = \lambda f_1^* + \mu f_2^*$ which proves the first claim. For the second assertion let be $0 \neq u^* \in U^*$; then there is a $u \in U$ such that $u^*(u) \neq 0$. Since f is surjective, there exists $v \in V$ such that $u = f(v)$. Then $f^*(u^*)(v) = u^*(f(v)) = u^*(u) \neq 0$, i.e. $f^*(u^*) \neq 0$.

2. For all $w^* \in W^*$ it holds by definition

$$(g \circ f)^*(w^*) = w^* \circ (g \circ f) = (w^* \circ g) \circ f = g^*(w^*) \circ f = f^*(g^*(w^*)) = (f^* \circ g^*)(w^*) \ .$$

■

Remark 5.1.4 *Later we will show that for finite dimensional spaces it holds that f^* is surjective if f is injective.*

For the remainder of this section we assume that $\dim V = n < \infty$.

Proposition 5.1.5 1. *Let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ be a basis of V. Then there is a unique basis (the dual basis) $\mathcal{B}^* = \{v_1^*, v_2^*, \ldots, v_n^*\}$ of V^* such that*

$$(*) \quad \forall\ 1 \leq i, j \leq n \ : \ v_i^*(v_j) = \delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

In particular, it holds $\dim V^* = \dim V$.

2. *Let $\mathcal{B}'$ be another basis of V and $(\mathcal{B}')^*$ its dual basis of V^*. If $A = (a_{ij})_{i,j=1}^n$ is the (invertible) transition matrix from $\mathcal{B}$ to $\mathcal{B}'$, then the transposed matrix $A^t = (a_{ji})_{i,j=1}^n$ is the transition matrix from $(\mathcal{B}')^*$ to $\mathcal{B}^*$.*

3. *The natural map $\delta : V \longrightarrow (V^*)^*$ is an isomorphism.*

Proof:

1. Since $\mathcal{B}$ is a basis of V, for every $1 \leq i \leq n$ there is a unique element $v_i^* \in V^*$ satisfying $v_i^*(v_j) = \delta_{ij}$ for all $1 \leq j \leq n$; we have to show that these v_i^* form a basis of V^*.

Assume that $\sum_{i=1}^{n} \lambda_i v_i^* = 0$, $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Then for every $1 \leq j \leq n$ it holds

$$0 = \left(\sum_{i=1}^{n} \lambda_i v_i^*\right)(v_j) = \sum_{i=1}^{n} \lambda_i \left(v_i^*(v_j)\right) = \sum_{i=1}^{n} \lambda_i \delta_{ij} = \lambda_j \ ;$$

this shows that the v_i^*'s are linearly independent.

For every $v^* \in V^*$ and all $1 \leq j \leq n$ it holds

$$\left(\sum_{i=1}^{n} v^*(v_i) \cdot v_i^*\right)(v_j) = \sum_{i=1}^{n} v^*(v_i) \cdot (v_i^*(v_j)) = \sum_{i=1}^{n} v^*(v_i) \cdot \delta_{ij} = v^*(v_j) \ ;$$

this implies that every $v^* \in V^*$ is a linear combination $v^* = \sum_{i=1}^{n} v^*(v_i) \cdot v_i^*$, i.e. that the v_i^*'s generate V^*.
2. By definition, for all i we have $v_i' = \sum_{j=1}^{n} a_{ij} v_j$. Let $B = (b_{ij})_{i,j=1}^n$ be the transition matrix from $\mathcal{B}^*$ to $(\mathcal{B}')^*$, so that it holds $(v_i')^* = \sum_{j=1}^{n} b_{ij} v_j^*$; then we have

$$\delta_{ij} = (v_i')^*(v_j') = \sum_{k,l=1}^{n} a_{ik} b_{jl} v_k^*(v_l) = \sum_{k,l=1}^{n} a_{ik} b_{jl} \delta_{kl} = \sum_{k=1}^{n} a_{ik} b_{jk} ,$$

i.e. $B = (A^t)^{-1}$. The claim follows since the transition matrix from $(\mathcal{B}')^*$ to $\mathcal{B}^*$ is B^{-1}.
3. By 1. we have

$$\dim (V^*)^* = \dim V^* = \dim V ;$$

hence it suffices to show that δ is injective.
Using notations as in 1., suppose that $V \in v = \sum_{i=1}^{n} \lambda_i v_i$ and $\delta(v) = 0$; then for all j it holds

$$\begin{aligned} 0 &= \delta(v)(v_j^*) = \left(\delta(\sum_{i=1}^{n} \lambda_i v_i)\right)(v_j^*) = \sum_{i=1}^{n} \lambda_i \left(\delta(v_i)(v_j^*)\right) \quad \text{since } \delta \text{ is linear} \\ &= \sum_{i=1}^{n} \lambda_i v_j^*(v_i) = \sum_{i=1}^{n} \lambda_i \delta_{ij} = \lambda_j , \end{aligned}$$

i.e. $v = 0$. ■

Lemma 5.1.6 *Let $\{v_1, \ldots, v_n\}$ be a basis of V and $\{v_1^*, \ldots, v_n^*\}$ the dual basis of V^*. Then it holds*

$$\begin{aligned} v &= \sum_{i=1}^{n} v_i^*(v) \cdot v_i \quad \text{for all} \quad v \in V , \\ v^* &= \sum_{i=1}^{n} v^*(v_i) \cdot v_i^* \quad \text{for all} \quad v^* \in V^* . \end{aligned}$$

Proof: Write $v = \sum_{j=1}^{n} \lambda_j v_j$ and $v^* = \sum_{j=1}^{n} \mu_j v_j^*$; then for each i it holds

$$v_i^*(v) = \sum_{j=1}^{n} \lambda_j v_i^*(v_j) = \sum_{j=1}^{n} \lambda_j \delta_{ij} = \lambda_i \ , \quad v^*(v_i) = \sum_{j=1}^{n} \mu_j v_j^*(v_i) = \sum_{j=1}^{n} \mu_j \delta_{ji} = \mu_i \ .$$

■

Proposition 5.1.7 *Let U be another vector space with $m := \dim U < \infty$, and $f : V \longrightarrow U$ a linear map.*

1. *Let $\mathcal{B}$ be a basis of V, $\mathcal{B}'$ a basis of U, and $\mathcal{B}^*$, $(\mathcal{B}')^*$ the dual bases of V^*, U^*. If A is the matrix of f with respect to $\mathcal{B}$ and $\mathcal{B}'$, then the transposed matrix A^t is the matrix of the dual map f^* with respect to $(\mathcal{B}')^*$ and $\mathcal{B}^*$.*
2. *If f is injective, then f^* is surjective, and vice versa (compare Proposition 5.1.3 and Remark 5.1.4).*

Proof: 1. Write $\mathcal{B} = \{v_1, \ldots, v_n\}$, $\mathcal{B}' = \{u_1, \ldots, u_m\}$, $\mathcal{B}^* = \{v_1^*, \ldots, v_n^*\}$, $(\mathcal{B}')^* = \{u_1^*, \ldots, u_m^*\}$, and $A = (a_{ij})_{\substack{1\le i\le n\\1\le j\le m}}$; then by definition it holds $f(v_i) = \sum_{j=1}^{m} a_{ij} u_j$ for all i and j. It follows

$$f^*(u_j^*)(v_i) = u_j^*(f(v_i)) = \sum_{k=1}^{m} a_{ik} u_j^*(u_k) = \sum_{k=1}^{m} a_{ik}\delta_{jk} = a_{ij} = \sum_{l=1}^{n} a_{lj}\delta_{li} = \left(\sum_{l=1}^{n} a_{lj} v_l^*\right)(v_i)$$

for all i and j. Since $\mathcal{B}$ is a basis, it follows $f^*(u_j^*) = \sum_{l=1}^{n} a_{lj} v_l^*$ for all j, i.e. the claim.
2. This follows from 1. since A and A^t have the same rank. ■

Remark 5.1.8 *For bases as in the proof of Proposition 5.1.7 1. it holds*

$$f^*(u_j^*) = \sum_{i=1}^{n} u_j^*(f(v_i)) v_i^* ,$$

because for all k we have

$$\left(\sum_{i=1}^{n} u_j^*(f(v_i)) v_i^*\right)(v_k) = \sum_{i=1}^{n} u_j^*(f(v_i)) v_i^*(v_k) = u_j^*(f(v_k)) = f^*(u_j^*)(v_k) .$$

5.2 Exterior powers

Notation: For $n \in \mathbb{N}$ we denote by $\mathcal{S}_n$ the set of permutations of $\{1, 2, \ldots, n\}$.

Let V be an n-dimensional vector space over $\mathbb{R}$, and $k \in \mathbb{N}$. The kth exterior power of V is the vector space $\Lambda^k V$ defined as follows:

The elements of $\Lambda^k V$ are finite sums of the form

$$\sum_{i_1, i_2, \ldots, i_k} a_{i_1 i_2 \ldots i_k} \cdot v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_k} \quad , \quad a_{i_1 i_2 \ldots i_k} \in \mathbb{R} \quad , \quad v_{i_1}, v_{i_2}, \ldots, v_{i_k} \in V .$$

Addition of two of these is obvious, and multiplication by a scalar $a \in \mathbb{R}$ is multiplication of the $a_{i_1 i_2 \ldots i_k}$ by a. Furthermore, elements of $\Lambda^k V$ obey the following rules:

$$v_1 \wedge \ldots \wedge (a \cdot v_i + a' \cdot v_i') \wedge \ldots \wedge v_k = a \cdot v_1 \wedge \ldots \wedge v_i \wedge \ldots \wedge v_k + a' \cdot v_1 \wedge \ldots \wedge v_i' \wedge \ldots \wedge v_k ,$$

for all $v_1, \ldots, v_i, v_i', \ldots, v_k \in V$, $a, a' \in \mathbb{R}$, $1 \leq i \leq k$ (the $\wedge$-product is multilinear), and

$$v_1 \wedge \ldots \wedge v_i \wedge \ldots \wedge v_j \wedge \ldots \wedge v_k = -v_1 \wedge \ldots \wedge v_j \wedge \ldots \wedge v_i \wedge \ldots \wedge v_k$$

for all $v_1, \ldots, v_k \in V$, $1 \leq i < j \leq k$ (the $\wedge$-product is alternating).

It is easy to see that the second rule is equivalent to

$$v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(k)} = \operatorname{sign}(\sigma) \cdot v_1 \wedge \ldots \wedge v_k$$

for all $\sigma \in \mathcal{S}_k$, and (assuming the first rule) also to

$$v_1 \wedge \ldots \wedge v_k = 0 \ \text{ if } \ v_i = v_j \ \text{ for some } \ i < j \ .$$

Other important properties of $\Lambda^k V$ are the following:
- If $\{b_1, \ldots, b_n\}$ is a basis of V and $k \leq n$, then $\{ b_{i_1} \wedge \ldots \wedge b_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq n \}$ is a basis of $\Lambda^k V$.
- $\Lambda^k V = \{0\}$ for $k > n$.

This has the following consequences:
- $\Lambda^1 V$ is naturally identified with V.
- For $1 \leq k \leq n$ it holds $\dim \Lambda^k V = \binom{n}{k}$; in particular, $\Lambda^n V$ is a 1-dimensional vector space with basis $b_1 \wedge \ldots \wedge b_n$.
- For $1 \leq k \leq n$, every vector in $\Lambda^k V$ can be written uniquely in the form

$$\sum_{1 \leq i_1 < \ldots < i_k \leq n} a_{i_1 \ldots i_k} \cdot b_{i_1} \wedge \ldots \wedge b_{i_k} \quad , \quad a_{i_1 \ldots i_k} \in \mathbb{R} \ \text{ for all } \ 1 \leq i_1 < \ldots < i_k \leq n \ .$$

We furthermore define $\Lambda^0 V := \mathbb{R}$. Then for all $k, l \in \mathbb{N}_0$ we define the wedge product

$$\wedge : \Lambda^k V \times \Lambda^l V \longrightarrow \Lambda^{k+l} V \quad , \quad (\alpha, \beta) \mapsto \alpha \wedge \beta \ ,$$

to be the bilinear extension of

$$(v_1 \wedge \ldots \wedge v_k \ , \ w_1 \wedge \ldots \wedge w_l) \mapsto v_1 \wedge \ldots \wedge v_k \wedge w_1 \wedge \ldots \wedge w_l \ ,$$

where in the case $k = 0$ resp. $l = 0$ we set

$$a \wedge w_1 \wedge \ldots \wedge w_l := a \cdot w_1 \wedge \ldots \wedge w_l \ \text{ resp. } \ v_1 \wedge \ldots \wedge v_k \wedge a := a \cdot v_1 \wedge \ldots \wedge v_k \ .$$

Then from the second rule above one deduces

$$\alpha \wedge \beta = (-1)^{k \cdot l} \beta \wedge \alpha$$

for $\alpha \in \Lambda^k V$, $\beta \in \Lambda^l V$.

Observe that we have a natural map

$$\nu^k : \ \underbrace{V \times V \times \ldots \times V}_{k \text{ times}} \ \longrightarrow \Lambda^k V \quad , \quad \nu^k(v_1, v_2, \ldots, v_k) := v_1 \wedge v_2 \wedge \ldots \wedge v_k \ .$$

This map has the following properties [3]:

i) the image of ν^k generates $\Lambda^k V$;

ii) ν^k is <u>k-linear</u>, i.e. linear in each of its arguments, by the first rule;

iii) ν^k is <u>alternating</u>, i.e. satisfying $\nu^k(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_k) = -\nu^k(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_k)$ for $i < j$, by the second rule;

iv) for every vector space W and every k-linear alternating map $\mu : V \times V \times \ldots \times V \longrightarrow W$ there exists a unique linear map $m : \Lambda^k V \longrightarrow W$ with $\mu = m \circ \nu^k$ (this is called the **Universal Property** of the exterior power).

Remark: Be warned that the map ν^k is in general <u>not</u> surjective, i.e. in general a vector in $\Lambda^k V$ is <u>not</u> of the form $v_1 \wedge v_2 \wedge \ldots \wedge v_k$. For example, ν^2 is surjective if and only if $\dim V \leq 3$.

Example 5.2.1 *We write vectors in $\mathbb{R}^n$ as column vectors; then the map*

$$\delta : \underbrace{\mathbb{R}^n \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{n \text{ times}} \longrightarrow \mathbb{R} \quad , \quad \delta(v_1, v_2, \ldots, v_k) := \det(v_1, v_2, \ldots, v_n) \ ,$$

is n-linear and alternating. Hence there exists a unique linear map $d : \Lambda^n \mathbb{R}^n \longrightarrow \mathbb{R}$ satisfying

$$d(v_1 \wedge v_2 \wedge \ldots \wedge v_n) = \det(v_1, v_2, \ldots, v_n)$$

for all $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$. Since

$$\dim \Lambda^n \mathbb{R}^n = \binom{n}{n} = 1 = \dim \mathbb{R}$$

and

$$d(e_1 \wedge e_2 \wedge \ldots \wedge e_n) = \det(e_1, e_2, \ldots, e_n) = 1 \neq 0$$

for the unit basis $\{e_1, e_2, \ldots, e_n\}$, it follows that d is an isomorphism.

Let W be another finite dimensional vector space, and $f : V \longrightarrow W$ a linear map. It is easy to see that, for each k, the map

$$\underbrace{V \times V \times \ldots \times V}_{k \text{ times}} \longrightarrow \Lambda^k W \quad , \quad (v_1, v_2, \ldots, v_k) \mapsto f(v_1) \wedge f(v_2) \wedge \ldots \wedge f(v_k)$$

is k-linear and alternating, hence there exists a unique linear map

$$\Lambda^k f : \Lambda^k V \longrightarrow \Lambda^k W$$

with

$$\Lambda^k f(v_1 \wedge v_2 \wedge \ldots \wedge v_k) = f(v_1) \wedge f(v_2) \wedge \ldots \wedge f(v_k)$$

for all $v_1, v_2, \ldots, v_k \in V$.

[3] In fact, Λ^k is characterized by these four properties in the following sense:
Let L be a vector space with a k-linear alternating map $\lambda : V \times V \times \ldots \times V \longrightarrow L$ such that the image of λ generates L, and such that for every vector space W and every k-linear alternating map $\mu : V \times V \times \ldots \times V \longrightarrow W$ there exists a unique linear map $m : L \longrightarrow W$ with $\mu = m \circ \lambda$. Then there exist a unique isomorphism $I : L \longrightarrow \Lambda^k V$ with $I \circ \lambda = \nu^k$.

Example 5.2.2 *1. Let $f : V \longrightarrow V$ be a linear map. Since $\dim \Lambda^n V = 1$, the induced linear map $\Lambda^n f : \Lambda^n V \longrightarrow \Lambda^n V$ is the multiplication by a real number; we claim that this number equals the determinant of f. To see this, let $\{b_1, \ldots, b_n\}$ be a basis of V and $A = (a_{ij})_{i,j=1,\ldots,n}$ the matrix of f with respect to this basis, i.e. $f(b_i) = \sum_{j=1}^{n} a_{ji} b_j$ for all $1 \leq i \leq n$; then $\det(f) = \det(A)$. Since $b_1 \wedge \ldots \wedge b_n$ is a basis of $\Lambda^n V$, it suffices to show that*

$$\Lambda^n f(b_1 \wedge \ldots \wedge b_n) = \det(A) \cdot b_1 \wedge \ldots \wedge b_n \ .$$

Since the $\wedge$-product is multilinear we have

$$\Lambda^n f(b_1 \wedge \ldots \wedge b_n) = \sum_{j_1, \ldots, j_n = 1, \ldots, n} a_{j_1 1} \cdot a_{j_2 2} \cdot \ldots \cdot a_{j_n n} \cdot b_{j_1} \wedge b_{j_2} \wedge \ldots \wedge b_{j_n} \ .$$

Since it is alternating, it holds

$$a_{j_1 1} \cdot a_{j_2 2} \cdot \ldots \cdot a_{j_n n} \cdot b_{j_1} \wedge b_{j_2} \wedge \ldots \wedge b_{j_n} = 0$$

if $\{j_1, \ldots, j_n\} \neq \{1, \ldots, n\}$, because then there are $k < l$ with $j_k = j_l$. The remaining terms are precisely those for which there exists a permutation $\sigma \in \mathcal{S}_n$ with $j_i = \sigma(i)$, $1 \leq i \leq n$, hence of the form

$$\begin{aligned} a_{j_1 1} \cdot \ldots \cdot a_{j_n n} \cdot b_{j_1} \wedge \ldots \wedge b_{j_n} &= a_{\sigma(1)1} \cdot \ldots \cdot a_{\sigma(n)n} \cdot b_{\sigma(1)} \wedge \ldots \wedge b_{\sigma(n)} \\ &= \operatorname{sign}(\sigma) \cdot a_{\sigma(1)1} \cdot \ldots \cdot a_{\sigma(n)n} \cdot b_1 \wedge \ldots \wedge \end{aligned}$$

$$b_n$$

where the last equality holds because $\wedge$ is alternating. Hence we get

$$\begin{aligned} \Lambda^n f(b_1 \wedge \ldots \wedge b_n) &= \left(\sum_{\sigma \in \mathcal{S}_n} \operatorname{sign}(\sigma) \cdot a_{\sigma(1)1} \cdot a_{\sigma(2)2} \cdot \ldots \cdot a_{\sigma(n)n} \right) \cdot b_1 \wedge b_2 \wedge \ldots \wedge b_n \\ &= \det(A) \cdot b_1 \wedge \ldots \wedge b_n \ . \end{aligned}$$

2. Let $\{b_1, \ldots, b_n\}$ and $\{c_1, \ldots, c_n\}$ be two bases of V, then $b_1 \wedge \ldots \wedge b_n$ and $c_1 \wedge \ldots \wedge c_n$ are both a basis of the 1-dimensional vector space $\Lambda^n V$, so they differ by a non-zero factor. To determine this factor, let $A = (a_{ij})_{i,j=1,\ldots,n}$ be the invertible matrix such that $c_i = \sum_{j=1}^{n} a_{ji} \cdot b_j$, $1 \leq i \leq n$. Then A is the matrix with respect to $\{b_1, \ldots, b_n\}$ of the linear map $f : V \longrightarrow V$ determined by $f(b_i) = c_i$, $1 \leq i \leq n$. Hence from i) it follows

$$\det(A) \cdot b_1 \wedge \ldots \wedge b_n = \Lambda^n f(b_1 \wedge \ldots \wedge b_n) = f(b_1) \wedge \ldots \wedge f(b_n) = c_1 \wedge \ldots \wedge c_n \ .$$

Excercise 5.2.3 *Let V be an n-dimensional real vector space and $1 \leq k \leq n$.*

1. *Show that there is a natural linear map*

$$\Delta^k : \Lambda^k(V^*) \longrightarrow \left(\Lambda^k V\right)^*$$

satisfying

$$\left(\Delta^k(v_1^* \wedge \ldots \wedge v_k^*)\right)(v_1 \wedge \ldots \wedge v_k) = \det\left((v_i^*(v_j))_{i,j=1,\ldots,k}\right)$$

for all $v_1^*, \ldots, v_k^* \in V^*$, $v_1, \ldots, v_k \in V$.

2. *Let* $\{b_1, \ldots, b_n\}$ *be a basis of* V *and* $\{b_1^*, \ldots, b_n^*\}$ *the dual basis of* V^*.
Let

$$\{\ (b_{i_1} \wedge \ldots \wedge b_{i_k})^* \mid 1 \le i_1 < \ldots < i_k \le k\ \}$$

be the basis of $(\Lambda^k V)^*$ *dual to the basis*

$$\{\ b_{i_1} \wedge \ldots \wedge b_{i_k} \mid 1 \le i_1 < \ldots < i_k \le k\ \}$$

of $\Lambda^k V$. *Show that*

$$\Delta^k(b_{i_1}^* \wedge \ldots \wedge b_{i_k}^*) = (b_{i_1} \wedge \ldots \wedge b_{i_k})^* ,$$

and conclude that Δ^k *is an isomorphism.*

5.3 Orientation of vector spaces

Let V be an n-dimensional vector space, $n > 0$, and $\mathbb{B} = \mathbb{B}(V)$ the set of bases of V. We define an equivalence relation $\sim$ in $\mathbb{B}$ as follows:

For $B, B' \in \mathbb{B}$, $B = \{b_1, b_2, \ldots, b_n\}$, $B' = \{b_1', b_2', \ldots, b_n'\}$, let $f_{B'B} : V \longrightarrow V$ be the linear isomorphism defined by $f_{B'B}(b_i) = b_i'$, $i = 1, 2, \ldots, n$. Then

$$B \sim B' \quad :\Longleftrightarrow \quad \det(f_{B'B}) > 0 .$$

Since $f_{B'B} = f_{BB'}^{-1}$ and $f_{B''B} = f_{B''B'} \circ f_{B'B}$, it is easily seen that $\sim$ is indeed an equivalence relation.

Definition 5.3.1 1. *An* <u>*orientation*</u> *in* V *is an equivalence class* $\mathfrak{o} \in {}^{\mathbb{B}}/_{\sim}$; *the pair* $(V, \mathfrak{o})$ *is then called an* <u>*oriented*</u> *vector space.*

2. *If* $(V, \mathfrak{o})$ *is an oriented vector space and* $B \in \mathbb{B}$, *the* B *is called a* (<u>*positively*</u>) <u>*oriented*</u> *basis if* $B \in \mathfrak{o}$.

3. *The* <u>*standard*</u> *orientation in* $\mathbb{R}^n$ *is the equivalence class of the unit basis.*

4. *We define the orientations of the zero vector space* $V = \{0\}$ *to be the numbers* ± 1.

Lemma 5.3.2 *Let* V *be an* n*-dimensional vector space,* $n > 0$.

1. V *has precisely two orientations.*

2. The map

$$\mathbb{B}(V) \longrightarrow \mathbb{B}(\Lambda^n V) \quad , \quad \{b_1, b_2, \ldots, b_n\} \mapsto \{b_1 \wedge b_2 \wedge \ldots \wedge b_n\} \ ,$$

induces a bijection between orientations in V and orientations in $\Lambda^n V$.

3. The map $\mathbb{B}(V) \longrightarrow \mathbb{B}(V^*)$, mapping a basis $\{b_1, \ldots, b_n\}$ of V to the dual basis $\{b_1^*, \ldots, b_n^*\}$ of V^*, induces a bijection between orientations in V and orientations in V^*.

Proof: 1. Let be $B = \{b_1, b_2, \ldots, b_n\} \in \mathbb{B}$; then $\{b_1, b_2, \ldots, b_n\} \not\sim \{-b_1, b_2, \ldots, b_n\}$, hence there are at least two orientations. On the other hand, if $B \not\sim B' \not\sim B''$, then $B \sim B''$, hence there are at most two.

2. This follows from the fact that for $B = \{b_1, b_2, \ldots, b_n\}$, $B' = \{b_1', b_2', \ldots, b_n'\}$ it holds

$$b_1 \wedge b_2 \wedge \ldots \wedge b_n = \det(f_{BB'}) \cdot b_1' \wedge b_2' \wedge \ldots \wedge b_n'$$

(see Example 5.2.2.2).

3. This follows from $\det(f_{BB'}) = \det(f_{(B')^* B^*})$, which is an easy consequence of Proposition 5.1.5.2.
■

Corollary 5.3.3 *The map*

$$\mathbb{B}(V) \longrightarrow \mathbb{B}(\Lambda^n V^*) \quad , \quad \{b_1, b_2, \ldots, b_n\} \mapsto \{b_1^* \wedge b_2^* \wedge \ldots \wedge b_n^*\}$$

induces a bijection between orientations in V and orientations in $\Lambda^n V^*$.
In particular, every $0 \neq \omega \in \Lambda^n V^*$ defines an orientation $\mathfrak{o}_\omega$ in V by

$$\{b_1, b_2, \ldots, b_n\} \in \mathfrak{o}_\omega \quad :\Longleftrightarrow \quad b_1^* \wedge b_2^* \wedge \ldots \wedge b_n^* = a \cdot \omega \quad \text{with} \quad a > 0 \ .$$

5.4 Tensor products

Let R be a commutative ring with unit, and M, N two R-modules.

Definition 5.4.1 *A tensor product of M and N (over R) is a pair $(M \otimes_R N, T)$ with the following properties.*

1. *$M \otimes_R N$ is a R-module, and $T : M \times N \longrightarrow M \otimes_R N$ is a bilinear map.*
2. *(Universal property) For every R-module L and bilinear map $F : M \times N \longrightarrow L$ there exists a unique linear map $f : M \otimes_R N \longrightarrow L$ such that $F = f \circ T$.*

Proposition 5.4.2 *A tensor product $(M \otimes_R N, T)$ of M and N exists.*
It is unique up to isomorphism of pairs, i.e. if $(M \otimes'_R N, T')$ is another tensor product, then there is a unique isomorphism $I : M \otimes_R N \longrightarrow M \otimes'_R N$ satisfying $T' = I \circ T$.

Because of the uniqueness statement in this proposition we may speak of the tensor product of M and N and write it just as $M \otimes N$. For $(m,n) \in M \times N$ we define $m \otimes n := T(m,n)$. $M \otimes N$ has the following properties:

1. $M \otimes N$ is generated by the image of T, i.e. every element of $M \otimes N$ is a finite linear combination of the form $\sum \lambda_i m_i \otimes n_i$ with $\lambda_i \in R,\ m_i \in M,\ n_i \in N$ for all i.
2. For all $\lambda, \mu \in R$, $m, m' \in M$, $n, n' \in N$ it holds
$$\lambda(m \otimes n) = (\lambda m) \otimes n = m \otimes (\lambda n)$$
and
$$(\lambda m + \mu m') \otimes n = \lambda m \otimes n + \mu m' \otimes n \ , \ m \otimes (\lambda n + \mu n') = \lambda m \otimes n + \mu m \otimes n' \ .$$
3. If M resp. N is freely generated by $\{m_1, m_2, \ldots, m_r\}$ resp. $\{n_1, n_2, \ldots, n_s\}$, then $M \otimes N$ is freely generated by $\{ \ m_i \otimes n_j \mid 1 \le i \le r \ , \ 1 \le j \le s \ \}$.
4. Let M', N' be two other R-modules and $f : M \longrightarrow M'$, $g : N \longrightarrow N'$ linear maps. Then there is a unique linear map $f \otimes g : M \otimes N \longrightarrow M' \otimes N'$ such that $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$ for all $m \in M$, $n \in N$.

Observe that the last property follows from the fact that the map
$$M \times N \longrightarrow M' \otimes N' \ , \ (m,n) \mapsto f(m) \otimes g(n)$$
is bilinear, and the universal property.

Let A resp. A' be a $(r \times s)$- resp. $(r' \times s')$-matrix with coefficients in R. Let $f : R^r \longrightarrow R^s$ resp. $f' : R^{r'} \longrightarrow R^{s'}$ be the linear maps such that A resp. A' is the matrix of f resp. f' with respect to the standard set of generators $\{m_1, \ldots, m_r\}$ of R^r, $\{n_1, \ldots, n_s\}$ of R^s resp. of f' with respect to the standard set of generators $\{m'_1, \ldots, m'_{r'}\}$ of $R^{r'}$, $\{n'_1, \ldots, n^{s'}\}$ of $R^{s'}$. Then we define the $(rs \times r's')$-matrix $A \otimes A'$ to be the matrix of $f \otimes f'$ with respect to the bases $\{ \ m_i \otimes m'_j \mid 1 \le i \le r, 1 \le j \le s \ \}$ of $R^r \otimes R^s$ and $\{ \ n_k \otimes n'_l \mid 1 \le k \le r\prime, 1 \le l \le s\prime \ \}$ of $R^{r\prime} \otimes R^{s\prime}$.

Let V resp. W be an n- resp. m-dimensional $\mathbb{R}$-vector spaces.

Lemma 5.4.3 *There is a natural isomorphism* $h : V^* \otimes_{\mathbb{R}} W \longrightarrow \mathrm{Hom}(V, W)$ *such that*
$$h(v^* \otimes w)(v) = v^*(v) \cdot w \quad (*)$$
for all $v \in V \ v^* \in V^* \ w \in W$.

Proof: The map $H : V^* \times W \longrightarrow \mathrm{Hom}(V, W)$ defined by $H(v^*, w)(v) := v^*(v) \cdot w$ is bilinear, hence by the universal property of the tensor product there exists a unique linear map h satisfying $(*)$; it remains to show that is is bijective. Let $\mathcal{B}_{\mathcal{V}} = (v_1, \ldots, v_n)$ resp. $\mathcal{B}_{\mathcal{W}} = (w_1, \ldots, w_m)$ be a basis of V

resp. W, and $\mathcal{B}_{\mathcal{V}}^* = (v_1^*, \ldots, v_n^*)$ the basis dual to $\mathcal{B}_V$. Let be $f \in \mathrm{Hom}(V, W)$ and (a_{ij}) the matrix of f with respect to $\mathcal{B}_{\mathcal{V}}$ and $\mathcal{B}_W$. Then it holds

$$h\left(\sum_{i,j} a_{ij} v_i^* \otimes w_j\right)(v_k) = \sum_j a_{kj} w_j = f(v_k) \ ;$$

this shows that h is surjective. Bijectivity follows because $V^* \times W$ and $\mathrm{Hom}(V, W)$ both have dimension nm. ∎

6 Vector bundles and connections

6.1 Vector bundles

Let X be an n-dimensional differentiable manifold.

Definition 6.1.1 *A* <u>*vector*</u> <u>*bundle*</u> <u>*of*</u> <u>*rank*</u> *r* <u>*over*</u> *X is a triple $(E, \pi, \mathcal{A}^E)$ with the following properties.*

1. *E is a differentiable manifold of dimension $n + r$, and $\pi : E \longrightarrow X$ is a differentiable map.*
2. *$\mathcal{A}^E$ is a* <u>*bundle*</u> <u>*atlas*</u> *for E, i.e. $\mathcal{A}^E = \{ \ (U_i^E, h_i^E, U_i \times \mathbb{R}^r) \mid i \in I \ \}$, where*
   - *(a) for all $i \in I$, $U_i \subset X$ is open, $U_i^E = \pi^{-1}(U_i)$ and $h_i^E : U_i^E \longrightarrow U_i \times \mathbb{R}^r$ is a diffeomorphism;*
   - *(b) it holds $X = \bigcup_{i \in I} U_i$, $E = \bigcup_{i \in I} U_i^E$;*
   - *(c) For all $i \in I$ let $p_i : U_i \times \mathbb{R}^r \longrightarrow U_i$ the projection onto the first factor; then it holds*

     $$\pi_{U_i^E} = p_i \circ h_i^E \ .$$

     In particular, for $p \in U_i$ and $E_p := \pi^{-1}(p)$, the <u>*fibre*</u> *of E at p, it holds that*

     $$h_{i,p} := h_i^E|_{E_p} : E_p \longrightarrow \{p\} \times \mathbb{R}^r = \mathbb{R}^r$$

     is a bijection.
   - *(d) For all $i, j \in I$ and $p \in U_i \cap U_j$ the composition*

     $$g_{ij}(p) := h_{i,p} \circ h_{j,p}^{-1} : \mathbb{R}^r \longrightarrow \mathbb{R}^r$$

     is linear, and hence (by (c)) an isomorphism.

A simple example of such a vector bundle (called the trivial bundle) is the product $E := X \times \mathbb{R}^r$ with the natural projection $\pi : X \times \mathbb{R}^r \longrightarrow X$ onto the first factor: take an atlas $\mathcal{A} = \{ (U_i, h_i, V_i) \mid i \in I \}$ for X, then it is easy to see that

$$\{ (U_i \times \mathbb{R}^r, h_i \times \mathrm{id}_{\mathbb{R}^r}, V_i \times \mathbb{R}^r) \mid i \in I \}$$

is the atlas for an $(n+r)$-dimensional differentiable structure in E such that π is differentiable. With $\mathcal{A}^E := \{X \times \mathbb{R}^r, \mathrm{id}_{X\times\mathbb{R}^r}, X \times \mathbb{R}^r\}$ the triple $(E, \pi, \mathcal{A}^E)$ satisfies all conditions.

Remark 6.1.2 *Let $(E, \pi, \mathcal{A}^E)$ be a differentiable vector bundle of rank r over X, with data as in the definition.*

1. *The map π is surjective.*
2. *For all $i, j \in I$ the gluing map $g_{ij} : U_i \cap U_j \longrightarrow Gl(r, \mathbb{R}) = \mathbb{R}^{r^2}$ is differentiable. For all $i, j, k \in I$ and $p \in U_i \cap U_j \cap U_k$ it holds*

$$g_{ik}(p) = g_{ij}(p) \circ g_{jk}(p) \ ; \qquad (*)$$

   this is called the cocycle condition. In particular, for all $i, j \in I$ and $p \in U_i \cap U_j$ it holds $g_{ij}(p) = g_{ji}(p)^{-1}$, and for all $i \in I$ and $p \in U_i$ it holds $g_{ii}(p) = \mathrm{id}_{\mathbb{R}^r}$.
3. *For $p \in X$ choose $i \in I$ with $p \in U_i$. The bijection $h_{i,p}$ induces the structure of an r-dimensional vector space in E_p, but by 5.4.1.2(d) this structure is independent of the choice of $i \in I$. This means that the given data define a natural structure of r-dimensional vector space in each E_p. In particular, there is a natural zero-element 0_p in each E_p.*
   In the case of the trivial bundle $X \times \mathbb{R}^r$ it obviously holds $0_p = (p, 0)$.

Definition 6.1.3 *Let $(E, \pi, \mathcal{A}^E)$ be a vector bundle over X of rank r. A section of E is a differentiable map $s : X \longrightarrow E$ such that $\pi \circ s = \mathrm{id}_X$, so in particular it holds $s(p) \in E_p$ for all $p \in X$.*
A zero of a section s is a point p such that $s(p) = 0_p$.

Excercise 6.1.4

1. *Show that the map*

$$0_E : X \longrightarrow E \quad , \quad p \mapsto 0_p \ ,$$

   is differentiable, and therefore rightfully called the zero section of E.
2. *Show that every section $s : X \longrightarrow E$ is an embedding.*
3. *Consider the trivial bundle $E := X \times \mathbb{R}^r$ with $r \geq 1$, and let $\{b_1, b_2, \ldots, b_r\} \in \mathbb{R}^r$ be a basis. Show that the maps*

$$s_i : X \longrightarrow E \quad , \quad p \mapsto (p, b_i) \ , \ i = 1, \ldots, n \ ,$$

   are differentiable, and hence sections of E without zeroes. Observe that for all $p \in X$ the set $\{s_1(p), \ldots, s_r(p)\}$ is a basis of E_p.

Consider the <u>tangent</u> <u>bundle</u> of X. i.e. the disjoint union

$$TX := \coprod_{p\in X} T_pX \ ;$$

We want to explain that TX has the structure of a vector bundle of rank n over X; for this we first observe that we have a natural projection

$$\pi : TX \longrightarrow X \quad , \quad \pi(v) = p \ :\Leftrightarrow \ v \in T_pX \ .$$

Let $\mathcal{A} = \{ \ (U_i, h_i, V_i) \mid i \in I \ \}$ be a differentiable atlas for X with coordinates $(x_1^i, \ldots, x_n^i)$ in V_i. For each $i \in I$ define

$$U_i^{TX} := \pi^{-1}(U_i) = \coprod_{p\in U_i} T_pX \ ,$$

and

$$\Psi_i : U_i \times \mathbb{R}^n \longrightarrow U_i^{TX} \quad , \quad \Psi_i(p, \lambda) := \sum_{k=1}^n \lambda_k \cdot \frac{\partial}{\partial x_k^i}(p) \ ;$$

then Ψ_i is bijective by the theory of manifolds. Observe that

$$\pi \circ \Psi_i = pr_1 \quad , \quad pr_1 : U_i \times \mathbb{R}^n \longrightarrow U_i \ \text{ the projection} \ . \qquad (**)$$

Observe further that $X = \bigcup_{i\in I} U_i$ implies that $TX = \bigcup_{i\in I} U_i^{TX}$.
Recall that for $p \in U_i \cap U_j$ and $\psi = (\psi_1, \ldots, \psi_n) := h_j \circ h_i^{-1}$ it holds

$$\frac{\partial}{\partial x_k^i}(p) = \sum_{l=1}^n \frac{\partial \psi_l}{\partial x_k^i}(h_i(p)) \cdot \frac{\partial}{\partial x_l^j}(p)$$

and hence

$$\Psi_i(p, \lambda) = \sum_{k=1}^n \lambda_k \cdot \frac{\partial}{\partial x_k^i}(p) = \sum_{k,l=1}^n \lambda_k \cdot \frac{\partial \psi_l}{\partial x_k^i}(h_i(p)) \cdot \frac{\partial}{\partial x_l^j}(p) = \sum_{l=1}^n \left(\sum_{k=1}^n \frac{\partial \psi_l}{\partial x_k^i}(h_i(p)) \cdot \lambda_k \right) \cdot \frac{\partial}{\partial x_l^j}(p) \ .$$

This implies

$$\Psi_i(p, \lambda) = \Psi_j(p, D\psi(h_i(p)) \cdot \lambda) \ ,$$

and thus

$$\Psi_j^{-1} \circ \Psi_i(p, \lambda) = (p, D\psi(h_i(p)) \cdot \lambda) \ . \qquad (***)$$

Using the fact that $D\psi(h_i(p) : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a linear isomorphism and hence a homeomorphism it's not hard to prove the following topological fact.

Excercise 6.1.5 *There is a unique topology in TX such that for all $i \in I$ the subset U_i^{TX} is open in TX, and $\Psi_i : U_i \times \mathbb{R}^n \longrightarrow U_i^{TX}$ is a homeomorphism, where $U_i \times \mathbb{R}^n$ is equipped with the product topology. This topology in TX is Hausdorff, second countable, and independent of the choice of the atlas $\mathcal{A}$ we started with.*

For $i \in I$ define

$$k_i^{TX} := (h_i \times \mathrm{id}_{\mathbb{R}^n}) \circ \Psi_i^{-1} : U_i^{TX} \longrightarrow U_i \times \mathbb{R}^n \longrightarrow V_i \times \mathbb{R}^n \quad ;$$

it follows from the exercise above that this is a homeomorphism. We have earlier seen that the U_i^{TX} form an open cover of TX, so $\{ (U_i^{TX}, k_i^{TX}, V_i \times \mathbb{R}^n) \mid i \in I \}$ is an $2n$-dimensional topological atlas for TX. For all $i, j \in I$ it holds on $k_i^{TX}(U_i^{TX} \cap U_j^{TX})$

$$k_j^{TX} \circ \left(k_i^{TX}\right)^{-1} = \left((h_j \times \mathrm{id}_{\mathbb{R}^n}) \circ \Psi_j^{-1}\right) \circ \left(\Psi_i \circ (h_i^{-1} \times \mathrm{id}_{\mathbb{R}^n})\right) \ ,$$

and using $(**)$ we get for all $(x, \lambda) \in h_i(U_i \cap U_j) \times \mathbb{R}^n$ (with $\psi = h_j \circ h_i^{-1}$ as above, and $p = h_i^{-1}(x)$)

$$\begin{aligned}\left(k_j^{TX} \circ \left(k_i^{TX}\right)^{-1}\right)(x, \lambda) &= \left((h_j \times \mathrm{id}_{\mathbb{R}^n}) \circ \Psi_j^{-1}\right)(\Psi_i(p, \lambda)) = (h_j \times \mathrm{id}_{\mathbb{R}^n}) \left(\Psi_j^{-1} \circ \Psi_i(p, \lambda)\right) \\ &= (h_j \times \mathrm{id}_{\mathbb{R}^n})\left((p, D\psi(h_i(p)) \cdot \lambda)\right) = (\psi(x), D\psi(h_i(p)) \cdot \lambda) \ .\end{aligned}$$

Since ψ and $D\psi$ are differentiable, it follows that $\{ (U_i^{TX}, k_i^{TX}, V_i \times \mathbb{R}^n) \mid i \in I \}$ is indeed a differentiable atlas.

We have

$$\begin{aligned}\left(h_i \circ \pi \circ \left(k_i^{TX}\right)^{-1}\right)(x, \lambda) &= (h_i \circ \pi \circ \Psi_i)\,(h_i^{-1}(x), \lambda) \\ &= h_i(h_i^{-1}(x)) \quad \text{by} \quad (**) \\ &= x \ ;\end{aligned}$$

this shows that π is differentiable.

Excercise 6.1.6 *Show that* $\{ (U_i^{TX}, \Psi_i^{-1}, U_i \times \mathbb{R}^n) \mid i \in I \}$ *is a bundle atlas for* TX *in the sense of Definition 5.4.1, and that the associated gluing maps (in the sense of Remark 5.4.2) are given by*

$$g_{ij}(p) = D(h_i \circ h_j^{-1})(h_j(p)) \ .$$

Now assume that $X = \bigcup_{i \in I} U_i$ is an open cover, and that we are given a set

$$\{ g_{ij} : U_i \cap U_j \longrightarrow GL(r, \mathbb{R}) \mid i, j \in I \}$$

of differentiable maps satisfying the cocycle condition $(*)$. Define

$$E := \coprod_{i \in I} U_i \times \mathbb{R}^r \Big/ \sim \ ,$$

where the equivalence relation $\sim$ is defined as follows: for $(p, \lambda) \in U_i \times \mathbb{R}^r$ and $(q, \mu) \in U_j \times \mathbb{R}^r$ it holds

$$(p, \lambda) \sim (q, \mu) \quad \Leftrightarrow \quad p = q \ \text{ and } \ \lambda = g_{ij}(p) \cdot \mu \ .$$

Let $\tilde{\pi} : \coprod_{i \in I} U_i \times \mathbb{R}^r \longrightarrow E$ be the natural projection, and define $\pi : E \longrightarrow X$ by $\pi(\tilde{\pi}(p, \lambda)) := p$.

Excercise 6.1.7 *Show that there is a unique topology in in E such that each $\tilde{\pi}|_{U_i\times\mathbb{R}^r}$ is a homeomorphism onto its image. Show further that $(E,\pi,\mathcal{A}^E)$ is a vector bundle of rank r over X, where*

$$\mathcal{A}^E = \{\ (\pi^{-1}(U_i), (\tilde{\pi}|_{U_i\times\mathbb{R}^r})^{-1}, U_i \times \mathbb{R}^r \mid i \in I\ \}\ .$$

As an example consider

$$S^1 = \{\ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 = 1\ \}\ ,\ U_1 := S^1 \setminus \{(1,0)\}\ ,\ U_2 := S^1 \setminus \{(-1,0)\}\ ;$$

then $U_1 \cap U_2 = U^+ \cup U^-$, where $U^{\pm} = \{\ (x,y) \in S^1 \mid \pm y > 0\ \}$.
Define $g_{ij} : U_i \cap U_j \longrightarrow GL(1,\mathbb{R}) = \mathbb{R}$ by

$$g_{11} \equiv 1\ \ ,\ \ g_{22} \equiv 1\ \ ,\ \ g_{12}|_{U^+} = g_{21}|_{U^+} \equiv 1\ \ ,\ \ g_{12}|_{U^-} = g_{21}|_{U^-} \equiv -1\ .$$

It is obvious that $\{\ g_{ij} \mid i,j \in \{1,2\}\ \}$ satisfies the cocycle condition $(*)$ and hence defines a vector bundle M of rank 1 on S^1. This is a (slightly abstract) form of the Möbius Strip.

Excercise 6.1.8

1. *Let X be a differentiable manifold, $X = \bigcup_{i\in I} U_i$ an open cover,*

   $$\{\ g_{ij} : U_i \cap U_j \longrightarrow GL(r,\mathbb{R}) \mid i,j \in I\ \}$$

   differentiable maps satisfying the cocycle condition $(*)$, and E the vector bundle defined by these data. Show that a section s of E is the same as a family $\{\ s_i : U_i \longrightarrow \mathbb{R}^r \mid i \in I\ \}$ such that for all $i,j \in I$ and $p \in U_i \cap U_j$ it holds $s_i(p) = g_{ij}(p)(s_j(p))$.
   Show further that $s(p) = 0_p \ \Leftrightarrow\ s_i(p) = 0$ for all $p \in U_i$.

2. *Show that every section of the Möbius strip M (constructed above) has a zero.*

Let $(E,\pi^E,\mathcal{A}^E)$ and $(F,\pi^F,\mathcal{A}^F)$ be vector bundles on X

Definition 6.1.9 *A bundle map between E and F is a differentiable map $f : E \longrightarrow F$ with the following properties:*

1. *f is fibre preserving, i.e. $\pi^F \circ f = \pi^E$, i.e. $f(E_p) \subset F_p$ for all $p \in X$.*
2. *$f_p := f|_{E_p} : E_p \longrightarrow F_p$ is linear for all $p \in X$.*
3. *$\mathrm{rk}(f) = \mathrm{rk}(f_p)$ is constant as a function of $p \in X$.*

A bundle map f is called bundle isomorphism if f_p is bijective for every $p \in X$.

Excercise 6.1.10 *Let $f : E \longrightarrow F$ be a bundle map, and r the rank of E.*

1. *Define*

   $$\ker(f) := \bigcup_{p\in X} \ker(f_p) \subset E\ .$$

   Show that $\ker(f)$ has a unique structure of vector bundle of rank $r-\mathrm{rk}(f)$ such that the inclusion $\ker(f) \hookrightarrow E$ is a bundle map.

2. Define

$$\mathrm{im}(f) := \bigcup_{p\in X} \mathrm{im}(f_p) \subset F \ .$$

Show that $\mathrm{im}(f)$ *has a unique structure of vector bundle of rank* $\mathrm{rk}(f)$ *such that the inclusion* $\mathrm{im}(f) \hookrightarrow F$ *is a bundle map.*

3. Assume that f *is a bundle isomorphism. Show that* f *is a diffeomorphism of manifolds, and that its inverse* f^{-1} *is a bundle isomorphism, too.*

Let $f : E \longrightarrow F$ be a bundle map and s a section of E. From the first condition in Definition 6.1.9 it follows that $f \circ s$ is a section of F, and the second condition implies that if p is a zero of s, then it is also a zero of $f \circ s$. From this observation, Exercise 6.1.4.3 and Exercise 6.1.8.2 we conclude

Corollary 6.1.11 *The Möbius strip* M *is not bundle isomorphic to a trivial bundle.*

Let E and F be vector bundles on X, given with respect to a suitable open cover by cocycles g_{ij} and h_ij. Then we get new vector bundles via standard algebraic operations as follows

bundle	*cocycle*	*fibre*
E^*	$(g_{ij}^{-1})^t$	$(E^*)_p \cong (E_p)^*$
$E \oplus F$	$\begin{pmatrix} g_{ij} & 0 \\ 0 & h_{ij} \end{pmatrix}$	$(E \oplus F)_p \cong E_p \oplus F_p$
$E \otimes F$	$g_{ij} \otimes h_{ij}$	$(E \otimes F)_p \cong E_p \otimes F_p$
$\mathrm{Hom}(E,F) = E^* \otimes F$	$(g_{ij}^{-1})^t \otimes h_{ij}$	$\mathrm{Hom}(E,F)_p \cong \mathrm{Hom}(E_p, F_p)$
$\Lambda^k E$	$\Lambda^k g_{ij}$	$(\Lambda^k E)_p \cong \Lambda^k(E_p)$

Observe that for the trivial bundle $X \times \mathbb{R}^k$ it holds $E \otimes (X \times \mathbb{R}^k) = E^{\oplus k}$, and that $\Lambda^0(E) = X \times \mathbb{R}$.
Let V be an $(n+1)$-dimensional $\mathbb{R}$-vector space, and denote $V^\times := V \setminus \{0\}$. We equip V with its natural topology and differentiable structure (see Chapter 2).

Definition 6.1.12 *The n-dimensional (real) projective space associated to* V *is the quotient*

$$\mathbb{P}(V) := {V^\times}/{\sim}$$

where the equivalence relation $\sim$ *in* $V^\times$ *is defined by*

$$x \sim y \quad :\Leftrightarrow \quad \exists\, \lambda \in \mathbb{R} \ : x = \lambda \cdot y \ .$$

Equivalently, we can view $\mathbb{P}(V)$ *as the set of lines through the origin in* V.

We denote by $pr : V^\times \longrightarrow \mathbb{P}(V)$ the natural projection, and equip $\mathbb{P}(V)$ with the quotient topology.

Now assume that V is equipped with an inner product with associated norm $\|.\|$. Just as in the case of $V = \mathbb{R}^{n+1}$ one shows (using e.g. coordinates) that

$$S(V) = \{ \ v \in V \mid \|v\| = 1 \ \}$$

is an n-dimensional submanifold of V, and thus is an n-dimensional manifold.

We state without proof

Theorem 6.1.13 *1. The topology in $\mathbb{P}(V)$ is second countable and Hausdorff.*

2. $\mathbb{P}(V)$ has a unique n-dimensional differentiable structure such that $pr|_{S(V)} : S(V) \longrightarrow \mathbb{P}_n$ is a local diffeomorphism. This implies that pr is a surjective submersion.

Consider the product bundle $\mathbb{P}(V) \times V$ with bundle projection $\pi_1 : \mathbb{P}(V) \times V \longrightarrow \mathbb{P}(V)$ the projection onto the first factor.[4] Now define

$$\mathcal{O}_V(-1) := \{ \ (p, v) \in \mathbb{P}(V) \times V \mid v = 0 \ \text{ or } \ pr(v) = p \ \} \ ;$$

then

$$\pi_1^{-1}(p) \cap \mathcal{O}_V(-1) = \{p\} \times L \ ,$$

where $L \subset V$ is the line through the origin corresponding to p. In fact, one can show that

Proposition 6.1.14 *$\mathcal{O}_V(-1)$ has a unique structure of a vector bundle of rank 1 such that the inclusion $\alpha : \mathcal{O}(-1) \hookrightarrow \mathbb{P}(V) \times V$ into the product bundle is an injective bundle map.*

The last item of this course is to explain in an informal way the following basic fact in algebraic geometry.

Theorem 6.1.15 *There is an exact sequence of vector bundles*

$$0 \longrightarrow \mathcal{O}_V(-1) \xrightarrow{\alpha} \mathbb{P}(V) \times V \xrightarrow{\beta} \mathcal{O}_V(-1) \otimes T\mathbb{P}(V) \longrightarrow 0 \ ,$$

called the <u>Euler</u> <u>sequence</u>.

Here by "exact sequence" we mean that α and β are bundle maps, α is injective, β is surjective, and $\mathrm{im}(\alpha) = \ker(\beta)$.

To explain the symbol "$\otimes$", called <u>tensor</u>, we consider a 1-dimensional vector space L and an n-dimensional vector space W. Let b be a basis vector of L; then as a set we define

$$L \otimes_b W := \{ \ b \otimes_b w \mid w \in W \ \} \ .$$

[4]That this is indeed a vector bundle follows after choice of coordinates in V as in the case of the trivial bundle.

With addition defined by

$$b \otimes_b w + b \otimes_b w' := b \otimes_b (w + w') \ ,$$

and scalar multiplication defined by

$$\lambda \cdot (b \otimes_b w) := b \otimes_b (\lambda \cdot w)$$

this becomes a vector space. If b' is another basis vector of L, then there exists a unique $\lambda \neq 0$ such that $b' = \lambda \cdot b$. This implies that we can identify $L \otimes_b W$ and $L \otimes_{b'} W$ in a natural way via

$$L \otimes_{b'} W \ni b' \otimes_{b'} w = (\lambda \cdot b) \otimes_{b'} w := b \otimes_b (\lambda \cdot w) \in L \otimes_b W \ .$$

Using this identification, we get a vector space (unique up to natural isomorphy etc.) $L \otimes W$ consisting of vectors $l \otimes w$, $l \in L$, $w \in W$, with in particular the following two properties.

1. If b is a basis vector of L and $b_1, \ldots, b_n$ is a basis of W, then $b \otimes b_1, \ldots, b \otimes b_n$ is a basis of $L \otimes W$.
2. For all $\lambda \in \mathbb{R}$, $l \in L$, $w \in W$ it holds

$$\lambda \cdot (l \otimes w) = (\lambda \cdot l) \otimes w = l \otimes (\lambda \cdot w) \ .$$

Now we go back to our manifold X. If E is a vector bundle over X of rank r and L a vector bundle over X of rank 1 (a line bundle), then there is a vector bundle $L \otimes E$ over X with the following properties.

1. For every $p \in X$, $(L \otimes E)_p$ is naturally isomorphic to $L_p \otimes E_p$.
2. If E resp. L is defined by the cocycle $\{g_{ij}\}$ resp $\{h_{ij}\}$, then $L \otimes E$ is defined by the cocycle $\{h_{ij} \cdot g_{ij}\}$.

To define the map β in the Euler sequence we first observe that we have the natural map

$$\nu : V^\times \longrightarrow \mathcal{O}(-1) \ , \ \nu(x) := (pr(x), x) \in \mathbb{P}(V) \times V \ .$$

Second, we define the map

$$pr_* : V^\times \times V \longrightarrow T\mathbb{P}_n \quad , \quad pr_*(x, v) := [\gamma_{x,v}] \in T_{pr(x)}\mathbb{P}_n \ ,$$

where the curve $\gamma_{x,v}$ through $pr(x)$ is given by $\gamma_{x,v}(t) = pr(x + t \cdot v)$ on a suitably small interval around zero.

It is an easy exercise to show that

$$\lambda \neq 0 \quad \Rightarrow \quad pr_*(\lambda \cdot x, v) = \frac{1}{\lambda} \cdot pr_*(x, v) \ . \quad (\dagger)$$

Now we define for $p \in \mathbb{P}(V)$

$$\beta_p : \{p\} \times V \longrightarrow \mathcal{O}(-1)_p \otimes T\mathbb{P}(V)_p \quad , \quad \beta_p(p, v) := \nu(x) \otimes pr_*(x, v) \ ,$$

with $x \in V^\times$ such that $pr(x) = p$. It follows from (†) that this is indeed independent of the choice of x in $pr^{-1}(p)$.

Finally one shows:

1. $\beta : \mathbb{P}(V) \times V \longrightarrow \mathcal{O}(-1) \otimes T\mathbb{P}(V)$, $\beta|_{\{p\}\times V} := \beta_p$ is a surjective bundle map. This follows in particular from the fact that pr is a surjective surjection.
2. $\mathrm{im}(\alpha) \subset \ker(\beta)$. This follows because $pr_*(\alpha(p,v))$ is the class of the constant curve in p, i.e. the zero in $T_p\mathbb{P}_n$.
3. $\mathrm{im}(\alpha) = \ker(\beta)$. This follows because from dimensional reasons: β_p is surjective and hence has rank n, so $\ker(\beta_p)$ has dimension 1, which equals the dimension of $\mathrm{im}(\alpha)_p$.

6.2 Connections in vector bundles

Let X be an n-dimensional differentiable manifold, and $\pi : E \longrightarrow X$ a differentiable vector bundle of rank r. For $U \subset X$ open we define $E|_U := \pi^{-1}(U) \subset E$, then

$$\pi|_{E|_U} : E|_U \longrightarrow U$$

is a vector bundle on U. We denote by $A^0(U,E)$ the vector space of sections of $E|_U$, and set $A^0(E) := A^0(X,E)$.

We further use the following notations:

$$A^q(U) := A^0(U, \Lambda^q TX^*) = \Omega^q U$$

$$A^q(U,E) := A^0(U, E \otimes \Lambda^q TX^*) = A^0(U,E) \otimes_{A^0(U)} A^q(U)$$

$$A^q(E) := A^q(X,E)$$

Observe that all these spaces are modules over $\mathcal{C}^\infty(U,\mathbb{R}) = A^0(U)$.

The elements of $A^0(U,TX)$ are called <u>vector</u> <u>fields</u> on U, and we have a bilineair pairing

$$A^0(U,TX) \times A^1(U) \longrightarrow A^0(U) \ , \ (v,\alpha) \mapsto \alpha(v)$$

where $\alpha(v)(p) = \alpha(p)(v(p))$ for all $p \in X$. Similarly, we have a bilinear pairing

$$A^0(U,TX) \times A^1(U,E) \longrightarrow A^0(U,E) \ , \ (v, s\otimes\alpha) \mapsto \alpha(v)\cdot s \ .$$

Therefore the elements of $A^1(U,E)$ are called 1-forms with values in E. Finally we have a bilinear map

$$A^p(U,E) \times A^q(U) \longrightarrow A^{p+q}(U,E) \ , \ (s\otimes\alpha, \beta) \mapsto s \otimes (\alpha \wedge \beta) \ .$$

The image of $\sigma \otimes \beta \in A^p(U,E) \otimes A^q(U)$ in $A^{p+q}(U,E)$ under the induced linear map we denote by $\sigma\wedge\beta$.

Definition 6.2.1 *A <u>connection</u> in E is an $\mathbb{R}$-linear map*

$$D : A(E) \longrightarrow A^1(E)$$

satisfying

$$D(f\cdot s) = s \otimes df + f \cdot D(s)$$

for all $f \in A^0(X)$, $s \in A^0(E)$. A section $s \in A^0(E)$ is called <u>parallel</u> if $D(s) = 0$.

An analogous definition holds for any open subset $U \subset X$.

Excercise 6.2.2 *Let D be a connection in E and $U \subset X$ open.*

1. *For $s \in A^0(E)$ show that $s|_U = 0$ implies $D(s) = 0$.*
2. *Show that there is a unique connection $D|_U$ in $E|_U$ such that for all $s \in A^0(E)$ it holds $D(s)|_U = D|_U(s|_U)$.*
   (This is not completely obvious because not every section of E over U is the restriction to U of a global section.)

We define $\mathbb{R}$-linear maps, also denoted D,

$$D : A^p(E) \longrightarrow A^{p+1}(E)$$

as $\mathbb{R}$-bilinear extensions of

$$D(s \otimes \alpha) := D(s) \wedge \alpha + s \otimes d\alpha$$

for all $s \in A^0(E)$, $\alpha \in A^p(X)$.

Excercise 6.2.3 *Show that for $\sigma \in A^p(E)$, $\alpha \in A^q(X)$ it holds*

$$D(\sigma \wedge \alpha) = D(\sigma) \wedge \alpha + (-1)^p \sigma \wedge d\alpha \ .$$

Definition 6.2.4 *The <u>curvature</u> of D is the $\mathbb{R}$-linear map*

$$F_D = D \circ D : A(E) \longrightarrow A^2(E) \ .$$

D is called <u>flat</u> if $F_D = 0$.

Lemma 6.2.5 *For all $f \in A^0(X)$, $s \in A^0(E)$ it holds $F_D(f \cdot s) = f \cdot F_D(s)$.*

Proof:

$$F_D(f \cdot s) = D(s \otimes df + f \cdot D(s)) = D(s) \wedge df + s \otimes d^2 f - D(s) \wedge df + f \cdot D^2(s) = f \cdot F_D(s) \ .$$

■

Excercise 6.2.6 *Show that the above Lemma implies that $F_D \in A^2(\mathrm{End}(E)) = A^2(E^* \otimes E)$.*

Example 6.2.7 *Consider the trivial bundle $E_0 = X \times \mathbb{R}^r$. Then we have*

$$A^0(E_0) = \{ \ (f_1, \ldots, f_r) \mid \forall \ 1 \leq i \leq r \ : \ f_i \in A^0(X) \ \} \ .$$

This is a free $A^0(X)$-module with basis $e_1, \dots, e_r$, where e_i is the constant section $e_i \equiv (0, \dots, 1, \dots, 0)$ with the 1 in the ith place:

$$A^0(E_0) \ni s = (f_1, \dots, f_r) = \sum_{i=1}^{r} f_i \cdot e_i \ .$$

We define $D : A^0(E_0) \longrightarrow A^1(E_0)$ by

$$D\left(\sum_{i=1}^{r} f_i \cdot e_i\right) := \sum_{i=1}^{r} e_i \otimes df_i \ ,$$

this is obviously $\mathbb{R}$-linear, and an easy calculation shows that D is indeed a connection in E_0. It is obvious that each e_i is parallel with respect to this connection, hence it holds for all sections of E_0

$$F_D\left(\sum_{i=1}^{r} f_i \cdot e_i\right) = D\left(\sum_{i=1}^{r} e_i \otimes df_i\right) = \sum_{i=1}^{r} e_i \otimes d^2 f_i = 0 \ ,$$

this means that D is flat. It is called the canonical flat connection in the trivial bundle.

Let $\pi : E \longrightarrow X$ a vector bundle of rank r and $U \subset X$ open.

Definition 6.2.8 *A (local) frame for E over U is a set $s_1, \dots, s_r \in A^0(U, E)$ that $s_1(p), \dots, s_r(p)$ is a basis of $E(p)$ for all $p \in U$.*

Excercise 6.2.9 *Show that the existence of a global frame $s_1, s_2, \dots, s_r \in A^0(E)$ is equivalence to E being isomorphic to the trivial bundle.*

Since every point $p \in X$ has an open neighborhood U such that $E|_U$ is isomorphic to the trivial bundle over U by a bundle chart, there always exists a local frame defined in an open neighborhood of p.

Theorem 6.2.10 *For a connection D in E are equivalent:*

1. *D is flat.*
2. *For every $p \in X$ exists an open neighborhood U of p and a local frame of E over U consisting of parallel sections.*

Remark 6.2.11
1. *Not every bundle admits a flat connection.*
2. *The existence of a flat connection in E does not imply that E is isomorphic to the trivial bundle.*

Proposition 6.2.12 *Let D be a connection in E. Then for every $p \in X$ and every $e \in E_p$ there exists an open neighborhood U of p and a section $s \in A^0(U, E)$ such that $s(p) = e$ and $D(s)(p) = 0$. If s' is another local section with $s'(p) = e$ and $D(s')(p) = 0$, then*

$$\mathrm{im}(\tilde{D}(s)(p)) = \mathrm{im}(\tilde{D}(s')(p)) \subset T_e E \ ,$$

where $\tilde{D}(s)(p)) : T_p X \longrightarrow T_e E$ denotes the tangent map of s at p.

Therefore, D determines at e the well defined horizontal subspace $T^h_e E := \mathrm{im}(\tilde{D}(s)(p)) \subset T_e E$. As vertical subspace at e we define the tangent space of the fibre, i.e. $T^h_e E := T_e E_e \subset T_e E$. Observe that $\pi \circ s = \mathrm{id}_X$ implies $\tilde{D}(\pi)(e) \circ \tilde{D}(s(p) = \mathrm{id}_{T_p X}$. In particular it holds that $\tilde{D}\pi(e)|_{\mathrm{im}(\tilde{D}(s)(p))}$ is injective and that $\tilde{D}\pi(e) \geq n$, hence $\dim \ker \tilde{D}\pi(e) \leq r$. On the other hand, since π maps every curve through e in E_e to the constant curve p in X, and since E_e has dimension r it holds $\ker \tilde{D}\pi(e) = T^h_e E$. We conclude that

$$T_e E = T^v_e E \oplus T^h_e E \ .$$

Proposition 6.2.13 *Let* $\gamma : [a,b] \longrightarrow X$ *be a differentiable curve and* $e \in E_{\gamma(a)}$ *. Then there exists a unique horizontal lift of* γ *starting at e, i.e. a differentiable curve* $\tilde{\gamma}_e : [a,b] \longrightarrow E$ *such that* $\tilde{\gamma}_e(a) = e$ *,* $\pi \circ \tilde{\gamma}_e = \gamma$ *and* $\dot{\tilde{\gamma}}_e(t) \in T^h_{\tilde{\gamma}_e(t)} E$ *for all* $t \in [a,b]$ *, where* $\dot{\tilde{\gamma}}_e(t)$ *denotes the class of the curve* $s \mapsto \tilde{\gamma}_e(t+s)$

This defines a map

$$E_{\gamma(a)} \longrightarrow E_{\gamma(b)} \ , \ e \mapsto \tilde{\gamma}_e(b)$$

which is a linear isomorphism and independent of the parametrization of γ*.*

Let be $p \in X$, $C(p)$ the set of closed continuous and piecewise differentiable curves $\gamma : [0,1] \longrightarrow X$ with start- and endpoint p, and $C^0(p)$ the subset of $C(p)$ consisting of curves homotopic to the constant curve. The proposition above implies that there is a well defined map

$$H : C(p) \longrightarrow GL(E_p) \ , \ \gamma \mapsto (e \mapsto \tilde{\gamma}_e(1))$$

Theorem 6.2.14 *Let* X *be connected.*

1. $\Phi(p) := H(C(p)$ *is a Lie subgroup of* $GL(E_p)$*, called the holonomy group of* D *at* p*. For all* $p, q \in X$ *, the subgroups* $\Phi(p)$ *and* $\Phi(q)$*are conjugated in and hence isomorphic.*
2. $\Phi^0(p) := H(C^0(p)$ *is a connected normal subgroup of* $\Phi(p)$ *such that* ${\Phi(p)}/{\Phi^0(p)}$ *is countable. In particular,* $\Phi_0(p)$ *is the identity component of* $\Phi(p)$*.*

A connection D in E induces a connection D^* in the dual bundle E^* as follows. First observe that $s \in A^0(E)$, $s^* \in A^0(E^*)$ define $s^*(s) \in A^0(E)$, and that there are unique well defined bilinear maps

$$A^1(E^*) \times A^0(E) \longrightarrow A^1(E) \ , \ A^0(E^*) \times A^1(E) \longrightarrow A^1(E)$$

satisfying

$$(s^* \otimes \alpha, s) \mapsto s^*(s) \cdot \alpha =: (s^* \otimes \alpha)(s) \ , \ (s^*, s \otimes \alpha) \mapsto s^*(s) \cdot \alpha =: s^*(s \otimes \alpha)$$

for all $s \in A^0(E)$, $s^* \in A^0(E^*)$, $\alpha \in A^1(X)$. Now D^* is uniquely determined by

$$d(s^*(s)) = D^*(s^*) + s^*(D(s))$$

for all $s \in A^0(E)$, $s^* \in A^0(E^*)$.

Let D_E resp. $D_{E'}$ be a connection in the vector bundle E resp. E' over X. Then we define the induced connection $D_{E\otimes E'}$ in $E \otimes E'$ by

$$D_{E\otimes E'}(s \otimes s') := D_E(s) \otimes s' + s \otimes D_{E'}(s') \text{ for all } s \in A^0(E) \ , \ s' \in A^0(E') \ ;$$

here we use the convention

$$(s \otimes \alpha) \otimes s' := (s \otimes s') \otimes \alpha =: s \otimes (s' \otimes \alpha) \text{ for all } s \in A^0(E) \ , \ s' \in A^0(E') \ \alpha \in A^k(X) \ .$$

Excercise 6.2.15 *Show that for the curvatures F, F' and $F^{\otimes}$ associated to D_E, $D_{E'}$ and $D_{E\otimes E'}$ in E, E' and $E \otimes E'$ it holds*

$$F^{\otimes}(s \otimes s') := F(s) \otimes s' + s \otimes F'(s') \text{ for all } s \in A^0(E) \ , \ s' \in A^0(E') \ .$$

Now let V be an n-dimensional real vector space, and $Bil(V)$ the vector space of bilinear maps $V \times V \longrightarrow \mathbb{R}$.

Excercise 6.2.16 *Show that there is a well defined linear isomorphism*

$$b : V^* \otimes V^* \longrightarrow Bil(V)$$

satisfying

$$b(v^* \otimes u^*)(v, u) = v^*(v) \cdot u^*(u) \text{ for all } v^*, u^* \in V^* \ , \ v, u \in V \ .$$

From now on we identify the two spaces by this isomorphism.

Let $b_1, \ldots, b_n$ be a basis of V, and $b_1^*, \ldots, b_n^*$ the dual basis of V^*. Using Exercise 6.2.16 it is easy to see that we can write every $h \in Bil(V)$ as $h = \sum_{i,j=1}^{n} h(b_i, b_j) \cdot b_i^* \otimes b_j^*$.

Definition 6.2.17 *A <u>metric</u> in a vector bundle E over X is a section $h \in A^0(E^* \otimes E^*)$ such that for every $p \in X$ the bilinear form $h(p) \in E_p^* \otimes E_p^*$ is symmetric and positive definite. The pair (E, h) is then called a <u>metric</u> <u>bundle</u>.*

Excercise 6.2.18 *Show, using a partition of unity, that every vector bundle admits a metric.*

Definition 6.2.19 *An <u>h-connection</u> in a metric bundle (E, h) is a connection D in E such that h is parallel with respect to the connection in $E^* \otimes E^*$ induced by D.*

Excercise 6.2.20 *Show that a connection D in E is an h-connection if and only if*

$$d(h(s, t) = h(D(s), t) + h(s, D(t)) \text{ for all } s, t \in A^0(E) \ ,$$

where we use the convention

$$h(s \otimes \alpha, t) := h(s, t) \otimes \alpha =: h(s, t \otimes \alpha) \text{ for all } s, t \in A^0(E) \ , \ \alpha \in A^k(X) \ .$$

A derivation on X is a linear map $\delta : A^0(X) \longrightarrow A^0(X)$ satisfying $\delta(f \cdot g) = \delta(f) \cdot g + f \cdot \delta(g)$ all $f, g \in A^0(X)$. The space $Der(X)$ of derivations obviously is a linear subspace of $Hom(A^0(X), A^0(X))$. Let $v \in A^0(TX)$ be a vector field, then it is easy to see that

$$\delta_v : A^0(X) \longrightarrow A^0(X) \ , \ \delta_v(f)(p) := v(p)([f, X]) = df(p)(v(p))$$

is a derivation on X.

Proposition 6.2.21 *The linear map* $A^0(TX) \longrightarrow Der(X)$, $v \mapsto \delta_v$, *is an isomorphism.*

For a derivation $\delta \in Der(X)$ we denote by v_δ the inverse image of δ under this map.

Excercise 6.2.22 *1. For* $\delta_1, \delta_2 \in Der(X)$ *show that* $\delta_1 \circ \delta_2 - \delta_2 \circ \delta_1 \in Der(x)$.

2. Give an example of X *and* $\delta_1, \delta_2 \in Der(X)$ *where* $\delta_1 \circ \delta_2 \notin Der(X)$.

By the above proposition and exercise, for $v, w \in A^0(TX)$ we can define the Lie-bracket

$$[v, w] := v_{(\delta_v \circ \delta_w - \delta_w \circ \delta_v)} \ .$$

Now let D be a connection in TX, and define a $\mathbb{R}$-bilinear map

$$A^0(TX) \times A^0(TX) \longrightarrow A^0(TX) \quad , \quad (v, w) \mapsto D_v w := D(w)(v) \ ,$$

where we use the convention $(u \otimes \alpha)(v) := \alpha(v) \cdot u$ for all $v, u \in A^0(TX)$, $\alpha \in A^1(X)$. $D_v w$ is called the D-covariant derivative of w in the direction of v. Observe that

$$D_{f \cdot v} w = f \cdot D_v w \quad , \quad D_v(f \cdot w) = df(v) \cdot w + f \cdot D_v w \ \ \text{for all} \ \ v, w \in A^0(TX) \ , \ f \in A^0(X) \ ,$$

i.e. that $D_v w$ is $A^0(X)$-linear in v but not in w.
The torsion of D is the map

$$T : A^0(TX) \times A^0(TX) \longrightarrow A^0(TX) \quad , \quad T(v, w) := D_v w - D_w v - [v, w] \ .$$

Excercise 6.2.23 *Show that*

$$T(f \cdot v, w) = f \cdot T(v, w) = T(v, f \cdot w) \ \ \text{for all} \ \ v, w \in A^0(TX) \ , \ f \in A^0(X) \ .$$

A Riemannian metric in X is a metric h in TX, the pair (X, h) is the called a Riemannian manifold. Using a partition of unity one can show that every manifold admits a Hermitian metric.

Theorem 6.2.24 *On a Riemannian manifold* (X, h) *exists a unique metric connection with vanishing torsion, this is called the Levi-Civita connection.*

Let $\pi : E \longrightarrow X$ be a vector bundle and D a connection in E.

Lemma 6.2.25 *For all $p \in X$ and $e \in E_p$ exists a section $s \in A^0(E)$ such that $s(p) = e$ and $D(s)(p) = 0$. If s' is another section with $s'(p) = e$ and $D(s')(p) = 0$, then it holds*

$$\operatorname{im}\left((\tilde{D}s(p)\right) = \operatorname{im}\left((\tilde{D}s'(p)\right) \subset T_eE ,$$

where $\tilde{D}s(p) : T_pX \longrightarrow T_eE$ denotes the tangent map associated to s.

Observe that $\pi \circ s = \mathrm{id}_X$ implies $\tilde{D}\pi(e) \circ \tilde{D}s(p) = \mathrm{id}_{T_pX}$, so $\tilde{D}s(p)$ and $\tilde{D}\pi(e)|_{\operatorname{im}\left((\tilde{D}s(p)\right)}$ are injective, and $\tilde{D}\pi(e)|_{\operatorname{im}\left((\tilde{D}s(p)\right)} : \operatorname{im}\left((\tilde{D}s(p)\right) \longrightarrow T_pX$ is an isomorphism by reason of dimensions. In particular, it holds $\dim \operatorname{im}\left((\tilde{D}s(p)\right) = n$. We call $T^h_eE := \operatorname{im}\left((\tilde{D}s(p)\right)$ the (with respect to D) horizontal tangent space of E at e.

On the other hand, we call $T^v_eE := T_pE_p$ the (with respect to D) vertical tangent space of E at e. Since π is constant on E_p, we have $T^v_eE \subset \ker \tilde{D}\pi(e)$, and hence $T^v_eE \cap T^h_eE = \{0\}$. Because of $\dim T^v_eE = r$ it follows, again by reason of dimensions, that $T^v_eE = \ker \tilde{D}\pi(e)$ and $T_eE = T^v_eE \oplus T^h_eE$.

Now let $\gamma : [a, b] \longrightarrow X$ a differentiable curve and $e \in E_{\gamma(a)}$.

Definition 6.2.26 *A (with respect to D) horizontal lift of γ to E with starting point e is a differentiable curve $\tilde{\gamma}_e : [a, b] \longrightarrow E$ with $\tilde{\gamma}_e(a) = e$ and*

$$\forall\, t \in [a, b] \ : \ \pi\left(\tilde{\gamma}_e(t)\right) = \gamma(t) \ , \ \dot{\tilde{\gamma}}_e(t) \in T^h_{\tilde{\gamma}_e(t)}E \ ,$$

where $\dot{\tilde{\gamma}}_e(t)$ is the equivalence class of the curve $s \mapsto \tilde{\gamma}_e(t + s)$.

From now on we will be sketchy without references. The interested reader can find the details in the standard literature on differential geometry, e.g. the ”Foundations of Differential Geometry” by Kobayashi and Nomizu.

By the theory of differential equations, a horizontal lift as above always uniquely exists,and we get a map

$$p_\gamma : E_{(a)} \longrightarrow E_{\gamma(b)} \ , \ e \mapsto \tilde{\gamma}_e(b) \ .$$

p_γ is called the parallel transport (with respect to D) along γ.

We define $\gamma^{-1} : [a, b] \longrightarrow X \ , \ t \mapsto \gamma(a + b - t)$, and for a differentiable curve $\tau : [b, c] \longrightarrow X$ with $\gamma(b) = \tau(b)$ we define $\tau * \gamma : [a, c] \longrightarrow X \ , \ t \mapsto \begin{cases} \gamma(t) & \text{if } a \le t \le b, \\ \tau(t) & \text{if } b \le t \le c; \end{cases}$

Proposition 6.2.27 *p_γ is a linear isomorphism which is independent of the parametrization of γ. It holds $p_\gamma^{-1} = p_{\gamma^{-1}}$, and for τ as above it holds $p_{\tau*\gamma} = p_\tau \circ p_\gamma$.*

We denote by $C(p)$ the set of piecewise differentiable loops at p, i.e. of continuous curves $\gamma : [0,1] \longrightarrow X$ with the following properties: $\gamma(0) = \gamma(1) = p$, and there exists $k \in \mathbb{N}$ and $0 < t_1 < t_2 < \ldots < t_k < 1$ such that $\gamma|_{[t_i,t_{i+1}]}$ is differentiable for all $1 \leq i \leq k-1$. By $C^0(p) \subset C(p)$ we denote the subset of simply connected curves.

For $\gamma, \tau \in C(p)$ we define

$$\gamma^{-1}, \tau \cdot \gamma : [0,1] \longrightarrow X \ , \ \gamma^{-1}(t) := \gamma(1-t) \ , \ (\tau \cdot \gamma)(t) := \begin{cases} \gamma(2t) & \text{if } t \leq 1/2, \\ \tau(2t-1) & \text{if } t \geq 1/2; \end{cases}$$

then $\gamma^{-1}, \tau \cdot \gamma \in C(p)$. Observe that $\tau \cdot \gamma$ is a reparametrization of $\tau * \gamma$ defined above.

Parallel transport p_γ along $\gamma \in C(p)$ is defined as follows: Let be $0 < t_1 < t_2 < \ldots < t_k < 1$ as above, then $p_\gamma := p_{\gamma_{k-1}} \circ \ldots \circ p_{\gamma_1}$.

From the proposition above it follows

Proposition 6.2.28 *The map* $H : C(p) \longrightarrow GL(E(p)$, $\gamma \mapsto p_\gamma$ *is well defined, and* $\Phi^0(p) := H(C^0)$ *and* $\Phi(p) := H(C(p)$ *are subgroups of* $GL(E_p)$.

$\Phi(p)$ is called the <u>holonomy</u> <u>group</u>, and $\Phi^0(p)$ the <u>restricted</u> <u>holonomy</u> <u>group</u> (with respect to D) of E at p.

Theorem 6.2.29 *If* X *is connected, then* $\Phi(p)$ *is a Lie subgroup of* $GL(E_p)$, *and* $\Phi^0(p)$ *is a connected normal Lie subgroup of* $\Phi(p)$ *such that* ${}^{\Phi(p)}/_{\Phi^0(p)}$ *is countable. In particular,* $\Phi^0(p)$ *is the identity component of* $\Phi(p)$.
For all $p, q \in X$, $\Phi(p)$ *and* $\Phi(q)$ *are conjugated in* $GL(E_p)$ *and hence isomorphic.*