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Unprejudiced measurements of free parameters in Neutrino Mass Models

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Abstract

Bayesian statistics are a necessary tool for statistical model comparisons in cosmology because we have only one universe to observe. This means that we need to choose a prior probability distribution on our parameter space, but the exact choice of prior has been proven to greatly affect results. It has been proposed that priors should be put as directly as possible on the underlying physical model parameters, for example on the entries of the neutrino mass matrix when studying the physical origin of neutrino masses. However, mathematical principles then result in an ultimately informative prior. We describe the physics beyond the Standard Model of Particle Physics needed to explain the observed mass of neutrinos and introduce the mathematical concepts needed to analyze the three main models statistically. These are the Dirac, Majorana and Seesaw models. We discuss the impact of the principles of concentration of measure and eigenvalue repulsion on random matrices, which we need to use while studying if one could discern between the Dirac and Majorana models and the Normal and Inverted hierarchy using priors on the individual matrix elements of the neutrino mass matrix. We present sampling distributions for the observable quantities $\sum m_\nu$, Δm_{\odot}^2 and Δm_{atm}^2 based on this model and explain where the named principles have an effect, leading to highly informative priors in the given context due to, possibly unexpected, interference of these mathematical phenomena.

Contents

Abstract	2
Introduction	5
1 Lagrangians and Neutrino Masses	8
1.1 Lagrangian Formalism	8
1.1.1 Classical	8
1.1.2 Field Theory	12
1.2 Euler-Lagrange equations	13
1.3 Noether's theorem	14
1.4 Free Fields	17
1.4.1 Scalar Fields	17
1.4.2 Vector Fields	18
1.4.3 Spinor (Dirac) Fields	19
1.5 Symmetry Breaking	20
1.5.1 Higgs Mechanism	23
1.6 Fermion Masses in the Standard Model	25
1.6.1 Neutrino Masses	30
2 Invariant Priors and Concentration of Measure	35
2.1 Priors	35
2.1.1 The Maximal Entropy Principle	36
2.1.2 Invariance of Distributions	39
2.1.3 Invariant Priors	40
2.1.4 Jeffreys' Rule	43
2.1.5 Reference priors	43
2.2 Concentration of Measure	45
2.2.1 Early Inequalities	45
2.2.2 Efron-Stein Inequality	48
2.2.3 Information inequalities and the Entropy method	51
2.3 Random Matrices	54
2.3.1 Gaussian Ensembles	54
2.4 Eigenvalue Repulsion	56
2.4.1 Analytical Method	57

3	Methods and Results	59
3.1	Previous Results and Phenomenology	59
3.1.1	Experimental and Phenomenological Details	59
3.1.2	Statistical Analysis	62
3.2	Methods	63
3.3	Results and Discussion	65
	Conclusion	71
	Bibliography	73
A	Additional Figures and Derivations	77
A.1	Chapter 1, Gauge Symmetry	77
A.2	Chapter 1, Neutrino Oscillations	79
A.3	Chapter 3, Figures alike figure 3.3	81
A.4	Chapter 3, Forecast likelihoods	86

Introduction

When wanting to understand the origin and the structure of the Universe, we need to understand the physics of the Universe at very early times. The most commonly accepted theory about these early times is nowadays known as the Hot Big Bang model, but even that theory leaves questions unanswered, for instance we can only guess what happened before the era of recombination (when the Cosmic Microwave Background originated), because we do not receive light from before that era. Additionally, we do not even know yet how to describe the state of the universe at scales of the order of the Planck scale theoretically[11].

Because the state of the Universe at early times, according to the Hot Big Bang model, is a hot plasma consisting of all types of elementary particles, an understanding of particle physics is of utmost importance for Cosmology and thus Astronomy. We do already have a model that describes many phenomena in particle physics correctly, known as the Standard Model of Particle Physics (SM). Many of the parameters in the SM also have an impact on cosmological parameters in our models. For example the densities of radiation and matter, which of course consist of particles, thus depend, fundamentally, on the properties of said particles. To be able to fully understand the current structure of the universe, we need to fully understand this plasma, so we need to understand not only what the properties of particles are, but also where these properties originate from physically. One of the more prominent particles in current research is the neutrino, which, according to the SM, should be massless[34], but observations have been made leading us to believe that it actually has a finite mass. These observations include mostly the solar neutrino problem, which is solved when introducing the concept of neutrino oscillations [46],[19]. In this thesis we try to find more details about the physical origin of neutrino mass by checking if we can use current data to discern between the current models using Bayesian inference. Additionally we try to show how important a good choice of prior probability distribution can be when performing this kind of inference.

The neutrino was first discovered by Pauli when he studied two-body β -decay, where a neutron decays into a proton and an electron. In a two-body decay like this you would expect the produced electron to have the same energy, but Pauli found an entire spectrum, so he proposed the existence of a third, electrically neutral particle, which he called the neutron. It had to be electrically neutral because he did not observe it directly. Then Chadwick discovered the particle we now know as the neutron, and Fermi renamed Pauli's particle to 'neutrino', meaning something along the lines of 'the little neutron'. To indirectly detect the neutrino, we see that the reverse reaction of β -decay (denoted by β^+) should be able to be caused by an energetic neutrino hitting a stable nucleus. When the neutrino hits a proton it should produce a neutron and a positron. The cross-section for this reaction was theorized to be a factor 10^{-24} smaller than the Thomson cross-section for photons scattering of non-relativistic electrons, so observing any reactions like this requires a huge source of neutrinos. Fermi thus built a nuclear reactor as source of a large number of neutrinos, which was directed to a container of water to invoke β^+ -decays, which create positrons, which should annihilate producing

two photons. Indeed, two photons and a neutron were observed as expected[9].

The SM describes our current observations very precisely, but there are some phenomena not explained by the SM. For example the theory of gravity, the existence of dark matter and energy and the observed fact that neutrinos have a mass are all evidence the SM does not provide the entire picture. In the SM, neutrinos have exactly zero mass, and just as for the charged leptons (the electron, muon and tauon), there are three generations of neutrinos, their so-called flavour eigenstates, the electron, muon and tauon neutrino.

The fact that the muon had its own flavor of neutrino was discovered in 1962 by Lederman, Schwartz and Steinberger by letting muons decay into an electron, electron-neutrino and, presumably, a muon-neutrino. Directing the resulting beam at neutrons produced protons and muons, but these muons would not be produced if only the electron-neutrino was involved in muon-decay. The tauon-neutrino was found in 2000.

We know of the existence of neutrino oscillations due to the proposed solution to the solar neutrino problem, which is a process where neutrinos change flavour without interacting with other particles. This happens in a periodic manner, which is why it is called 'oscillations'. Indeed, the predictions of this theory solved the neutrino problem. These oscillations could not happen if neutrinos were indeed massless as the SM dictates[34].

Recently, many experiments have been done to get ever better constraints on these neutrino masses, but the physical origin of them will need to be inferred by comparing possible models with the observational evidence. Because we only have one sky to observe, this has to be done in a Bayesian manner for we can not use the frequentist method of collecting many independent pieces of information from many trials and theoretically finding the limit of the probability distribution. The Bayesian method is that of assuming a prior probability distribution on the parameter space, including information about prior knowledge, which we combine with observational results via Bayes' theorem to end up with the posterior probability distribution. This posterior gives us new information about the probability for the parameters of our models taking certain values in the parameter space. The choice of the prior probability distribution could introduce too much assumptions into the model, which would give false results after calculations.

The best-known models are those describing the neutrino as a Dirac or a Majorana particle, while more sophisticated models exist, such as the Seesaw mechanism, and even models simultaneously used to explain open questions in Cosmology, such as the Growing Neutrino Quintessence model.

Another important characteristic is the hierarchy of the neutrinos. Neutrino oscillations give two independent values of mass differences between the three mass eigenstates, which are superpositions of the aforementioned flavour eigenstates. One of the mass differences is of order $\Delta m^2 \sim 10^{-5}$ and one is of order $\Delta m^2 \sim 10^{-3}$, so two possibilities hold. These are the cases where $m_3 \gg m_2 > m_1$ (normal hierarchy) or where $m_2 > m_1 \gg m_3$ (inverted hierarchy), where m_i denotes the mass of the mass eigenstate ν_i .

All of these models ultimately depend on how a neutrino interacts with other particles, which is defined in its Lagrangian. The SM Lagrangian is not sufficient for describing how neutrinos interact with the three leptons, or why neutrino oscillations occur. The difference between the Dirac and Majorana model for neutrinos is encoded in the mass-term of said Lagrangian, which has a different form in both formalisms. The physical origin of neutrino masses will have to be inferred from measurements, so after constraining the masses well enough, we can ask whether we can infer the properties of neutrinos directly from their Lagrangian, as the information about the mass should be encoded in it all along

[39], [13], [23], [18], [22], [17], [3].

One important recurring problem in the practice of inferring neutrino masses is which choice we need to make in specifying the prior to discern between models. Previous research has shown that a least informative prior can be constructed and it was actually proven to be least informative, ensuring that data dominates over possible prior beliefs when constraining the masses and hierarchy of the neutrinos [27]. The use of priors has since been tested and improved, many results have been shown to be dominated by priors. Other methods and theories for neutrino masses have been developed to try to make Bayesian analysis more effective, but even they were dominated by prior knowledge when compared to the least informative prior method [43], [16], [44].

It has recently been proposed to put priors directly on the mass terms in the Lagrangian, but then mathematical phenomena may still drastically alter the results, because the number of dimensions and independent matrix-elements change during the mathematical process distilling the mass from the matrix [32].

Unfortunately, when wanting to discern between the Majorana and Dirac models to understand the exact physical origin of the mass of neutrinos, we do have to put the prior on the mass matrix.

The first chapter of this thesis will introduce the theory and phenomenology of neutrino masses in the context of the SM and Lagrangian theory. To do this effectively, we start by introducing the concept of Lagrangians in Classical mechanics, and generalize to field theory, showing examples from particle physics along the way. Then we explain the concept of symmetry breaking and how it can be used to generate masses for the leptons of the theory. Lastly, we show how we can use these mass terms to produce a viable model for neutrino masses, and explain exactly why that model is viable. In the second chapter, we will explain prior probability distributions and how we can choose or construct priors such that they encode the minimal amount of prejudice possible. These priors are called invariant and are often based upon symmetries of the underlying theory. We will study Jeffreys' prior and reference priors to conclude the topic of priors. Afterwards, we will introduce the concept of concentration of measure, and how it occurs when working with random matrices, resulting in the phenomenon of eigenvalue splitting.

The third chapter then describes previous results in this area and how our methods and results add to those, while presenting our findings.

A conclusion with possible follow-up research will be presented at the end.

Chapter 1

Lagrangians and Neutrino Masses

As we explained in the Introduction, neutrino masses are not included in the SM Lagrangian. To understand why this happens and how to solve it, we need to understand the physics behind the leptonic section of the SM Lagrangian. We will start this discussion by introducing the concept of the Lagrangian formalism in Classical Mechanics. We do this by deriving it naturally from Hamilton's principle, stating that the path chosen by particles is such that it minimizes the action. Afterwards, we can easily generalize this to field theoretical Lagrangians which we need to describe particle physics, where the Lagrangian becomes more axiomatic instead, for we can less easily write down equations of motion for fields from nothing, and we have to start somewhere. But how can we write down a Lagrangian without information? This is where symmetries of the theory play an important role. If the system has a certain symmetry (i.e. time or translation invariance), it should be reflected in the Lagrangian, and even correspond to conservation laws, which we will show. Both global and local (gauge) symmetries are of importance. It seems that some of the current SM Lagrangian components violate certain conservation laws. This is where the concept of symmetry breaking comes in play. We will introduce the Higgs mechanism and how it is connected to the production of lepton masses. Finally we will then have the ingredients needed to introduce neutrino masses and the concepts behind them, introducing the Dirac and Majorana mass terms and the see-saw mechanism, and the concept of neutrino oscillations. Afterwards we thus know enough about this theory to be able to talk about priors in the context of this theory, which will be done in the next chapter.

1.1 Lagrangian Formalism

Let us thus start by introducing the Lagrangian Formalism in Classical Mechanics and Field Theory. We will present the classical formalism by deriving it from Hamilton's principle of least action, which is the most natural way of approaching the theory.

1.1.1 Classical

This discussion is mostly based on [41].

Classical Mechanics can be described in many ways. The earliest viable method for describing forces and motions formally were Newton's laws. This method has many disadvantages though, for forces are not always explicitly known, except in the way they constrain motion, and choice of coordinates can make the discussion significantly more difficult. Also when more and more forces start acting, we have to write down more and more terms, which becomes way too tedious for us physicists, so there

must be a simpler way. The Lagrangian formalism is nothing more than a more formal reformulation of Newtonian mechanics, making it more elegant and easier to apply. It suggests a deeper level of the theory, which was developed by Hamilton in 1833. Hamilton gave the following postulate, the now called **Hamiltonian principle**:

”Of all the possible paths along which a dynamical system may move from one point to another within a specific time interval (consistent with any constraints), the actual path followed is that which minimizes the time integral of the difference between kinetic and potential energies.”

As far as we are currently aware, this principle cannot be derived from any deeper or more fundamental principle. Hamilton’s principle has been tested in multiple ways. For example the path taken by light in refraction between two media is that path which minimizes the time of travel of the light, so Snell’s law (as derived from the Fermat principle, which is stated along the same lines as Hamilton’s postulate) can be seen as one of the consequences of Hamilton’s principle. Hamilton described how the known laws of classical mechanics follow from his postulate.

As already demonstrated with this example in the case of Snell’s law, the postulate can be applied to a much wider class of physical phenomena. It is for instance applied to electromagnetic fields and fields governing the interactions of elementary particles, of which we will see more shortly.

The starting point in a mathematical discussion of the Hamiltonian principle is the **Lagrangian function** $L = L(q_1, \dots, q_{3N-k}, \dot{q}_1, \dots, \dot{q}_{3N-k}, t)$. The q here denote the $3N - k$ **generalized coordinates**, which describe the system in as little coordinates as possible working with the known constraints. N then is the number of particles and k the amount of **holonomic constraints**, which are constraints that can be expressed as functions f_i of the non-generalized coordinates of the particles \mathbf{r}_i ($i = 1, \dots, N$) as $f_i(\mathbf{r}_1, \dots, \mathbf{r}_N, t) = a_i$ for $i = 1, \dots, k$, where a_i are fixed constants. In classical mechanics, as stated in Hamilton’s postulate, $L = T - U$, where T is the kinetic, and U the potential energy of a system.

If the system starts at t_1 and is at a new point at t_2 , it may have followed many different paths in **phase space**, but Hamilton’s principle states that the actual path followed is the path for which:

$$S = \int_{t_1}^{t_2} L(q_1, \dots, q_{3N-k}, \dot{q}_1, \dots, \dot{q}_{3N-k}, t) dt, \quad (1.1)$$

also known as the **action**, is minimal.

We will make use of the variational principle to derive the Lagrangian equations from equation (1.1). We will first do this in one degree of freedom, so $L = L(q, \dot{q}, t)dt$. When we add to a given $q_0(t)$ a function $\eta(t)$, which has to be continuous and differentiable, and should be zero at t_1 and t_2 , we can write $q(t) = q_0(t) + \lambda\eta(t)$, where we take λ to be small, such that $q(t)$ is only different from $q_0(t)$ slightly¹. Consequently, we find $\dot{q}(t) = \dot{q}_0(t) + \lambda\dot{\eta}(t)$. We will use this to find how S changes with λ , and then to find how S changes with respect to small variations in $q(t)$:

$$\frac{\partial S}{\partial \lambda} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \frac{\partial q}{\partial \lambda} + \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial \lambda} \right) dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \eta(t) + \frac{\partial L}{\partial \dot{q}} \frac{d\eta}{dt} \right) dt, \quad (1.2)$$

¹More mathematically, λ has to be small such that $|\lambda\eta(t)| \ll |q_0(t)|$ for all $t_1 \leq t \leq t_2$. Then it holds that $q(t) = q_0(t) + \lambda\eta(t) + (O)(\lambda^2)$ when expanding, and we neglect these higher order terms during most discussions.

where we can integrate the second term by parts to find:

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}} \frac{d\eta}{dt} dt = \left[\frac{\partial L}{\partial \dot{q}} \eta(t) \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) \eta(t) dt. \quad (1.3)$$

This can be used to find:

$$\frac{\partial S}{\partial \lambda} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \eta(t) dt. \quad (1.4)$$

When we want S to be minimal, we want its derivative with respect to λ to be 0 for any $\eta(t)$, which can only happen if:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = 0, \quad (1.5)$$

which are known as the **Lagrange equations**, because these were the equations derived by Lagrange when rewriting Newtonian mechanics.

This derivation works not only if $L = T - U$, for it only makes use of the fact $L = L(q, \dot{q}, t)$, and i.e. not $L = L(\ddot{q})$. We do not even have to know what kind of mechanical system L describes.

We can generalize the result to higher dimensions with $3N - k$ degrees of freedom, and introduce the more canonical notation for the **variation** of a quantity, δ , which rewrites Hamilton's requirement to the form:

$$\begin{aligned} \delta S &= \delta \int_{t_1}^{t_2} L(q_1, \dots, q_{3N-k}, \dot{q}_1, \dots, \dot{q}_{3N-k}, t) dt = 0 \\ &= \int_{t_1}^{t_2} \sum_{i=1}^{3N-k} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i dt = 0 \end{aligned} \quad (1.6)$$

The variation δq_i here represents all possible small deviations from $(q_0)_i$ that are zero at t_1 and t_2 . Because we need to consider all independent possible variations δq_i of all generalized coordinates, all terms within brackets should vanish, which gives $3N - k$ Lagrangian equations.

Symmetries and the Hamiltonian

We want to describe the motions of particles, and another way of doing this is by defining a generalized momentum $p_i = \frac{\partial L}{\partial \dot{q}_i}$, conjugate to the generalized coordinate q_i , as analogue of the momentum in the 'standard' coordinates. We can then rewrite Lagrange's equations as:

$$\dot{p}_i = \frac{\partial L}{\partial q_i}, \quad (1.7)$$

and when L does not explicitly depend on q_i , we find that $\dot{p}_i = 0$, so p_i is a constant of the motion. This shows that constants of motion can be connected to symmetries of the Lagrangian. If, for instance, a system is rotationally invariant, the total angular momentum is conserved, and if a system is spherically symmetric, we already know all components of the angular momentum are conserved. Usually we only consider additive conserved quantities such as (angular) momentum and energy. This gives a maximum of independent conserved quantities in a given situation. For example in three-dimensional

space we have translation and rotation symmetry, as well as independence of time, so we have seven constants of the motion.

These constants of motion lead us to believe there should be conservation of energy as well. To find it, observe the total time derivative of L :

$$\frac{dL}{dt} = \sum_i \left(\frac{\partial L}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} \right) + \frac{\partial L}{\partial t} = \sum_i \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} \right) + \frac{\partial L}{\partial t} = \sum_i \frac{d}{dt} \left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) + \frac{\partial L}{\partial t}, \quad (1.8)$$

where in the second step we used the fact that L satisfies Lagrange's equations (1.5) for each coordinate. To check for conservation of energy, we need a way to express the energy of the system, so we define an **energy function**:

$$h = h(q_1, \dots, q_{3N-k}, \dot{q}_1, \dots, \dot{q}_{3N-k}, t) = \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L. \quad (1.9)$$

Now we can write $dh/dt = -\frac{\partial L}{\partial t}$, so h is a conserved quantity if L does not explicitly depend on t . Note that this is more general than conservation of total energy. h only equals $T + U = E$ if the potential energy does not depend on generalized velocities and if the coordinate transformations back from generalized coordinates to 'normal' coordinates do not depend on time explicitly.

Hamilton expressed h in terms of p_i instead of \dot{q}_i , to be able to quantify more explicitly the dependence of h on the position in phase space. We then write H instead of h , and call it the **Hamiltonian**, which can already be compared with the Hamiltonian known from quantum mechanics. Using the definition of the generalized momentum we can show that:

$$H = \sum_i p_i \dot{q}_i - L, \quad (1.10)$$

which can be seen as the Legendre transformation of L .

The total derivative dH of the Hamiltonian leads to the canonical equations of motion, or Hamilton's equations of motion:

$$dH = \sum_{i=1}^{3N-k} (\dot{q}_i dp_i + p_i d\dot{q}_i) - dL = \sum_{i=1}^{3N-k} \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right) + \frac{\partial H}{\partial t} dt. \quad (1.11)$$

The R.H.S. denotes the general total derivative of a function dependent on q_i and p_i . Now in addition, we take the total derivative of L from its definition, substituting the definitions for the conjugate momentum and its time derivative:

$$dL = \sum_{i=1}^{3N-k} (p_i d\dot{q}_i + \dot{p}_i dq_i) + \frac{\partial L}{\partial t} dt, \quad (1.12)$$

which allows us to compare the two expressions for dH :

$$dH = \sum_{i=1}^{3N-k} (\dot{q}_i dp_i - \dot{p}_i dq_i) - \frac{\partial L}{\partial t} dt = \sum_{i=1}^{3N-k} \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i \right) + \frac{\partial H}{\partial t} dt. \quad (1.13)$$

The resulting **canonical equations of motion** are:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}; \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}; \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}. \quad (1.14)$$

Why would we, in certain situations, choose the Hamiltonian over the Lagrangian or vice versa?

The Hamiltonian forms the basis for the more advanced theories of matter, including quantum mechanics, although often we need to first find the Lagrangian before knowing the Hamiltonian. The other reason for using the Hamiltonian is that q_i and p_i drop out of the problem completely when q_i is a coordinate along a symmetry of the system.

On the other hand, the Lagrangian is independent of the choice of generalized coordinates, it is Lorentz invariant (which is a property we want when we talk about particle physics especially), and the Hamiltonian is not always Lorentz invariant. For example introducing a change of coordinates involving time will allow for conservation of H to be violated, in contrast with what was discussed above. Therefore, in particle physics and general relativity especially, we construct most of the theory using Lagrangians.

Now we have discussed in detail the main concepts of the Lagrangian formalism in Classical Mechanics we can use those concepts to construct the same kind of theory for fields, which are the basis of the SM of particle physics.

1.1.2 Field Theory

The results presented on the field theoretic matter have been based on discussions in [28], [6], [42] and [31].

A classical field theory is a mechanical system with a continuous set of degrees of freedom. Examples of field theories in general include the density of a fluid $\rho(x)$ and the electric field $\vec{E}(x)$. Field theories are often defined in terms of the Hamiltonian or Lagrangian densities \mathcal{H} and \mathcal{L} , such that we can write (in three dimensions):

$$H = \int \mathcal{H} d^3x; \quad L = \int \mathcal{L} d^3x. \quad (1.15)$$

The Lagrangian density is defined as the Legendre transform of the Hamiltonian density, and vice versa, but one of them should thus be proposed before being able to construct and derive more statements.

Let $\phi(\vec{x})$ denote a scalar field, and define $\pi(\vec{x}) = \partial_t \phi(\vec{x})(t=0)$. From a more Quantummechanical point of view we can see $\phi(\vec{x})$ as the analog of the \hat{x} operator, and $\pi(\vec{x})$ as the analog of the \hat{p} operator. Even their commutator equals: $[\phi(\vec{x}), \pi(\vec{y})] = i\delta^3(\vec{x} - \vec{y})$, encapsulating the uncertainty principle in field theory.

In terms of these defined fields we write:

$$\mathcal{L}[\phi, \dot{\phi}] = \pi[\phi, \dot{\phi}] \dot{\phi} - \mathcal{H}[\phi, \pi[\phi, \dot{\phi}]] \quad (1.16)$$

$$\mathcal{H}[\phi, \pi] = \pi \dot{\phi}[\phi, \pi] - \mathcal{L}[\phi, \dot{\phi}[\phi, \pi]], \quad (1.17)$$

Note that in the first equation $\pi[\phi, \dot{\phi}]$ is implicitly defined by $\frac{\partial \mathcal{H}[\phi, \pi]}{\partial \pi} = \dot{\phi}$, and in the second $\dot{\phi}[\phi, \pi]$ by $\frac{\partial \mathcal{L}[\phi, \dot{\phi}]}{\partial \dot{\phi}} = \pi$.

A more concrete example is found by looking at a neutral particle and writing:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \mathcal{V}[\phi] = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\vec{\nabla}\phi)^2 - \mathcal{V}[\phi], \quad (1.18)$$

where $\mathcal{V}[\phi]$ is the **potential density**. In this particular case $\pi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}$, which we use to write:

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\phi)^2 + \mathcal{V}[\phi]. \quad (1.19)$$

There are of course expressions for \mathcal{L} which do not have a nice closed form like this².

We define the types of terms in the Lagrangian as kinetic terms or interaction terms. **Kinetic** terms are defined to be bilinear, so these components have exactly two fields. They tell about non-interacting behaviour of a field, and fields with kinetic terms are said to be **propagating**. **Interaction terms** are defined as the terms having three or more fields, for which it is helpful if the coefficients are small in some sense, allowing the fields to be weakly interacting, and allowing us to do perturbation theory.

Now we have introduced the concept of Lagrangians for fields, we can derive some important relations for them. These include the Euler-Lagrange equations of motion and Noether's theorem about symmetries and conservation laws, discussed below.

1.2 Euler-Lagrange equations

As for the classical case, we can construct equations of motion from the Lagrangian. In field theory, these equations are known as the Euler-Lagrange equations, which we will now derive.

The dynamics are still found by Hamilton's principle, but now flavored into the principle of least action $\delta S = 0$ as follows:

$$\begin{aligned} 0 = \delta S &= \delta \int L dt = \delta \int \mathcal{L} d^3x = \int \left[\frac{\partial\mathcal{L}}{\partial\phi} \delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \delta(\partial_\mu\phi) \right] d^3x \\ &= \int \left\{ \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \right] \delta\phi + \partial_\mu \left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \delta\phi \right] \right\} d^3x. \end{aligned} \quad (1.20)$$

The last term here is a total derivative, and thus only depends on the boundary values. These boundaries are often taken to be at spatial and temporal infinity, where we assume fields vanish, so we can integrate by parts as in the classical case without consequences. Consequently, in a Lagrangian $A\partial_\mu B = -(\partial_\mu A)B$ holds, which we use to see, when $\delta\phi$ denotes an arbitrary (although continuous and vanishing on boundaries) variation to ϕ , that:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0 \quad (1.21)$$

²There are even situations where the Legendre transform does not exist, because it just changes $\dot{\phi}$ for a new variable called π , at each value of ϕ . Holding ϕ fixed and writing $\mathcal{L}[\dot{\phi}]$ only, nothing goes wrong in writing $\dot{\phi} = \pi$ as long as $\pi = \frac{\partial\mathcal{L}[\dot{\phi}]}{\partial\dot{\phi}}$ and $\dot{\phi}$ are one-to-one. For any function $f(x)$, the x and $\partial_x f(x)$ are in one-to-one correspondence as long as they are both convex. Thus only if \mathcal{L} is a convex function of $\dot{\phi}$ at every value of ϕ and \mathcal{H} is a convex function of π we can perform the transformation.

has to hold.

These are the **Euler-Lagrange equations**, giving the equations of motion for fields following from a given Lagrangian. We can arrive at the same results when replacing q_i with $\phi(\vec{x})$, \dot{q}_i with $\partial_\mu\phi(\vec{x})$ and $L(q_1, \dots, q_{3N-k}, \dot{q}_1, \dots, \dot{q}_{3N-k}, t)$ with $\mathcal{L}(\phi, \partial_\mu\phi)$ from our classical Lagrangian equations of motion in equation (1.5).

The Lagrangian for a particular system is clearly not unique, for multiplying by a constant and adding constant terms or the divergence of an arbitrary vector function of ϕ and $\partial_\mu\phi$ will not alter the result of the Euler-Lagrange equations. Therefore setting up a field theoretical Lagrangian is very much a practice of trial and error and dependent on searching for terms that satisfy the wanted symmetries of the theory. Only disagreement with observations can disprove a proposal for a certain Lagrangian.

Which principles we often use for setting up a Lagrangian is discussed in the next section, where we discuss Noether's theorem, linking symmetries to conservation laws.

1.3 Noether's theorem

It may happen that the Lagrangian is invariant under a specific variation $\phi = \phi_0 + \delta_\epsilon\phi$, which is then called a symmetry of the Lagrangian under certain conditions discussed below. In a generalization of the discussion about symmetries for Classical Mechanics above, there exists a theorem written by Noether in 1918, stating that when an action (and thus Lagrangian) has a symmetry, we can derive a conserved quantity.

To formulate this in a more mathematical manner we observe a scalar field $\phi(x)$ and a Lagrangian density:

$$\mathcal{L}(x) = \mathcal{L}(\phi(x), \partial_\mu\phi(x)), \quad (1.22)$$

and make a change $\phi(x) \rightarrow \phi(x) + \delta\phi(x)$ that is infinitesimal³. This implies a change in the Lagrangian $\mathcal{L}(x) \rightarrow \mathcal{L}(x) + \delta\mathcal{L}(x)$, again infinitesimal⁴. This transformation is then called a **symmetry transformation** when it can be shown, without using the equations of motion, that $\delta\mathcal{L}(x) = \partial_\mu\Lambda^\mu(x)$, where $\Lambda^\mu(x)$ is a vector.

³ A more rigid way of stating what the infinitesimal change is, is to look at a smooth one-parameter family of fields $\hat{\phi}(\alpha, x)$ for which $\hat{\phi}(0, x) = \phi(x)$, a so-called **flow of ϕ** . We define the variation of the field as:

$$\delta\phi(x) = \frac{\partial\hat{\phi}}{\partial\alpha}(0, x), \quad (1.23)$$

such that we can expand $\hat{\phi}$ around $\alpha = 0$ as:

$$\hat{\phi}(\alpha, x) = \phi(x) + \alpha\delta\phi(x) + \mathcal{O}(\alpha^2), \quad (1.24)$$

but in the main text we incorporate the α into the $\delta\phi(x)$, with α small such that $|\delta\phi(x)| \ll |\phi(x)|$.

⁴More mathematically put, the variation of the Lagrangian density is:

$$\delta\mathcal{L}(\phi(x), \partial_\mu\phi(x)) = \frac{\partial}{\partial\alpha}\mathcal{L}(\hat{\phi}(\alpha, x), \partial_\mu\hat{\phi}(\alpha, x))|_{\alpha=0}. \quad (1.25)$$

These symmetries are called **continuous** symmetries. The opposite of these are **discrete** symmetries, of which we only know three: **parity** (\hat{P} , $x \rightarrow -x$), **charge conjugation** (\hat{C} , $e^- \rightarrow e^+$) and **time reversal** (\hat{T} , $t \rightarrow -t$).

Note that symmetry variations should not be confused with the variations used in deriving the Euler-Lagrange equations, because those had to vanish at initial and final times, while the ones discussed here are usually nonzero everywhere.

Also, the exact coordinates play no role, for the action is a functional of $\phi(x)$, and the coordinates are integrated out. As we show for translations in example 1.3.2, a symmetry associated to the coordinates can be written as some transformation acting on the field.

Lastly, if an infinitesimal transformation as above is a symmetry, we may apply arbitrarily many of them to recover invariance of S under finite transformations. This allows for an entire symmetry group to exist, with generators equal to the infinitesimal symmetries found. These groups are formally known as **Lie groups**, which are mathematical groups that are also a differentiable manifold with smooth group operations. In the context of many Lagrangian theories, the generators of the Lie group are called **charges**. Later we will show that a symmetry implies a conserved current, and the thing 'flowing' in that current is this charge. For example, electric charge can be seen as the generator of the $U(1)$ symmetry of electromagnetism, where thus the electric current is conserved. The eigenvalue of the generator is referred to as the **quantum number**⁵.

We call these symmetry transformations, because we only change the Lagrangian by a surface term, and the action of our fields vanishes at the infinite boundary. Otherwise said, the equations of motion derived from the principle of least action are invariant under such a symmetry:

$$0 = \delta S = \int \delta \mathcal{L}(x) d^4x = \int \partial_\mu \Lambda^\mu(x) d^4x, \quad (1.26)$$

where due to the symmetry transformation we see a surface term as discussed, so it does not add to the action in any way, and the principle of least action in equation (1.20) still holds and produces the same Euler-Lagrange equations of motion.

Example 1.3.1. Take for example the case where $\mathcal{L} = \lambda \phi^* \phi$, where $*$ denotes complex conjugation. The transformation $\delta \phi(x) = i\alpha \phi(x)$, where $\alpha \in \mathbb{R}$, is a symmetry transformation for which $\delta \mathcal{L} = 0$, so $\Lambda^\mu = 0$. This is seen by writing:

$$\begin{aligned} \mathcal{L}(\phi + \delta \phi) &= \lambda(\phi + \delta \phi)^*(\phi + \delta \phi) = \lambda(\phi^* \phi + \phi^* \delta \phi + (\delta \phi)^* \phi + (\delta \phi)^* \delta \phi) \\ &= \mathcal{L}(\phi) + \lambda(\phi^* i\alpha \phi - i\alpha \phi^* \phi + \mathcal{O}(|\delta \phi|^2)) = \mathcal{L}(\phi), \end{aligned} \quad (1.27)$$

where we used in the last step that complex numbers commute with fields, and that $\phi^* \phi = |\phi|^2$. ◀

Example 1.3.2. To make a translation $x'^\mu = x^\mu + \epsilon^\mu$, where ϵ^μ is an infinitesimal four-vector, we introduce a $\delta \phi(x)$. Assuming $\phi(x)$ to be a scalar field, we know that $\phi'(x') = \phi(x)$ has to hold, but power expansion along ϵ^μ gives:

$$\phi'(x') = \phi'(x + \epsilon) = \phi'(x) + \epsilon^\mu \partial_\mu \phi(x), \quad (1.28)$$

⁵In representation theory, these quantum numbers correspond to the highest weight module of a given representation of the group.

when only retaining first order in ϵ^μ . This gives that $\delta\phi(x) = \phi'(x) - \phi(x) = -\epsilon^\mu \partial_\mu \phi(x)$. If for example the Lagrangian is given as:

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi \partial_\mu \phi - \frac{m^2}{2} \phi^2, \quad (1.29)$$

it follows that:

$$\delta\mathcal{L} = \partial^\mu \phi \partial_\mu \delta\phi - m^2 \phi \delta\phi = -\epsilon^\mu \partial_\mu \left(\frac{1}{2} \partial^\lambda \phi \partial_\lambda \phi - \frac{m^2}{2} \phi^2 \right) = -\epsilon^\mu \partial_\mu \mathcal{L} = \partial_\mu (-\epsilon^\mu \mathcal{L}), \quad (1.30)$$

because ϵ^μ is a constant. ◀

Now using the equations of motion to find the theoretical expression for $\delta\mathcal{L}(x)$ (already seen in equation (1.20)):

$$\begin{aligned} \delta\mathcal{L}(x) &= \frac{\partial\mathcal{L}(x)}{\partial\phi(x)} \delta\phi(x) + \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi(x))} \delta(\partial_\mu\phi(x)) \\ &= \left[\frac{\partial\mathcal{L}(x)}{\partial\phi(x)} - \partial_\mu \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi(x))} \right] \delta\phi(x) + \partial_\mu \left[\frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi(x))} \delta\phi(x) \right] \\ &= \partial_\mu \left(\frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi(x))} \delta\phi(x) \right), \end{aligned} \quad (1.31)$$

which, combined with the definition for a symmetry transformation, gives us:

$$\partial_\mu j^\mu \equiv \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \delta\phi - \Lambda^\mu \right) = 0. \quad (1.32)$$

This object j^μ is also called the **Noether current** associated to the symmetry $\delta\phi(x)$, which is thus conserved. Thus knowing which particles to describe and which symmetries to fulfill is enough to write down the complete non-interacting Lagrangian for a theory.

Example 1.3.3. For the second example above, we had the kinetic term and mass term in the Lagrangian, and found $\Lambda^\mu = -\epsilon^\mu \mathcal{L}$, and $\delta\phi(x) = -\epsilon^\mu \partial_\mu \phi(x)$, which we can insert in our expression for j^μ to find:

$$j^\mu = -\epsilon^\mu \mathcal{L}(x) + \epsilon^\nu \partial_\nu \phi(x) \frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi(x))} = \epsilon^\nu \left(\frac{\partial\mathcal{L}(x)}{\partial(\partial_\mu\phi(x))} \partial_\nu \phi(x) - \delta_\nu^\mu \mathcal{L}(x) \right), \quad (1.33)$$

where δ_ν^μ is the Kronecker delta. The term between brackets is often called the **Energy-momentum tensor** of the Lagrangian. ◀

Example 1.3.4. Consider another symmetry of the Lagrangian from the last example, now written somewhat differently:

$$\mathcal{L} = -\partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi, \quad (1.34)$$

signifying a complex scalar field with an interaction term.

This Lagrangian is left invariant by the $U(1)$ -transformation (Unitary 1-dimensional matrix, thus all complex numbers with absolute value equal to 1) $\phi \rightarrow e^{-i\alpha} \phi$ for any real number α . Infinitesimally, (retaining only terms of first order in α) this looks like:

$$\begin{aligned} \phi(x) &\rightarrow \phi(x) - i\alpha\phi(x) \Rightarrow \delta\phi(x) = -i\alpha\phi(x); \\ \phi^*(x) &\rightarrow \phi^*(x) + i\alpha\phi^*(x) \Rightarrow \delta\phi^*(x) = i\alpha\phi^*(x), \end{aligned} \quad (1.35)$$

and because $\delta\mathcal{L} = 0$ up to first order in α in this case, the Noether current becomes:

$$\alpha j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\delta\phi^* = (-\partial^\mu\phi^*)(-i\alpha\phi) + (-\partial^\mu\phi)(i\alpha\phi^*) = i\alpha[(\partial^\mu\phi^*)\phi - \phi^*\partial^\mu\phi]. \quad (1.36)$$

We can see that:

$$\partial_\mu j^\mu = i\alpha[\phi\Box\phi^* + \partial_\mu\phi\partial^\mu\phi^* - \partial_\mu\phi^*\partial^\mu\phi - \phi^*\Box\phi] = i\alpha[\phi\Box\phi^* - \phi^*\Box\phi] = 0, \quad (1.37)$$

where \Box denotes the d'Alembertian, $\partial^\mu\partial_\mu$, when $\Box\phi = -m^2\phi$ and $\Box\phi^* = -m^2\phi^*$ are satisfied, which are just the results from the Euler-Lagrange equations for these fields. \blacktriangleleft

To discuss how we practice physics with Lagrangians and, more importantly, how we know which terms to put in our Lagrangians, we will discuss some common fields in particle physics and how we deal with setting up a Lagrangian theory for them in the next section.

1.4 Free Fields

There are three main types of fields that are of interest in the SM. These are scalar fields (of which we have seen more in the previous sections), vector fields and spinor fields, or Dirac fields. Scalar fields describe spin-0 particles, vector fields spin-1 particles and Dirac fields describe spin-1/2 fermions⁶. We will discuss main results about these fields shortly. For a more in-depth discussion see [28].

1.4.1 Scalar Fields

The Lagrangian for a free real scalar field describing neutral spinless particles with mass m has been discussed in the examples in the last section, and this Lagrangian yields the **Klein-Gordon equations** as field equation:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 \Rightarrow (\Box + m^2)\phi = 0. \quad (1.38)$$

This form is a generalization of the Lagrangian we have from Classical Mechanics, and incorporates exactly the same symmetries we want, so it represents the most easy way to describe our system. This form will be generalized for other fields as well, with the same train of thought.

A complex scalar field describes spinless particles with charge, and the two degrees of freedom (real and imaginary parts) can be interpreted as particles and antiparticles. The same Klein-Gordon equation holds for the Lagrangian:

$$\mathcal{L} = (\partial_\mu\phi)^*(\partial^\mu\phi) - m^2\phi^*\phi. \quad (1.39)$$

Both have solutions that we can expand in terms of a complete set of plane waves, with coefficients that can be compared to the annihilation and creation operators from quantum mechanics.

⁶The reason why these objects describe the given fields with the given charges and spins is a more complex mathematical concept outside the scope of this thesis. It is heavily dependent on representation theory of groups, where in this case the groups in consideration are the Lie groups mentioned above.

1.4.2 Vector Fields

A vector field $A_\mu(x)$ describes particles with spin 1, whose states can be described by a momentum k and a helicity. This helicity can be $\lambda = \pm 1, 0$ for massive particles, and can only be $\lambda = \pm 1$ for massless particles. We want Lorentz invariance to hold, so we can not do the same as for the scalar field Lagrangian. We introduce $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, which is Lorentz invariant, to be able to write:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m^2}{2}A_\mu A^\mu, \quad (1.40)$$

where this enforces that $\partial_\mu A^\mu = 0$ has to hold, eliminating one degree of freedom from A_μ , thus there are three for massive spin 1 left⁷.

This Lagrangian (also known as the Proca Lagrangian) then yields the field equations of motion:

$$[(\square + m^2)g^{\mu\nu} - \partial^\mu \partial^\nu]A_\nu = 0, \quad (1.46)$$

where $g^{\mu\nu}$ is the metric tensor of the underlying space-time.

Again plane waves of the form $\epsilon_\mu^{(\lambda)} e^{\pm i k x}$ where $\epsilon_\mu^{(\lambda)}$ can be one of three linearly independent polarization vectors, form a complete set of solutions. The polarization vectors have to be transverse and can be chosen orthogonal, normalized and fulfilling the polarization sum:

$$\epsilon^{(\lambda)} \cdot k = 0; \quad \epsilon^{(\lambda)*} \cdot \epsilon^{(\lambda')} = -\delta^{\lambda\lambda'}; \quad \sum_{\lambda=1}^3 \epsilon_\mu^{(\lambda)*} \epsilon_\nu^{(\lambda)} = -g_{\mu\nu} + \frac{k_\mu k_\nu}{m^2}. \quad (1.47)$$

In the massless case we regain Maxwell's equations:

$$(\square g^{\mu\nu} - \partial^\mu \partial^\nu)A_\nu = 0, \quad (1.48)$$

where we know from electrodynamics that there are only two polarization vectors, thus we have an additional degree of freedom, which can be removed by a gauge transformation. Because of this, we need to fix a gauge by adding a term to the Lagrangian. More on gauge symmetry is discussed below.

⁷The more natural guess would have been to say:

$$\mathcal{L} = -\frac{1}{2}\partial_\nu A_\mu \partial^\nu A^\mu + \frac{1}{2}m^2 A_\mu^2, \quad (1.41)$$

but finding the energy density gives:

$$\mathcal{E} = \frac{\partial \mathcal{L}}{\partial(\partial_t A_\mu)} \partial_t A_\mu - \mathcal{L}, \quad (1.42)$$

which yields negative signs for the energy, so this Lagrangian would not produce a physical theory. There is one more Lorentz-invariant two-derivative kinetic term: $A_\mu \partial^\mu \partial_\nu A^\nu$, or $A_\mu \square A^\mu$. Writing

$$\mathcal{L} = \frac{a}{2}A_\mu \square A^\mu + \frac{b}{2}A_\mu \partial^\mu \partial_\nu A^\nu + \frac{1}{2}m^2 A_\mu^2, \quad (1.43)$$

we find equations of motion as:

$$a \square A_\mu + b \partial_\mu \partial_\nu A^\nu + m^2 A_\mu = 0, \quad (1.44)$$

and taking ∂^μ gives

$$[(a+b)\square + m^2]\partial^\mu A_\mu = 0, \quad (1.45)$$

and thus when $a = -b = 1$ and $m \neq 0$ this reduces to $\partial_\mu A^\mu = 0$, which is Lorentz invariant, thus we arrive on the given Lagrangian.

1.4.3 Spinor (Dirac) Fields

Particles that have spin 1/2 (like electrons and neutrinos) are described by spinor fields $\psi(x) = (\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x))$, which consist of four components. These four components can be interpreted as the spin up and spin down states of the particle, and the spin up and spin down states of the antiparticle. This theory was proposed by Dirac when trying to find a relativistic version of Schrödinger's equation. The Klein-Gordon equation from above seemed not to reproduce the same interpretation as we knew from Quantum Mechanics beforehand, and thus Dirac proposed a new Hamiltonian (for particles with mass m):

$$\mathcal{H} = \alpha \cdot \hat{\mathbf{p}} + \beta m, \quad (1.49)$$

where, when trying plane wave solutions, we find conditions that:

$$\alpha_i \alpha_j + \alpha_j \alpha_i = 2\delta_{ij} \mathbf{1}; \quad \alpha_i \beta + \beta \alpha_i = 0; \quad \beta^2 = \mathbf{1}, \quad (1.50)$$

which cannot be satisfied if α_i or β are numbers, they should even be 4×4 matrices. To then restore the symmetry between ∂_t and ∂_x derivatives as in Quantum mechanics, the gamma-matrices are introduced:

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix}; \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad (1.51)$$

where σ_i are the Pauli matrices and $\mathbf{1}$ represents the 2×2 identity matrix.

Again, intuitively, one would write down a Lagrangian of the sort:

$$\mathcal{L} = \partial_\mu \bar{\psi} \partial^\mu \psi - m \bar{\psi} \psi, \quad (1.52)$$

where $\bar{\psi} = \psi^\dagger \gamma^0 = (\psi^*)^T \gamma^0$ ($.^T$ meaning the transpose) signifies the Dirac conjugate, and we can state $\bar{\psi} \psi \sim \psi^2$. This Lagrangian, however, describes not the interpretation we offered for a spinor field above when we look at the details, for it leads to a Hamiltonian unbounded from below.

To get a bounded Hamiltonian, the kinetic term needs to contain ψ and ψ^\dagger . A candidate thus is the Lorentz invariant and hermitian (up to a total divergence) term: $i\bar{\psi} \gamma^\mu \partial_\mu \psi$. If we use the newly proposed interaction term to get:

$$\mathcal{L} = i\bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi \quad (1.53)$$

we regain the Dirac equations:

$$(i\gamma^\mu \partial_\mu - m) \psi = 0 \quad (1.54)$$

again, which confirms this to be one viable Lagrangian.

Two types of solutions exist, corresponding to particle and antiparticle wave functions, which are often denoted by u and v respectively, and \bar{u} and \bar{v} then form the other two components of the four-vector.

Now we have discussed the main fields of importance in particle physics briefly, we will go on to discuss symmetry breaking. We will thus refer the reader to the Appendix for more information on Gauge symmetries, which is another type of symmetry. These can be linked to local transformations, thus the transformation can explicitly depend on the coordinates.

We know which symmetries we discern between and how we use them to be able to construct 'convincing' Lagrangians, and we will continue to see what happens when this symmetry is 'broken' in the next section, which is of importance for how lepton masses appear in the SM.

1.5 Symmetry Breaking

The discussions on symmetries, as well as gauge symmetries in this thesis (appendix included) are based on [14], [36], [10] and [42].

Oftentimes a breaking of symmetry goes alongside an addition of a new term to the Lagrangian, for instance when new interactions start to play a role. We will show how symmetry breaking can generate masses for fermions without affecting gauge invariance in the next section, but first we need to explain the concept of symmetry breaking.

There are two kinds of symmetry breaking. First is explicit symmetry breaking, where terms in the equations of motion do not respect the symmetry. This often happens in situations where the symmetry-breaking terms are relatively small. One example of this is the Zeeman effect, where, due to an added magnetic interaction perturbation in the Hamiltonian, the spectral lines of the atoms involved split. In spontaneous symmetry breaking, the ground state, or vacuum, of the theory breaks the symmetry. When the system goes to one of these vacuum solutions, the symmetry is broken for perturbations around that vacuum even though the Lagrangian retains the symmetry.

First we will discuss the case where global symmetries are broken, and then we will discuss the Higgs mechanism, which comes into play when local (gauge) symmetries are broken. We will do this by looking at multiple examples again.

Example 1.5.1. One of the most common examples of symmetry breaking is found in the transition from paramagnetism to ferromagnetism in a simple spin-spin interaction model like the Ising model. This system has a critical temperature T_C above which all spins are disordered completely, making the system (and thus the vacuum-state) $SO(3)$ -invariant ($SO(n)$ is the **special orthogonal group**, containing $n \times n$ orthogonal matrices with determinant 1, also called the rotation group). A spontaneous magnetization of this system occurs below T_C , aligning all spins along some axis, making the vacuum not invariant under $SO(3)$ anymore. Instead, the system is $SO(2)$ -invariant ($SO(2)$ is isomorphic to $U(1)$), signifying rotations along the spin-axis. ◀

We will observe a more general example of a discrete symmetry breaking, and then for a continuous symmetry breaking.

Example 1.5.2. Take for example a scalar, self-interacting real field with the general Lagrangian:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi); \quad V(\phi) = \frac{1}{2} \mu^2 \phi^2 + \frac{1}{4} \lambda \phi^4. \quad (1.55)$$

In the Ising model discussed above, ϕ takes the role of the average magnetization M , and $V(\phi)$ is known as the Gibbs free energy in statistical physics.

This Lagrangian is invariant under $\phi \rightarrow -\phi$, but we do not yet know if the vacuum state, ϕ_0 is. A constant ϕ_0 at the minimum of $V(\phi)$ implies:

$$\frac{\partial V(\phi)}{\partial \phi} \Big|_{\phi=\phi_0} = 0 \Rightarrow \mu^2 \phi_0 + \lambda \phi_0^3 = 0 \Rightarrow \phi_0(\mu^2 + \lambda \phi_0^2) = 0 \Rightarrow \phi_0^\pm = \pm \sqrt{-\frac{\mu^2}{\lambda}} \text{ if } \mu^2 < 0. \quad (1.56)$$

To keep the Hamiltonian bounded in this case (in the same sense as in the discussion about Vector Fields above, the Hamiltonian would otherwise be unbounded from below), λ should stay positive, thus the exact minimum depends on the sign of μ^2 . If $\mu^2 > 0$, there is only a minimum at $\phi_0 = 0$,

and it is invariant under the given transformation. However, when $\mu^2 < 0$, two vacuum states exist, corresponding to the given minima above. Of course when $\mu^2 < 0$, μ itself is complex, so this case corresponds to a wrong sign when μ should signify a mass. The choice between ϕ_0^+ and ϕ_0^- seems to be irrelevant because the Lagrangian is invariant under the given symmetry. Nevertheless one of the states is chosen, and thus the symmetry is spontaneously broken since \mathcal{L} is invariant, but the vacuum is not.

We may define a new field $\phi' = \phi - v$ by shifting the old field by $v = \sqrt{-\mu^2/\lambda}$, which has $\phi'_0 = 0$, giving a theory suitable for small oscillations around the vacuum state, and we get a new Lagrangian:

$$\mathcal{L}' = \frac{1}{2}\partial_\mu\phi'\partial^\mu\phi' - \frac{1}{2}(\sqrt{-2\mu^2})^2\phi'^2 - \lambda v\phi'^3 - \frac{1}{4}\lambda\phi'^4, \quad (1.57)$$

describing a scalar field ϕ' with real and positive mass $\sqrt{-2\mu^2}$, but having lost the original symmetry because of the ϕ'^3 -term. ◀

This same sort of phenomenon also occurs when a continuous symmetry is broken spontaneously as seen in the next example, which introduces a particular application of Goldstone's theorem:

Example 1.5.3. Consider a charged, self-interacting scalar field with Lagrangian:

$$\mathcal{L} = \partial_\mu\phi^*\partial^\mu\phi - V(\phi^*\phi); \quad V(\phi^*\phi) = \mu^2\phi^*\phi + \lambda(\phi^*\phi)^2, \quad (1.58)$$

with a similar potential as the last example. We already know this to be invariant under the global transformation $\phi \rightarrow e^{-i\alpha}\phi$. We can redefine the field in terms of two real ones as follows:

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}}, \quad (1.59)$$

after which the Lagrangian becomes invariant under $SO(2)$ -rotations, for $\theta \in [0, 2\pi]$:

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \quad (1.60)$$

because we can write:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi_1\partial^\mu\phi_1 + \partial_\mu\phi_2\partial^\mu\phi_2) - V(\phi_1, \phi_2). \quad (1.61)$$

Again, if $\mu^2 > 0$, the vacuum exists at $\phi_1 = \phi_2 = 0$ and we have two scalar fields with mass $m^2 = \mu^2 > 0$. However, when $\mu^2 < 0$, there is a continuum of distinct vacua at:

$$|\phi|^2 = \frac{\phi_1^2 + \phi_2^2}{2} = -\frac{\mu^2}{2\lambda} = \frac{v^2}{2}. \quad (1.62)$$

Drawing a contour plot of this shows that these minima are invariant under $SO(2)$ as well, but it is spontaneously broken when we choose a specific vacuum, i.e. $\phi_1 = v$ and $\phi_2 = 0$. Then the new fields for small perturbations are $\phi'_1 = \phi_1 - v$ and $\phi'_2 = \phi_2$, which means the new Lagrangian becomes:

$$\mathcal{L}' = \frac{1}{2}\partial_\mu\phi'_1\partial^\mu\phi'_1 - \frac{1}{2}(-2\mu^2)\phi'^2_1 + \partial_\mu\phi'_2\partial^\mu\phi'_2 + \mathcal{L}_{int}, \quad (1.63)$$

where \mathcal{L}_{int} incorporates the interaction terms. Thus ϕ'_1 describes a scalar field with real and positive mass and ϕ'_2 is a massless scalar field signifying a boson. ◀

This is an example of what Goldstone's theorem predicts. For a set of N_G real scalar fields $\phi^i(x)$ contained in an N_G -dimensional vector Φ the Lagrangian has the following form:

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi - V(\Phi), \quad (1.64)$$

where a minimum of the potential is a constant $\Phi_0 = (\phi_0^1, \phi_0^2, \dots, \phi_0^{N_G})$ for which:

$$\left. \frac{\partial}{\partial \phi^i} V(\Phi) \right|_{\Phi=\Phi_0} = 0 \quad \forall i. \quad (1.65)$$

Now Taylor-expanding this potential around this minimum results in:

$$V(\Phi) = V(\Phi_0) + \frac{1}{2} (\phi^i - \phi_0^i) (\phi^j - \phi_0^j) \left(\frac{\partial^2 V}{\partial \phi^i \partial \phi^j} \right) \Big|_{\Phi=\Phi_0} + \dots, \quad (1.66)$$

where we define:

$$M_{ij}^2 = \left. \frac{\partial^2 V(\Phi)}{\partial \phi^i \partial \phi^j} \right|_{\Phi=\Phi_0}, \quad (1.67)$$

making a symmetric matrix (due to the commutation of partial derivatives) with mass terms (due to the structure of the potential) for the fields as eigenvalues, all non-negative since it is evaluated at Φ_0 , [36]. Now a general continuous symmetry T due to a group G with N_G generators T^i is given by:

$$\delta \Phi = -i \alpha_a T^a \Phi, \quad (1.68)$$

where α_a are corresponding infinitesimal parameters and T^a can be functions of the ϕ^i .

Now **Goldstone's theorem** states:

*When an exact global symmetry is broken spontaneously (thus it is not a symmetry of the physical vacuum) the theory contains one massless (and spinless) scalar particle for each broken generator of the original symmetry group. These massless particles are called **Goldstone Bosons**.*

In terms of the mathematical definitions above this becomes:

Theorem 1.5.1. (Goldstone, 1961): *Let \mathcal{L} be a Lagrangian left invariant under a group G with N_G generators T^a . Let such a transformation be given as in eq.(1.68). If, after 'choosing' a new vacuum when the symmetry is spontaneously broken, a sub-group A of G with dimension N_A remains a symmetry of the new vacuum, for each generator of A it need not necessarily be true that $M_{ij}^2 = 0$, while for the $N_G - N_A$ generators that break the symmetry it has to be, meaning that there are $N_G - N_A$ zero eigenvalues of the mass matrix.*

Proof: Since the potential is invariant under G , it needs to hold that:

$$V(\Phi) = V(\Phi + \delta \Phi) = V(\Phi) + \delta V(\Phi) = V(\Phi) + \frac{\partial V(\Phi)}{\partial \phi^i} \delta \phi^i = V(\Phi) - i \frac{\partial V(\Phi)}{\partial \phi^i} \alpha_a (T^a)_{ij} \phi^j, \quad (1.69)$$

where the parameters α_a are arbitrary and we thus get N_G equations:

$$\frac{\partial V(\Phi)}{\partial \phi^i} (T^a)_{ij} \phi^j = 0, \quad (1.70)$$

which we differentiate with respect to ϕ^k , to obtain:

$$0 = \frac{\partial^2 V}{\partial \phi^k \partial \phi^i} (T^a)_{ij} \phi^j + \frac{\partial (T^a)_{ij}}{\partial \phi^k} \frac{\partial V}{\partial \phi^i} \phi^j + \frac{\partial V}{\partial \phi^i} (T^a)_{ij} \delta_k^j. \quad (1.71)$$

Evaluating this at $\Phi = \Phi_0$, which minimizes the potential, we are left with only:

$$0 = \left. \frac{\partial^2 V(\Phi)}{\partial \phi^k \partial \phi^i} \right|_{\Phi=\Phi_0} (T^a)_{ij} \phi_0^j = M_{ki}^2 (T^a)_{ij} \phi_0^j. \quad (1.72)$$

If the N_A generators of G also contained in A remain a symmetry, then $(T^a)_{ij} \phi_0^j = 0$ for $a = 1, \dots, N_A \leq N_G$, meaning that M_{ki}^2 need not necessarily be zero for those dimensions. For the $N_G - N_A$ generators that break the symmetry we have the reverse case, for there $(T^a)_{ij} \phi_0^j \neq 0$, thus M_{ik}^2 should be zero there, meaning that there are $N_G - N_A$ zero eigenvalues of the mass matrix. \square

1.5.1 Higgs Mechanism

For local gauge theories that are spontaneously broken, no Goldstone bosons have been observed to appear. A system undergoing such a phenomenon is said to be in the Higgs mode instead of the Goldstone mode. In 1964 multiple authors were able to independently devise a method around this discrepancy. This method, the so-called **Higgs mechanism**, gives massive gauge bosons as a result. This is done by requiring that the Lagrangian exhibiting spontaneous symmetry breaking is not only invariant under the broken global symmetry, but also invariant under local gauge transformations. This is done in the SM by incorporating a new field, the **Higgs field**, with a mexican-hat like term in the Lagrangian.

This fits with the observation that electroweak interactions need, on short-range, a very massive intermediate particle which only locally breaks symmetry. An even more general discussion can be found in [36].

Suppose we have a set of N scalar fields ϕ_i appearing in a Lagrangian invariant under a group of continuous symmetries G generated by t^i :

$$\phi_i \rightarrow (1 + i\alpha t^a)_{ij} \phi^j. \quad (1.73)$$

We can rewrite this to $2N$ real-valued fields describing the system as was done before, and then it follows that the t^a should all be purely imaginary and antisymmetric, so we write them as $t_{ij}^a = iT_{ij}^a$ where T^a is real and antisymmetric. By applying the gauge principle to get a gauge symmetry from this global symmetry:

$$D_\mu \phi = (\partial_\mu - ieA_\mu^a t_a) \phi = (\partial_\mu + eA_\mu^a T_a) \phi, \quad (1.74)$$

making the kinetic term in the Lagrangian equal to:

$$\frac{1}{2} (D_\mu \phi_i)^2 = \frac{1}{2} (\partial_\mu \phi_i)^2 + eA_\mu^a (\partial^\mu \phi_i T_a^{ij} \phi_j) + \frac{1}{2} e^2 A_\mu^a A^{b\mu} (T_a \phi)_i (T_b \phi)^i. \quad (1.75)$$

Letting ϕ_i acquire a non-trivial lowest energy configuration by 'choosing' a new vacuum, breaking some symmetries and expanding fields around this new specific ground state leads to a mass-term already seen in the last equation:

$$\Delta \mathcal{L} = \frac{1}{2} m_{ab}^2 A_\mu^a A^{b\mu}, \quad m_{ab}^2 = e^2 (T_a \phi_0)_i (T_b \phi_0)^i. \quad (1.76)$$

This mass matrix is positive-semidefinite⁸ since all diagonal elements are squares of real numbers. Consequently, all eigenvalues are non-negative.

If T^a left the ground state invariant the generator does not contribute to the mass matrix, meaning the corresponding boson is massless, but for generators of the broken symmetry it acquires mass. This principle is explored in the following example:

Example 1.5.4. Take into account the same Lagrangian as in example 1.5.3 with the corresponding symmetry breaking and 'choice' of new vacuum. The spontaneous symmetry breaking again occurs at $\mu^2 < 0$, with the vacuum again at $\phi_0 = v^2/2$. The new Lagrangian, after writing the field as sum of two real fields and using q to denote electrical charge:

$$\begin{aligned} |D_\mu\phi|^2 &= |(\partial_\mu - iqA_\mu)\phi|^2 \\ &= \frac{1}{2} [(\partial_\mu\phi'_1 + qA_\mu\phi'_2)^2 + (\partial_\mu\phi'_2 - qA_\mu\phi'_1)^2] - qvA^\mu(\partial_\mu\phi'_2 + qA_\mu\phi'_1) + \frac{q^2v^2}{2}A^\mu A_\mu. \end{aligned} \quad (1.77)$$

Thus the Lagrangian becomes:

$$\mathcal{L} = \frac{1}{2} [\partial_\mu\phi'_1\partial^\mu\phi'_1 + 2\mu^2\phi'^2_1 + \partial_\mu\phi'_2\partial^\mu\phi'_2] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{q^2v^2}{2}A_\mu A^\mu + qvA_\mu\partial^\mu\phi'_2 + \text{interact.} \quad (1.78)$$

This represents a scalar field ϕ'_1 with mass $\sqrt{-2\mu^2}$, a massless scalar boson ϕ'_2 and a massive vector boson A_μ with mass ev .

However, the term $qvA_\mu\partial^\mu\phi'_2$ mixes A_μ and ϕ'_2 , making the interpretation less clear. To remove this mixing term, we can choose (fixing the gauge), for small perturbations around $\phi'_1 = v$ and $\phi'_2 = 0$ (meaning $e^{i\phi'_2/v} \approx 1$):

$$\alpha(x) = -\frac{1}{qv}\phi'_2(x) \Rightarrow \phi' = e^{-iq\frac{\phi'_2}{qv} + i\frac{\phi'_2}{v}} \frac{\phi'_1 + v}{\sqrt{2}} = \frac{1}{\sqrt{2}}(\phi'_1 + v). \quad (1.79)$$

This results in the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi'_1\partial^\mu\phi'_1 - \frac{1}{2}(-2\mu^2)\phi'^2_1 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{q^2v^2}{2}A'_\mu A'^\mu + \frac{1}{2}q^2(\phi'_1 + 2v)\phi'_1 A'_\mu A'^\mu - \frac{\lambda}{4}\phi'^3_1(\phi'_1 + 4v). \quad (1.80)$$

The term ϕ'_2 has disappeared, and when checking where the degrees of freedom for the theory are, we see that A'_μ has three degrees of freedom, where A_μ had 2, so the Goldstone boson has been absorbed by the massive gauge boson.

In Cheng-Li (the following is modulo minus signs) we see that we can remove the mixing term by parametrizing the complex field in polar variables instead, only shifting the modulus term:

$$\phi(x) = \frac{1}{\sqrt{2}}[v + \eta(x)]e^{i\xi(x)/v} \approx \frac{1}{\sqrt{2}}[v + \eta(x) + i\xi(x) + \dots], \quad (1.81)$$

where the approximation holds for 'small' oscillations η and ξ , meaning they are, in the infinitesimal regime, really ϕ'_1 and ϕ'_2 . The free Lagrangian thus retains the same form, and we can fix the gauge by defining:

$$\phi^v(x) = e^{-i\xi/v}\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x)); \quad B_\mu(x) = A_\mu(x) - \frac{1}{qv}\partial_\mu\xi(x), \quad (1.82)$$

⁸ $z^T M z \geq 0$ ($z^* M z \geq 0$) for all real (complex) non-zero n -dimensional vectors z and M a real (Hermitian) $n \times n$ matrix

giving:

$$\begin{aligned}
D_\mu\phi &= e^{i\xi/v}(\partial_\mu\phi^v - iqB_\mu\phi^v) = e^{-i\xi/v}(\partial_\mu\eta - iqB_\mu(v + \eta))/\sqrt{2}; \\
|D_\mu\phi|^2 &= \frac{1}{2}|\partial_\mu\eta - iqB_\mu(v + \eta)|^2; \\
F_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu; \\
\mathcal{L} &= \frac{1}{2}|D_\mu\phi|^2 + \frac{\mu^2}{2}(v + \eta)^2 - \frac{\lambda}{4}(v + \eta)^4 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = \mathcal{L}_0 + \mathcal{L}_1; \\
\mathcal{L}_0 &= \frac{1}{2}(\partial_\mu\eta)^2 - \mu^2\eta^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}q^2v^2B_\mu B^\mu; \\
\mathcal{L}_1 &= \frac{1}{2}q^2B_\mu B^\mu\eta(2v + \eta) - \lambda v^2\eta^3 - \frac{1}{4}\lambda\eta^4,
\end{aligned} \tag{1.83}$$

where \mathcal{L}_0 is the free Lagrangian for a massive vector boson with mass ev and a scalar field with mass $\sqrt{-2\mu^2}$. The field $\xi(x)$ has disappeared. This is less surprising when counting degrees of freedom. Before the breaking we had two scalar fields ϕ_1 and ϕ_2 and a massless boson A_μ (two polarization states). After breaking, one field is left, along with a massive boson with three polarization states. ◀

We can now present a general way of constructing a gauge theory which uses both the gauge principle and the principle of symmetry breaking or the Higgs principle. This procedure follows a few steps: First we choose the gauge group G with N_G generators and add N_G vector fields in a specific representation of this gauge group, giving the gauge bosons. Then we add scalar fields to give mass to some of these vector fields and define a covariant derivative coupling all these fields. Then we can write the most general Lagrangian, and can afterwards shift the fields such that the minimum of the potential is again at 0. Then the predictions found from the new Lagrangian can be checked with reality by experiment, and new models should be constructed along this same way but with other numbers if these predictions do not agree with reality.

Having introduced the general way of approaching gauge theories, we can introduce in the final section how fermions get their masses in the SM. Building on that knowledge we can introduce our possible models for massive neutrinos.

1.6 Fermion Masses in the Standard Model

The discussion from this section onward is based mainly on [8], [19], [46], [36], [34] and [21].

Now we have introduced the concepts of symmetry breaking and the Higgs mechanism, we can discuss how fermions get their masses in the Lagrangian of the SM. This can not be done by using a conventional mass term due to their underlying structure with respect to other symmetries, which we will discuss now. Having done this we can continue to introduce the possible mass terms for neutrinos based on the rules and concepts we find along the way.

We start again with the spinor fields from before, describing spin-1/2-particles. As in Quantum Mechanics, we represent the spin up (+1/2) and spin down (-1/2) states of a fermion along a direction in a complex doublet $|\psi\rangle = \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix}$. As discussed, this can be seen back in the Dirac spinor as **left- and right handed chiral components**⁹, where each of the two components⁹ can be interpreted as a

⁹What is the difference between chirality, helicity, and what does it have to do with parity? If the spin has the same orientation of your right hand (When putting your thumb along the direction of movement your fingers wrap in the

particle and an antiparticle respectively. We encountered the gamma matrices there, and will now need a fifth one:

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad (1.84)$$

which we use to define a **chiral projection operator** to project a spinor on its left- and right-handed components¹⁰. We thus introduce: $\psi_{R/L}(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix}$ for the left- and righthanded states. The projection operator has the following form, and the following conditions hold¹¹:

$$P_{L/R} = \frac{1}{2}(1 \mp \gamma_5); \quad P_L P_R = P_R P_L = 0; \quad P_{L/R}^2 = P_{L/R}; \quad P_L + P_R = 1. \quad (1.85)$$

and we want them to act on ψ as follows:

$$\psi_L = P_L \psi; \quad \psi_R = P_R \psi; \quad \bar{\psi}_L = \bar{\psi} P_R; \quad \bar{\psi}_R = \bar{\psi} P_L; \quad \psi = P_L + P_R. \quad (1.86)$$

The relation for the conjugate spinors can be found by working through:

$$\bar{\psi}_L = (P_L \psi)^\dagger \gamma_0 = \psi^\dagger P_L^\dagger \gamma_0 = \psi^\dagger P_L \gamma_0 = \psi^\dagger \gamma_0 P_R = \bar{\psi} P_R \quad (1.87)$$

Recall the Dirac Lagrangian in equation (1.53), where we introduce the slash notation for ∂ : $\gamma^\mu \partial_\mu = \not{\partial}$, and rewrite in terms of ψ_L and ψ_R :

$$\mathcal{L} = (\bar{\psi}_L + \bar{\psi}_R)(i\not{\partial} - m)(\psi_L + \psi_R) = i(\bar{\psi}_L \not{\partial} \psi_L + \bar{\psi}_R \not{\partial} \psi_R) - m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L). \quad (1.88)$$

The other terms are all equal to zero:

$$\begin{aligned} \bar{\psi}_L \psi_L &= \bar{\psi} P_R P_L \psi = 0 = \bar{\psi}_R \psi_R; \\ \bar{\psi}_R \not{\partial} \psi_L &= \bar{\psi} P_L \not{\partial} P_L \psi = \bar{\psi} \not{\partial} P_R P_L \psi = 0 = \bar{\psi}_L \not{\partial} \psi_R. \end{aligned} \quad (1.89)$$

As you can see in this new formula for the Lagrangian, the mass term, which is also the mass term for fermions, mixes the left- and right-handed components.

Experimentally, as found in the timeline in [34], Fermi wanted to describe the observed β -decay of the neutron ($n \rightarrow p + e^- + \nu_e$) by a current-current Lagrangian:

$$\mathcal{L}_{Fermi} = \frac{G_F}{\sqrt{2}} (\bar{\psi}_p \gamma_\mu \psi_n) (\bar{\psi}_e \gamma^\mu \psi_\nu) \rightarrow \mathcal{L}_{Gamow-Teller} \frac{G_F}{\sqrt{2}} \sum_i C_i (\bar{\psi}_p \Gamma^i \psi_n) (\bar{\psi}_e \Gamma^i \psi_\nu). \quad (1.90)$$

direction the particle spins.) the **helicity** is 'right-handed'. If you now only flip the direction of movement (cq. use the parity operator), you end up with a particle of the opposite handedness, for the spin does not transform along. When a particle has mass, another particle could catch up to it and pass it, making the first particle look as if it goes the other way, so helicity is not intrinsic to the particle. Therefore we introduce **chirality**, which is equivalent to helicity in the massless limit, and is something all observers would measure to be the same. In the Dirac spinor we contain the left- and right helicity states of the particle and the antiparticle, and thus it is helicity based, whereas thus the chiral projection operators project onto chiral components.

¹⁰The structure with these gamma-matrices can be derived from using symmetry arguments, and is dependent on specific representations of the Lorentz group containing three rotations and three boosts. This is also where the structure of left- and right-handedness arises. For a clearer digression see [42] page 166.

¹¹There are different bases to represent the gamma matrices in, and all have different advantages. We have given them in the Dirac basis, most appropriate for acting on Dirac spinors, while the Weyl (or chiral) basis is more suitable to find a simple form for the chiral projections.

The Lagrangian on the R.H.S. was proposed by Gamow and Teller, to generalize to more structures than only vectors like scalars and tensors, and Γ^i then depends on the structure being considered. G_F is the Fermi interaction constant. Note that this current does not mix the left- and right-handed components:

$$\bar{\psi}\gamma^\mu\psi = \bar{\psi}_R\gamma^\mu\psi_R + \bar{\psi}_L\gamma^\mu\psi_L. \quad (1.91)$$

Then parity violation was proven to happen. As discussed, parity is the operator $\hat{P} : x \rightarrow -x$, and symmetry under parity was one of the fundamental symmetries scientists thought to hold before Wu, and later Fraunfelder, discovered it did not in weak β -decays. There was a big preference for left handed electrons emitted in this process, meaning that γ^5 should appear in the weak interaction Lagrangian, and the simplest way describing these observations was found to be:

$$\mathcal{L}_{V-A} = \frac{G_F}{\sqrt{2}}(\bar{\psi}_p\gamma^\mu(1 - \gamma^5)\psi_n)(\bar{\psi}_e\gamma^\mu(1 - \gamma^5)\psi_\nu), \quad (1.92)$$

for which we see:

$$\bar{\psi}\gamma^\mu(1 - \gamma^5)\psi = 2\bar{\psi}\gamma^\mu P_L\psi = 2\bar{\psi}P_R\gamma^\mu P_L\psi = 2\bar{\psi}_L\gamma^\mu\psi_L, \quad (1.93)$$

so only left-handed components take part in the weak interactions. Thus we need to specify that something is different between the left- and right-handed portions of the field.

In electrodynamics, this does not seem to be a problem, and gauge transformations could only represent rotating the phase of the left- and right-handed particles equally. Thus to be able to unify electrodynamics and the weak interaction, we need to build a Lagrangian that is invariant under rotations between the left handed fields (which we thus put in a doublet like $L = (\nu_{l,L}, l_L)^T$ for each flavor $l = e, \mu, \tau$ of lepton for example¹²), but that leave right-handed fields untransformed (thus we put them in a singlet $R = e_R$). This, however, forces us to have massless particles, for these fields are not able to mix as they do in the mass term of the Dirac Lagrangian (the last term of equation (1.88)).

How do we get the leptons to have mass? Therefore we have to study the symmetry we want our lepton Lagrangian to have. Due to the doublet/singlet structure, we postulate invariance under $SU(2)_L \times U(1)_Y$, where $SU(2)_L$ is for the doublet only (denoted by the \cdot_L), and $U(1)_Y$ for the singlet and doublet. In general both the left- and right-handed parts can have different $U(1)_Y$ -charge, but the charge associated to $SU(2)$ must be the same for all fields in a doublet by definition. Note that in this case, the $U(1)_Y$ -charge is another one than the electric charge we already know from electrodynamics, which is also generated by a $U(1)_Y$ -symmetry. This charge is known as the **hypercharge**, and represented by Y . This means the covariant derivatives for the left- and right-handed components will look different. Introducing g as the $SU(2)$ -charge, g' as the $U(1)_Y$ charge and $\tau_i = \sigma_i/2$, where σ_i are again the Pauli matrices, and introducing W_i^μ as the $SU(2)$ gauge boson field and B^μ as the $U(1)_Y$ gauge boson field, we can introduce a covariant derivative:

$$D_L^\mu = \partial^\mu - ig\tau_i W_i^\mu - ig'\frac{Y_L}{2}B^\mu; \quad D_R^\mu = \partial^\mu - ig'\frac{Y_R}{2}B^\mu. \quad (1.94)$$

Now requiring a gauge symmetry for the Dirac Lagrangian without mass terms we find for one of the flavors of lepton only (the complete Lagrangian is then the sum over all lepton terms, and can also be

¹²When comparing to the current from the Lagrangian, which we are now just trying to rewrite, this is how we need to interpret the components for it to look the same. Furthermore, we know from the experiments that only left-handed neutrinos were observed

denoted by writing the fields for all flavors in a vector for brevity: $\nu_L = (\nu_{e,L}, \nu_{\mu,L}, \nu_{\tau,L})^T$:

$$\mathcal{L}_e = \bar{L} \left(i\gamma^\mu \left(\partial_\mu - ig\tau_\nu W_\mu^\nu - ig' \frac{Y_L}{2} B_\mu \right) \right) L + \bar{R} \left(i\gamma^\mu \left(\partial_\mu - ig' \frac{Y_R}{2} B_\mu \right) \right) R - \frac{1}{4} W_{\mu\nu} W^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}, \quad (1.95)$$

where $W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + g\epsilon_{abc} W_\mu^b W_\nu^c$, with the extra term due to this particular symmetry being non-Abelian¹³.

Rewriting this and redefining some of the W -terms we find the gauge boson terms representing the W^\pm and Z -bosons known from weak interactions, which start to matter when the symmetry of this gauge group is broken as shown. clearer discussed in [21] we can define the explicit couplings/quantum numbers Y_L , Y_R and $T_3 = \tau_3 = \sigma_3/2$ for all fields now. These gauge bosons couple to our other fields and currents in such a way that masses can be introduced again, but we will see why we need to be cautious with that. For instance, assume that a massive vector field A^μ couples to some current j^μ :

$$\begin{aligned} \mathcal{L} &= -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu - j^\nu A_\nu \\ \frac{\partial \mathcal{L}}{\partial A_\nu} &= m^2 A^\nu - j^\nu; \quad \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} = -F^{\mu\nu} \\ 0 &= -j^\nu + m^2 A^\nu + \partial_\mu F^{\mu\nu} \Rightarrow (\square^2 + m^2) A^\nu - \partial^\nu (\partial_\mu A^\mu) = j^\nu, \end{aligned} \quad (1.96)$$

where we immediately filled in the Euler-Lagrange equations. If $m = 0$, this looks exactly like Maxwell's equations with a source, but if $m \neq 0$, we take the derivative to find:

$$(\square^2 + m^2) \partial_\nu A^\nu - \square^2 (\partial_\mu A^\mu) = \partial_\nu j^\nu \Rightarrow m^2 \partial_\nu A^\nu = \partial_\nu j^\nu, \quad (1.97)$$

consequently, current is not conserved. Having massive gauge bosons thus seemingly breaks gauge invariance.

The existence of the massive bosons can be explained by the Higgs mechanism, where the breaking of this symmetry allows these masses to exist, which is why the Higgs field is added to the Lagrangian, charged under $SU(2)_L$. This symmetry is then broken. We want terms coupling ψ_R and ψ_L in a gauge invariant way, which we can now do, for example with a term like:

$$\bar{L}\phi R = \begin{pmatrix} \nu_{e,L} \\ e_L \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_1 + i\phi_2 \\ \phi_3 + i\phi_4 \end{pmatrix} e_R \rightarrow \begin{pmatrix} \nu_{e,L} \\ e_L \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h(x) \end{pmatrix} e_R, \quad (1.98)$$

where on the R.H.S. of the " \rightarrow " is the term describing the situation after the breaking of symmetry. The term before the breaking is clearly invariant under the $SU(2)_L$ transformations, which we can represent arbitrarily as:

$$\begin{aligned} \bar{L} &\rightarrow \bar{L} e^{\frac{i}{2} \theta_\mu(x) \sigma^\mu}; \quad \phi \rightarrow e^{-\frac{i}{2} \theta_\mu(x) \sigma^\mu} \phi, \\ \therefore \bar{L}\phi R &\rightarrow \bar{L} e^{\frac{i}{2} \theta_\mu(x) \sigma^\mu - \frac{i}{2} \theta_\mu(x) \sigma^\mu} R = \bar{L}\phi R. \end{aligned} \quad (1.99)$$

Also the hypercharge of the Higgs field is set equal to 1, and with the definitions for Y_L and Y_R above equation (1.95), we find that $Y_R + Y_\phi + Y_L = 0$, so this term is also $U(1)_Y$ -invariant, for it preserves

¹³We do not discuss this theory in detail, meaning all our other examples have been Abelian only, but you can find discussions in the books cited in this chapter.

hypercharge¹⁴.

The adjoint of the L.H.S. in equation (1.98) is given by:

$$(\bar{L}\phi R)^\dagger = R^\dagger \phi^\dagger \gamma_0^\dagger L = \bar{R}\phi^\dagger L, \quad (1.100)$$

meaning that we can write a mass term (with a constant g_e known as the Yukawa coupling):

$$\mathcal{L}_{mass} = -g_e[\bar{L}\phi R + \bar{R}\phi^\dagger L]. \quad (1.101)$$

Now we can expand around the new minimum after symmetry breaking $\phi = \frac{1}{\sqrt{2}}(0, v + h(x))^T$, getting:

$$\mathcal{L}_{\phi,fermion} = -\frac{g_e}{\sqrt{2}}[v(\bar{e}_L e_R + \bar{e}_R e_L) + h(\bar{e}_L e_R + \bar{e}_R e_L)] \Rightarrow m_e = \frac{g_e v}{\sqrt{2}}, \quad (1.102)$$

where m_e is the mass of the electron.

There are other ways to form $SU(2)_L$ -invariant terms, for instance with a term like $\tilde{\phi} = i\sigma_2\phi^*$, which transforms as:

$$i\sigma_2\phi^* \rightarrow i\sigma_2 \left(e^{-\frac{i}{2}\sigma_\mu\theta^\mu} \right)^* \phi^* = i\sigma_2 \left(\cos(\theta/2) + i\frac{\theta_\mu\sigma^{\mu*}}{\theta} \sin(\theta/2) \right) \phi^* = e^{-\frac{i}{2}\sigma_\mu\theta^\mu} i\sigma_2\phi^*, \quad (1.103)$$

because:

$$\sigma_2\theta_\mu\sigma^{\mu*} = \sigma_2(\theta_1\sigma^1 - \theta_2\sigma^2 + \theta_3\sigma^3) = -\theta_\mu\sigma^\mu\sigma_2, \quad (1.104)$$

thus we can combine $\tilde{\phi}$ with a barred term to again find $SU(2)_L$ -invariance as before.

However, this term has a different $U(1)_Y$ charge than ϕ , thus $\bar{L}\tilde{\phi}R$ is not allowed due to hypercharge conservation, and it seems all other terms we can write down will not give fermions their mass terms.

Here we have already seen once again how neutrinos have to be massless in the SM, for the symmetry breaking of the Higgs field only retains the second element of L , which was interpreted as the left-handed part of the electron field. However, if uncharged right-handed neutrinos exist, they must have hypercharge equal to $2(Q - T_3) = 0$, allowing for $\bar{L}\tilde{\phi}\nu_R$, for which we can perform a gauge transformation to the Higgs field as $\tilde{\phi} \rightarrow \frac{1}{\sqrt{2}}(v + h(x), 0)^T$, and giving a new mass term:

$$\mathcal{L}_{mass\nu} = -g_\nu[\bar{L}\tilde{\phi}\nu_R + \bar{\nu}_R\tilde{\phi}^\dagger L] = -\frac{g_\nu}{\sqrt{2}}(v + h)[\bar{\nu}_L\nu_R + \bar{\nu}_R\nu_L]. \quad (1.105)$$

We assume here that mass eigenstates are the same as the weak interaction eigenstates. We know this is not the case however, due to us observing neutrino oscillations, which is discussed in the next section.

¹⁴Gell-Mann and Nishijima found a relation between electric charge, hypercharge and T_3 : $Q = T_3 + Y/2$, which we can use to find $Y_R = -2$, $Y_L = 1$ and $Y_\phi = 1$

1.6.1 Neutrino Masses

We now have all the ingredients and context to talk about possible mass terms for neutrinos. We only touch on the three main concepts currently accepted, which are the Dirac and Majorana mass, and the see-saw mechanism. After having introduced these concepts we can talk about how neutrino oscillations work and how they can explain the solar neutrino problem.

As demonstrated above, charged fermions can only have **Dirac-type masses**, which are mass terms that look like:

$$m\bar{\psi}\psi. \quad (1.106)$$

A more general theory is based upon the charge conjugation operator C (which turns a particle state into its antiparticle state by flipping the sign of all relevant quantum numbers). This term was discovered by Ettore Majorana when searching for a way to construct a mass term using only the left-handed chiral state.

It can be shown that C can be expressed in the Dirac representation of the gamma matrices and it then has the following properties:

$$C = i\gamma^2\gamma^0; \quad C^\dagger = C^{-1}; \quad C^T = -C; \quad C\gamma_\mu^T C^{-1} = -\gamma_\mu; \quad C\gamma_5^T C^{-1} = \gamma_5; \quad \psi^c = C\psi^*. \quad (1.107)$$

This will be useful later on.

Using the Euler-Lagrange equations on equation (1.88) we find two coupled Dirac equations, where the coupling happens due to the mass term:

$$i\gamma^\mu\partial_\mu\psi_L = m\psi_R; \quad i\gamma^\mu\partial_\mu\psi_R = m\psi_L. \quad (1.108)$$

Usually, as is done in the SM, the neutrino is described using only two independent two-component spinors turning out to be helicity eigenstates with definite and opposite helicity. Then the right-handed one is neglected to only describe the neutrino with a single massless left-handed field, decoupling the Dirac equations from above.

Now taking the second Dirac equation from above, we want to make it look like the first one by finding an expression for ψ_R in terms of ψ_L . To do this, we take the Hermitian conjugate of the second equation and multiply on the right by γ^0 to find:

$$-i\partial_\mu\psi_R^\dagger\gamma^{\mu\dagger} = m\psi_L^\dagger \Rightarrow -i\partial_\mu\psi_R^\dagger\gamma^0\gamma^\mu = m\psi_L^\dagger\gamma^0 \Rightarrow -i\partial_\mu\bar{\psi}_R\gamma^\mu = m\bar{\psi}_L, \quad (1.109)$$

where in the last step we use that $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$, or $\gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu$. Then we take the transpose, and using that $C\gamma^{\mu T} = -\gamma^\mu C$ we get:

$$-i\gamma^{\mu T}\partial_\mu\bar{\psi}_R^T = m\bar{\psi}_L^T \Rightarrow i\partial_\mu C\bar{\psi}_R^T = mC\bar{\psi}_L^T. \quad (1.110)$$

Thus then we would define $\tilde{\psi}_R = C\bar{\psi}_L^T$, but note that it assumes that this term is indeed the right-handed component, which should mean that $P_L\tilde{\psi}_R = 0$. Using that $P_L C = CP_L^T$ we find:

$$P_L(C\bar{\psi}_L^T) = CP_L^T\bar{\psi}_L^T = C(\bar{\psi}_L P_L)^T = 0, \quad (1.111)$$

because:

$$\bar{\psi}_L P_L = (P_L\psi)^\dagger\gamma_0 P_L = \psi^\dagger P_L^\dagger\gamma_0 P_L = \psi^\dagger P_L\gamma_0 P_L = \psi^\dagger\gamma_0 P_R P_L = 0, \quad (1.112)$$

where we use $P_L^\dagger = P_L$, $\gamma_5\gamma_0 + \gamma_0\gamma_5 = 0$ and $P_L P_R = 0$.

Now we have defined a right-handed field in terms of the left-handed one and can write the Dirac equation in only terms of the left-handed field, and the new field becomes:

$$\psi = \psi_L + \psi_R = \psi_L + C\overline{\psi_L}^T = \psi_L + \psi_L^C; \quad \psi^C = (\psi_L + \psi_L^C)^C = \psi_L^C + \psi_L = \psi, \quad (1.113)$$

thus the field is its own charge conjugate, and a so-called **Majorana particle** should thus be its own antiparticle. It should even be electrically neutral, and the only known fermion that is electrically neutral is the neutrino.

Note that a particle being its own antiparticle would violate lepton number conservation, for Majorana particles can not get a conserved lepton number. Indeed, it is observed neutrinos violate lepton number conservation by ± 2 . Depending on which part of the Majorana neutrino interacts with which gauge boson, a negatively (left-handed with W^+) or a positively (right-handed with W^-) charges lepton is produced.

We can build a mass term for Majorana particles as follows.

Instead of $-m\bar{\nu}_R\nu_L$ we build $-\frac{1}{2}m\overline{\nu_L^C}\nu_L$, where $\nu_L^C = C\bar{\nu}_L^T$, and the factor one-half is to account for the fact that we count this term twice because the hermitian conjugate, which is also contained in the Lagrangian, is identical to this term. This inherently means that, when working with a Majorana mass matrix involving the mass terms for all three flavors at once (as has been introduced as a concept before and will be done for the neutrino oscillations below), this matrix should be symmetric.

Seesaw mechanism

If only the left-handed chiral field ν_L exists, there can be no Dirac mass term, and only a Majorana mass term. In the SM however, the Majorana mass term can not actually exist. This is due to the Higgs mechanism, which would produce interactions like $\overline{\nu_L^C}\phi\nu_L$, where ϕ is the aforementioned Higgs field. The left-handed neutrino field has **weak isospin** (T_3) equal to $1/2$ and hypercharge -1 . To be gauge invariant, quantum numbers should cancel out, and thus the Higgs field should have weak isospin -1 and hypercharge -2 , which can not exist in the SM due to the possible representations of the gauge group. On the other hand, a right-handed neutrino field could exist.

Hence there is something really odd going on that we haven't thought of yet, or a right-handed neutrino field exists, only interacting with gravity and the Higgs field. Let us collect the following mass terms in one Lagrangian:

$$\mathcal{L}_L^D = m_D\bar{\nu}_R\nu_L + h.c.; \quad \mathcal{L}_R^D = m_D\overline{\nu_L^C}\nu_R^C + h.c.; \quad \mathcal{L}_L^M = \frac{1}{2}m_L\overline{\nu_L^C}\nu_L + h.c.; \quad \mathcal{L}_R^M = \frac{1}{2}m_R\overline{\nu_R^C}\nu_R + h.c., \quad (1.114)$$

where a D or M represents if the mass term is Dirac or Majorana, and $h.c.$ stands for adding the hermitian conjugates of the terms to the Lagrangian too. Note that the Dirac mass should be the same for both terms, and we can now write:

$$2\mathcal{L}_{mass\nu} = L_L^D + L_R^D + L_L^M + L_R^M + h.c. = m_D\bar{\nu}_R\nu_L + m_D\overline{\nu_L^C}\nu_R^C + m_L\overline{\nu_L^C}\nu_L + m_R\overline{\nu_R^C}\nu_R + h.c. \\ \mathcal{L}_{mass\nu} \sim \begin{pmatrix} \overline{\nu_L^C} & \overline{\nu_R^C} \end{pmatrix} \begin{pmatrix} m_L & m_D \\ m_D & m_R \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_R^C \end{pmatrix} + h.c. \quad (1.115)$$

We of course have expressed the Lagrangian here in terms of the chiral components, which do clearly not have a definite mass because of the existence of the off-diagonal m_D -term. Thus the mass eigenstate of the neutrino is another one than the one that interacts with intermediate gauge bosons. There are thus mass eigenstates (call them ν_1 and ν_2 for now), and rewriting the Lagrangian such that the mass matrix M is diagonal (denoted by M') we find the masses of ν_1 and ν_2 as eigenvalues. To do this we search for a unitary matrix U , that transforms left- and right-handed chiral fields into left-handed components with a definite mass as:

$$\begin{pmatrix} \nu_L \\ \nu_R^c \end{pmatrix} = U \begin{pmatrix} \nu_{1,L} \\ \nu_{2,L} \end{pmatrix} \Rightarrow U^\dagger M U = M'. \quad (1.116)$$

The fact that U always exists is a theorem in linear algebra, and using the characteristic equation for eigenvalues of a square matrix we find:

$$m_{1,2} = \frac{1}{2} \left[m_L + m_R \pm \sqrt{(m_L - m_R)^2 + 4m_D^2} \right]. \quad (1.117)$$

The **see-saw mechanism** is now the choice of $m_L = 0$ and $m_R \gg m_D$, giving that:

$$m_1 = \frac{m_D^2}{m_R}; \quad m_2 = m_R \left(1 + \frac{m_D^2}{m_R^2} \right) \approx m_R, \quad (1.118)$$

thus if m_2 is very large, m_1 will be very small. Following the same logic we can write the quantities in matrix form M_D, M_R, M_L which contain information about all three flavors. Then up to order of M_D^2/M_R^2 , we find that the light (in the other theories the left-handed) state is given by $M_D M_R^{-1} M_D^T$, while the right-handed, and heavier state is M_R .

This structure is known as the see-saw mechanism because when m_2 increases, m_1 decreases and vice versa. When m_R thus is very large,

$$\nu_1 \sim \nu_L + \nu_L^c - \frac{m_D}{m_R^2} (\nu_R + \nu_R^c); \quad \nu_2 \sim \nu_R + \nu_R^c + \frac{m_D}{m_R^2} (\nu_L + \nu_L^c), \quad (1.119)$$

thus ν_1 is mostly the left-handed Majorana neutrino, and ν_2 mostly the right-handed, heavy, sterile partner. This is a really natural explanation of the relative smallness of the neutrino mass we currently have. These heavy particles would have been created very early in the universe, and now no longer exist as stable particles, but they could exist for long enough to generate mass due to the uncertainty principle.

Neutrino Oscillations

Neutrino Oscillations are a mechanism that was proposed, using the concepts we built for the see-saw mechanism, to explain problems we now encounter regarding the masslessness of neutrinos, for instance, the solar neutrino problem, which is as follows. Due to the standard model of fusion in the sun, we can predict the amount of neutrinos produced in the sun. In 1968, Davis et al. reported the first experiment to detect solar neutrinos, but they found only a third of what was expected from theory. More and more experiments like this have been performed, and more and more gave the same result. Pontecorvo then suggested that neutrino species oscillate into other species (for which the experiments were not sensitive) during their flight. To see how this mechanism works, we have to know the true stationary states (the mass states) and how they couple to the interaction-states. It so happens that

we now have all the information and ingredients to be able to write down a general Lagrangian for neutrinos, in terms of Dirac spinors ν_l where $l = e, \mu, \tau$ and their masses:

$$\mathcal{L} = \begin{pmatrix} \bar{\nu}_e \\ \bar{\nu}_\mu \\ \bar{\nu}_\tau \end{pmatrix} \left[i\cancel{\partial} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} V_F^e & 0 & 0 \\ 0 & V_F^\mu & 0 \\ 0 & 0 & V_F^\tau \end{pmatrix} \right] \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} + \begin{pmatrix} \bar{\nu}_e \\ \bar{\nu}_\mu \\ \bar{\nu}_\tau \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix} \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}, \quad (1.120)$$

where V_F^l is a constant for the Fermi theory of interactions for fermions. This gives a kinetic and an interaction term on the left, and a mass term on the right. We can then define mass eigenstates ν_i for $i = 1, 2, 3$ such that the kinetic and the mass term are diagonal, but the interaction term is not anymore. A unitary transformation rotates between the two choice of basis, as we showed before. The principle behind neutrino oscillations is more easily shown for two flavors, which we do now. Afterwards, we will generalize to more dimensions.

In two dimensions, U can be represented by one parameter, θ , in a simple rotation transformation matrix, giving:

$$\nu_e = \cos(\theta)\nu_1 + \sin(\theta)\nu_2; \quad \nu_\mu = \cos(\theta)\nu_2 - \sin(\theta)\nu_1. \quad (1.121)$$

If at time $t = 0$, the state is $\psi_0 = \nu_e$, then at time t :

$$\psi_t = e^{-iE_1 t} \cos(\theta)\nu_1 + \sin(\theta)\nu_2 e^{-iE_2 t}, \quad (1.122)$$

because the time evolution is only controlled by the energy eigenvalues of a state in this case. We can now calculate the probability of conversion $\nu_e \rightarrow \nu_\mu$, which is, quantum mechanically, equal to¹⁵:

$$P(\nu_e \rightarrow \nu_\mu) = |\langle \nu_\mu | \psi_t \rangle|^2 = \sin^2(2\theta) \sin^2\left(\frac{(E_2 - E_1)t}{2}\right). \quad (1.123)$$

Assuming both mass eigenstates propagate with the same momentum $p \gg m_i$, we write:

$$E_i = \sqrt{p^2 + m_i^2} = p\sqrt{1 + \frac{m_i^2}{p^2}} \approx p + \frac{m_i^2}{2p} \Rightarrow E_1 - E_2 = \frac{m_1^2 - m_2^2}{2p} = \frac{\Delta m_{12}^2}{2p}, \quad (1.124)$$

Thus neutrinos oscillate, and the probability of observing a given flavor changes with the distance travelled, and depends on $\Delta m_{ij}^2 = m_i^2 - m_j^2$, where i and j indicate the mass eigenstates.

When we have three mass eigenstates, the mixing matrix can be characterized by 3 angles and one phase, and it proves convenient to relate these angles to observables. Because the numbering of the mass states is arbitrary, we choose to identify Δm_{21}^2 with the smaller of the two independent neutrino mass squared differences ($\Delta m_{31}^2 = \Delta m_{21}^2 + \Delta m_{32}^2$), and number $m_1 < m_2$ so that $\Delta m_{21}^2 > 0$. Then there are two possibilities. Either $m_1 < m_2 < m_3$ or $m_3 < m_1 < m_2$, which are known as the **Normal** and the **Inverted** hypothesis respectively. The larger difference Δm_{31}^2 or Δm_{32}^2 can be associated with the observed oscillations in atmospheric neutrinos, because the effect of Δm_{21}^2 in those processes is relatively small. For the same reason but the other way around we identify Δm_{21}^2 with solar neutrino oscillations. For a more in-depth discussion, see [46].

The current best-fit values for the mass splittings, mixing angles and the phase δ can be found in [18].

¹⁵see Appendix for mathematical discussion

Furthermore, in 2016 the Planck Collaboration published updated values on an upper bound on the sum of the neutrino masses. Assuming three light massive neutrinos exist, and that the Λ CDM (Cold Dark Matter) model is valid for our universe they used many sets of data. These included data on the CMB temperature power spectrum anisotropies, polarization, gravitational lensing effects. Adding data on the Baryon Acoustic Oscillations lowers this limit to $\sum_i m_i < 0.170$ electronvolt, with a confidence limit of 95%. Currently, this upper bound is set at 0.12eV.

Now that we know about the possible models for describing massive neutrinos, we will introduce the statistical toolset needed to check which of these is more valid. To summarize our models: we can have the situation where neutrinos are Dirac or Majorana, and they can have Normal or Inverted ordering.

We have, in this chapter, introduced the Lagrangian formalism and how we can use symmetries of the theory to build a convincing Lagrangian. Using symmetries and the breaking of symmetry in these theories, we have described possible models, and thus physical origins, for neutrino masses. Because the neutrino masses and ordering can be derived from the matrix-elements of the mass matrix, and the amount of observables is way smaller than the amount of possible matrix elements, we will have to perform statistical analysis and simulate many possible mass matrices and check how well they fit our current observations. This will thus produce a posterior distribution for our matrix elements, which we can use to analyze the structure of the matrix to see which of the models most correctly produces the current data. Our statistical and mathematical methodology is explained in the next chapter.

A further interesting source is [47].

Chapter 2

Invariant Priors and Concentration of Measure

In this chapter we introduce the reason why we have to construct priors when searching for the correct model to describe neutrino masses, and what the difficulties are when working with priors. We discuss the basic concept of priors and so-called ignorant, invariant or uninformative priors and how they can be constructed by considering symmetries of the theory, or by calculating directly from the likelihood (Jeffreys' prior) or from entropy considerations (reference priors). We then discuss the concept of concentration of measure and how it can impact the practice of inferring neutrino masses from measurements. We also touch lightly on some linear algebra needed to work with random matrices, which we will need in our discussion. This concludes with a statement on eigenvalue splitting in random matrices.

2.1 Priors

When searching for the best model to describe the mass of neutrinos, what we want to do is to infer a statement about reality from observations. In this case, we have only one subject to look at, our Universe, which produces many samples produced by the same mechanism. The only thing limiting us is our own ability of observing these samples without error. Because neutrino masses are so small, we need very precise and very many measurements to be able to say anything conclusive about models, and luckily research is being done right now, and measurements are only getting better with time. Inference is a very common practice, and one of the most common procedures for this is Bayesian inference, which is most applicable to our situation. We assume that some true values of the parameters for our model exist, and that they describe our observations. Then we use different and continuously bigger data sets to find the probability on specific combinations of these model parameters representing our observations. Therefore we can interpret the probability for a specific point in parameter space as a measure of how realistic a model with those parameters is. We can of course not only compare sets of parameters for one model, but also compare multiple models against each other in this way.

If we denote the true parameter with a subscript t : θ_t , the data with x and the model with p , we can model x using the probability density function $p(x|\theta_t)$ given the true parameter θ_t . Because we can only infer the true parameter, we can only find a distribution $P(\theta|x)$ of probabilities on our parameter space for possible parameters θ given our data. We can then use **Bayes' theorem** to relate

these two distributions:

$$P(\theta|x) = \frac{p(x|\theta)\pi(\theta)}{f(x)}. \quad (2.1)$$

Here P is known as the **posterior** distribution for θ given our data and $f(x)$ is called the **evidence**, but since for now we are mostly interested in parameter value and $f(x)$ does not reference them, we can neglect it in most discussions and applications of Bayes' theorem. It does, however, provide a normalizing constant such that P integrates to 1 over its domain, which is of course a fundamental axiom of probability theory. $f(x)$ can be written as the marginal distribution of x when integrating over all possible parameters:

$$f(x) = \int p(x|\theta)\pi(\theta)d\theta. \quad (2.2)$$

Of interest for this section is $\pi(\theta)$, known as the **prior** probability distribution. It can be interpreted as a description of our ignorance about θ_t . The choice of prior can heavily affect the values of the posterior, so a choice of a specific prior should be argued as concisely as the specification of the data normally is. However, there seems to be an ambiguity in assigning the prior, and often a personal preference defines the choice of specific scientists. Often the uniform or Gaussian distributions are chosen, the uniform because it seems to not prefer specific parameters over others, and the Gaussian due to the common occurrence of this distribution in nature.

Actually priors should represent our prior information and be constructed by the logical analysis of our a priori knowledge. We will discuss how we try to make inference as objective as possible, in the sense that two persons with the same prior information should assign the same prior probabilities. A principle known as the maximal entropy principle, described by Shannon in the discrete case, and by Jaynes in the continuous case, is a valid way of setting up priors. The problem with the continuous case is that it requires more analysis than the discrete one, as we will see below. One way of incorporating prior knowledge to get around this problem is to use symmetries of the theory in constructing the prior. How this can be done will be discussed later.

2.1.1 The Maximal Entropy Principle

This discussion is based mostly on [30].

Suppose we only know the mean of a certain process, and want to infer the underlying probability distribution from it. For example, when an entire stack of 100 plates falls over, we can count the pieces, and find a mean of 8.65, meaning that there were 865 pieces. If we only know this mean, what would be the probability a plate broke into x pieces? We want to incorporate all our info, but not jump to conclusions inconsistent with our knowledge. The most general choice would be a uniform distribution, as spread out as possible, to not disfavor any possibility. Thus, can we find a reasonable measure of how uniform a probability distribution is, so we can maximize it subject to the constraints of available information. One measure for this would be the variance, and to maximize it subject to the information. This would, however, allow us to give a very large probability of not breaking, and an arbitrarily small probability of breaking into arbitrarily many pieces. Another proposal would be to minimize the sum of squares of the probabilities, where we know that the sum of probabilities should be 1 and the mean should be 8.65. This, however, seems to lack the property of non-negativity of probabilities. It does introduce a variational problem that can be solved by use of Lagrange multipliers, which seems as a good method. It leads us to believe that we can find a more consistent measure of uniformity. Shannon described the conditions such a measure would have to satisfy. Firstly, we assume it exists, and should be a continuous function of all the probabilities in the problem. Furthermore, it should correspond to common sense in the way that we are more uncertain if there are more possibilities. This means, when

denoting the **measure of uniformity** $U(p_1, \dots, p_n)$, it should monotonically increase in n if $p_i = 1/n$ for all i . Lastly, it should be consistent. So if we find more than one way of working out the value, we must get the same answer in all of these methods.

Discrete distributions

Assume we have n mutually exclusive propositions A_i with probabilities p_i . Instead of finding all probabilities directly, we can group the first k together as $B_1 = A_1 + \dots + A_k$ and assign $w_1 = p_1 + \dots + p_k$, and the next q have probability $w_2 = p_{k+1} + \dots + p_{k+q}$ etc. The uncertainty to the composite propositions is $U(w_1, \dots, w_r)$ if we make r groups. If we give conditional probabilities $p_1/w_1, \dots, p_k/w_1$ to A_1, \dots, A_k assuming B_1 is true, we get an additional uncertainty, encountered with probability w_1 , of $U(p_1/w_1, \dots, p_k/w_1)$. Thus we arrive at:

$$U(p_1, \dots, p_n) = U(w_1, \dots, w_r) + w_1 U\left(\frac{p_1}{w_1}, \dots, \frac{p_k}{w_1}\right) + w_2 U\left(\frac{p_{k+1}}{w_2}, \dots, \frac{p_{k+q}}{w_2}\right) + \dots \quad (2.3)$$

Since $U(p_1, \dots, p_n)$ should be continuous it suffices to determine it for rational values $p_i = n_i / \sum_i n_i$ for integers n_i . Then we can rewrite U in terms of $u(n) = U(1/n, \dots, 1/n)$, where we fill in $1/n$ for all parameters of U , giving all propositions equal probability. This gives:

$$u\left(\sum_i n_i\right) = U(p_1, \dots, p_n) + \sum_i p_i u(n_i). \quad (2.4)$$

If we choose all n_i to be equal to N , this becomes $u(nN) = u(n) + u(N)$, which is solved by setting $u(n) = K \log(n)$. The proof that this solution is unique in both the continuous and discrete cases can be found in [30]. Another K just amounts to choosing another base for the logarithm, and substituting this into equation (2.3) we get **Shannon's theorem**: The only function U that satisfies the imposed conditions as a reasonable measure of the amount of uncertainty is:

$$U(p_1, \dots, p_n) = - \sum_{i=1}^n p_i \log(p_i). \quad (2.5)$$

The distribution of p_i maximizing U will then represent the 'most ignorant' description of what we know about the propositions. This function U is called the **entropy**¹ of the distribution of p_i 's, which is why this is called the maximal entropy principle.

To find the needed distribution we introduce a somewhat new terminology, but keeping in mind the result in the last equation. Suppose a variable x can take on values x_1, \dots, x_n corresponding to n propositions A_i . Suppose we have m functions of x $f_k(x)$ for $1 \leq k \leq m < n$, wanting them to have expectations F_k , which are given to us in the statement of the problem. We introduce Lagrange multipliers to find the set of p_i 's assigned to the x_i 's that has maximum entropy subject to the constraint that:

$$F_k = \langle f_k(x) \rangle = \sum_{i=1}^n p_i f_k(x_i); \quad \sum_i p_i = 1. \quad (2.6)$$

¹This is not in any way the same as the *experimental entropy* of thermodynamics. The entropy from the text is often called *information entropy*.

This amounts to:

$$\begin{aligned}
0 &= \delta \left[U - (\lambda_0 - 1) \sum_i p_i - \sum_{j=1}^m \lambda_j \sum_i p_i f_j(x_i) \right] \\
&= \sum_i \left[\frac{\partial U}{\partial p_i} - \lambda_0 + 1 - \sum_{j=1}^m \lambda_j f_j(x_i) \right] \delta p_i.
\end{aligned} \tag{2.7}$$

From equation (2.5) we then find for our solution:

$$p_i = \exp \left[-\lambda_0 - \sum_{j=1}^m \lambda_j f_j(x_i) \right]; \quad 1 = e^{-\lambda_0} \sum_i \exp \left[-\sum_{j=1}^m \lambda_j f_j(x_i) \right]. \tag{2.8}$$

Defining the **partition function** we can reduce this to:

$$Z(\lambda_1, \dots, \lambda_m) = \sum_{i=1}^n \exp \left[-\sum_{j=1}^m \lambda_j f_j(x_i) \right]; \quad \lambda_0 = \log(Z); \quad F_k = -\frac{\partial Z}{\partial \lambda_k}, \tag{2.9}$$

because F_k must be equal to the expected value of f_x over the probability distribution: $\sum_i f_k(x_i) p_i$. Filling in the p_i 's we find $U_{max} = \lambda_0 + \sum_{j=1}^m \lambda_j F_j$. Of course, we derived this using the standard results of the variational arguments for Lagrange multipliers, and we can have missed a possible global maximum. Therefore, a small extra argument is needed, which is discussed in [30] as well.

Continuous distributions

As mentioned before, we can not quite use the same principle as above for continuous distributions, but we can pass to the limit from a discrete distribution. Thus we generate more and more points x_i and so take the limit $n \rightarrow \infty$ of the average number of points $\#_{[a,b]}$ in a certain interval $[a, b]$, which is then just the integral over a certain measure $m(x)$ (we only call it a measure to suggest the possible appropriate generalization):

$$\lim_{n \rightarrow \infty} \frac{1}{n} \#_{[a,b]} = \int_a^b m(x) dx. \tag{2.10}$$

If this limit is well-behaved², adjacent differences around any particular x will tend to zero such that:

$$\lim_{n \rightarrow \infty} [n(x_{i+1} - x_i)] = m(x_i)^{-1}. \tag{2.11}$$

Now we obtain a continuous probability $p(x)$, such that $p_i = p(x_i)(x_{i+1} - x_i)$, which in the limit of $n \rightarrow \infty$ becomes $p(x_i)[nm(x_i)]^{-1}$, thus we can write a continuous form of the discrete entropy as:

$$U \rightarrow_{n \rightarrow \infty} \int p(x) \log \left[\frac{p(x)}{nm(x)} \right] dx. \tag{2.12}$$

²Extension to infinite sets is only permitted if this is the result of a well-defined and well-behaved limiting process. Different processes could lead to different limits, and the safe procedure is to apply the ordinary processes of analysis to finite cases, and observe how the resulting finite expressions behave as the number of terms increases indefinitely. If this is such that the functions involved stay smooth and integrable, it is well-behaved.

This however contains a $\log(n)$ -term, which we can subtract to find a possible entropy for continuous distributions:

$$U \equiv \lim_{n \rightarrow \infty} [U - \log(n)] - \int p(x) \log \left[\frac{p(x)}{m(x)} \right] dx. \quad (2.13)$$

Since $p(x)$ and $m(x)$ transform similarly under a change of variables (of which we will see more in the next section about invariance of distributions), U is invariant under those changes of variables.

The solution we seek should be normalized and constrained by a priori information fixing the mean values of m different functions $f_k(x)$. Again, the solution is known from theory about Lagrange multipliers, and equal to:

$$p(x) = \frac{1}{Z} m(x) e^{\lambda_1 f_1(x) + \dots + \lambda_m f_m(x)}; \quad Z(\lambda_1, \dots, \lambda_m) = \int m(x) e^{\lambda_1 f_1(x) + \dots + \lambda_m f_m(x)} dx, \quad (2.14)$$

where we can find λ_k from $F_k = -\frac{\partial \log(Z)}{\partial \lambda_k}$.

Rather than just integrating the same expression as we had in the discrete case, we thus introduce the measure $m(x)$, which allows results to be invariant under reparametrizations. However, the problem of which choice for m we have to make remains, and the choice will still, as mentioned before, alter our results, and in some cases this will be quite significant. Intuitively, the measure is also the prior distribution showing complete ignorance of x . This is for example seen in one dimension by taking $x \in (a, b)$, but knowing nothing else. Then there are no Lagrange multipliers, and we get $p(x) = \frac{m(x)}{\int_a^b m(x) dx}$.

Consequently, we still ask ourselves how to find a prior showing true ignorance, independent of parameters, also called an **invariant prior**. Comparing this to the idea of the invariant line element in Riemannian geometry, which is also independent of choice of coordinates³, we see that symmetry principles will play a role, as we have seen in the last chapter as well. We discuss this in the next section.

2.1.2 Invariance of Distributions

We will introduce the concept of transformations and what we mean by invariant distributions to introduce the concepts of symmetries in priors and the concept of the least informative prior. As mentioned, generally a prior $\pi = \text{constant}$ seems a non-informative prior, but it is not invariant under transformations as we will see.

We will define a **reparametrization** as a transition from θ to another variable λ through a **transformation** T : $\lambda = T\theta$. We want this transformation such that it leaves probabilities invariant, for those are the values impacting our decisions. This means that they do not leave densities invariant, but should leave functions invariant. All in all, for the **transformed density** $\pi_T(\theta)$, it should hold that:

$$\pi_T(\lambda) = \pi(\theta) \left| \frac{d\theta}{d\lambda} \right|, \quad (2.15)$$

³A very important geometric property, known to anyone who read about General Relativity.

where the absolute values are instated because we want probabilities to be non-negative. If θ is multidimensional the derivative should be replaced by the Jacobian:

$$\left| \det \left(\frac{\partial \theta}{\partial \lambda} \right) \right| \equiv \left| \frac{d\theta}{d\lambda} \right|. \quad (2.16)$$

As we can see this means that, generally, a uniform distribution is changed to a non-uniform one. For example transforming θ to $\lambda = \theta^2$, we get that $\pi_T(\lambda) \propto \lambda^{-1/2}$. If we transform a constant function from Cartesian to polar coordinates, we get $\pi_T(\lambda) \propto \lambda$ if λ represents the radial coordinate.

For functions $f(\theta)$ it should hold that $f_T(\lambda) = f(\theta)$, to keep the probabilities invariant.

It should be noted that Bayes' theorem is invariant under transformations like these, for both θ or x . This can be seen by setting $\lambda = T\theta$, leaving the integral in equation (2.2) invariant (because $d\theta = d\lambda(d\theta/d\lambda)$ thus transforms with the inverse of the term π transforms with). Then P thus transforms as π , because the transformation is not of impact on p being a distribution for x and not θ . Transforming $x' = Tx$ instead, both p in the numerator and the denominator transform alike, and the Jacobians thus drop out, making P transform as p .

In the next section we will discuss how symmetries impact our choice of prior, to avoid subjectiveness in the choice, and make a formal definition of an invariant prior dependent on the symmetries.

2.1.3 Invariant Priors

Before we explain some examples of invariant priors under certain symmetries, we will define what we mean by invariance under symmetries. As in the last chapter, we represent symmetries by groups and their elements. Generally, the groups describing symmetries of conditional probabilities are groups with a manifold of elements, also known as the Lie groups which we encountered before. Consequently the index θ labeling the transformation T_θ is a real number or a vector of real numbers, and the transformation is a function of θ . A simple example would be $U(1)$, of which we represent the elements now as T_θ with $\theta \in [0, 2\pi]$, and can write them as $\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$, rotating by an angle θ about the origin. Because of the group structure given we have a surjective mapping of the domain onto itself.

We call the model $p(x|\theta)$ **invariant** under transformations T_θ of x if:

$$p(x|\theta) = f(T_\theta^{-1}x) \left| \frac{\partial T_\theta^{-1}x}{\partial x} \right|, \quad (2.17)$$

where we require f to be normalized. This implies that $f(x) = p(x|\theta = \epsilon)$, where ϵ is the unit element of the group. We can thus also state that we define p to be invariant if it remains unchanged under simultaneous transformation of x and θ with T_ρ :

$$p(x|\theta) = p(T_\rho x | T_\rho \theta) \left| \frac{\partial T_\rho x}{\partial x} \right|. \quad (2.18)$$

This is proven by:

$$\begin{aligned} p(T_\rho x | T_\rho \theta) \left| \frac{\partial T_\rho x}{\partial x} \right| &= f((T_\rho T_\theta)^{-1} T_\rho x) \left| \frac{\partial (T_\rho T_\theta)^{-1} T_\rho x}{\partial T_\rho x} \right| \left| \frac{\partial T_\rho x}{\partial x} \right| \\ &= f(T_\theta^{-1} x) \left| \frac{\partial T_\theta^{-1} x}{\partial x} \right| = p(x|\theta). \end{aligned} \quad (2.19)$$

If f is normalized, it follows that p is. Writing $x' = T_\theta^{-1}$ we see that:

$$\begin{aligned} \int p(x|\theta)dx &= \int f(T_\theta^{-1}x) \left| \frac{\partial T_\theta^{-1}x}{\partial x} \right| dx = \int f(T_\theta^{-1}x)d(T_\theta^{-1}x) \\ &= \int w(x')dx' = 1, \end{aligned} \tag{2.20}$$

because we still integrate over the entire range of x because T_θ^{-1} is a transformation of the domain in which x is defined. One common example is the Gaussian distribution:

Example 2.1.1. Consider a model with the structure $p(x|\theta) = f(x - \theta)$ for $-\infty < x < \infty$. The Gaussian is such a model, and then:

$$f(x - \theta) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x - \theta)^2}{2\sigma^2}}. \tag{2.21}$$

The group of translations, with elements $T_\theta x = x + \theta$, for $-\infty < \theta < \infty$, is the symmetry group in this case. ◀

When we have N events from the same invariant distribution, their joint distribution is invariant too, which is argued in [26], so our considerations in general hold for higher dimensions as well.

A Lie group can define an **invariant measure** $\mu(\theta)$ in the space of the group parameter, which is the only one left unchanged under all transformations of the group, meaning:

$$\mu(\theta) = \mu(T_\rho\theta) \left| \frac{\partial T_\rho\theta}{\partial \theta} \right|. \tag{2.22}$$

With this measure, the volume of an area A in the space of θ does not change when A is shifted to $T_\rho A$, meaning that the image of every point $\theta \in A$ is obtained by applying T_ρ to it. Thus these shifts are generalizations of translations of an interval in the space of real numbers, and the volume should thus be invariant under translations in an invariant measure. We see indeed that:

$$V = \int_{T_\rho A} \mu(T_\rho\theta)dT_\rho\theta = \int_{T_\rho A} \mu(\theta)d\theta = \int_A \mu(\theta)d\theta. \tag{2.23}$$

The invariant measure can be written as:

$$\mu(\theta) = \mu(\epsilon) \left| \frac{\partial \Phi(\theta, \rho)}{\partial \rho} \right|_{\rho=\epsilon}^{-1}, \tag{2.24}$$

where $\Phi(\theta, \rho)$ labels the index of the product $T_\Phi = T_\theta T_\rho$. The why for this is given in [26]. In group theory, this measure is called the left invariant **Haar measure**, which vanishes nowhere. This is seen by the fact that $\Phi(\theta, \rho)$ is a transformation of ρ , so the Jacobian given in the expression does not vanish, and when $\mu(\epsilon) \neq 0$ it is even nowhere zero. It is thus unique up to a constant factor⁴. We

⁴The Haar measure always exists for Lie groups, and could be seen as the measure induced by a left-invariant n -form on the manifold, if n is the dimension. It is only defined on a locally compact topological group, but in our cases, the Lie groups and parameter spaces, we satisfy these conditions. The naming of the left-invariant Haar measure is because it is invariant under actions of the group from the left, as considered in this text, in other words $\mu(T_\rho\theta) = \mu(\theta)$ should hold. We can specify the Haar measure uniquely by wanting it normalized on the given group, which we do here, for we want it to be linked to a prior distribution. There have been some papers on Haar measures in particle physics which thus link these two discussions: [33], [40], [38] and [4].

identify the invariant measure of the symmetry group with the prior distribution.

As a consequence of the symmetries of p and $\mu = \pi$, the distribution of the posterior from Bayes' theorem is invariant under actions of the group. This can be seen by rewriting f from equation (2.2) with the change of integration variable:

$$\begin{aligned} f(x) &= \int p(x|\theta)\pi(\theta)d\theta = \int p(x|T_\rho\theta)\pi(T_\rho\theta)dT_\rho\theta = \int p(x|T_\rho\theta)\pi(\theta)d\theta \\ &= \int p(T_\rho^{-1}x|\theta)\pi(\theta) \left| \frac{\partial T_\rho^{-1}x}{\partial x} \right| d\theta = f(T_\rho x) \left| \frac{\partial T_\rho^{-1}x}{\partial x} \right|. \end{aligned} \quad (2.25)$$

The invariance of π gives the second step, and the invariance of p the third. Note that the fact that f is invariant under transformations T_ρ does not mean it is the invariant measure in the space of x , for in general it is impossible to map x to θ one to one.

From this invariance we find:

$$\begin{aligned} P(\theta|x) &= \frac{p(x|\theta)\pi(\theta)}{f(x)} = \frac{p(T_\rho x|T_\rho\theta)\pi(\theta)}{f(x)} \left| \frac{\partial T_\rho x}{\partial x} \right| = \frac{p(T_\rho x|T_\rho\theta)\pi(\theta)}{f(T_\rho x)} \\ &= \frac{p(T_\rho x|T_\rho\theta)\pi(T_\rho\theta)}{f(T_\rho x)} \left| \frac{\partial T_\rho\theta}{\partial \theta} \right| = P(T_\rho\theta|T_\rho x) \left| \frac{\partial T_\rho\theta}{\partial \theta} \right|. \end{aligned} \quad (2.26)$$

The first step is given by invariance of p , the second by invariance of f and the third by invariance of π . Thus indeed, the posterior is also invariant.

We will now discuss a few simple examples of invariant measures, and thus priors.

Example 2.1.2. Consider the last example 2.1.1. We find that $\Phi(\theta, \theta') = \theta + \theta'$, thus the invariant measure is uniform, $\mu(\theta) = \mu(\epsilon)$, thus symmetries under translations result in a uniform invariant measure. ◀

Example 2.1.3. Consider the structure $p(x|\theta) = \theta^{-1}f(x/\theta)$, for $0 < x < \infty$, invariant under rescaling, where $T_\rho x = \rho x$, for $0 < \rho < \infty$. We find $\Phi(\theta, \theta') = \theta\theta'$, meaning that the invariant measure gives $\mu(\theta) \propto \theta^{-1}$. By reparametrizing to $x' = \ln(|x|)$ and $\lambda = \ln(\theta)$, this form takes the form from the last example, thus again, the Gaussian model is an example, as well as the Exponential model, given by $p(x|\theta) = \theta^{-1}e^{-x/\theta}$. ◀

Example 2.1.4. Suppose we have a structure $p(x|\theta, \sigma) = \sigma^{-1}f((x - \theta)/\sigma)$, as in the Gaussian model again, or as in the Students t -distribution $p(x|\theta, \sigma) = (B\sigma)^{-1}(1 + ((x - \theta)/\sigma)^2)^{-(\alpha+1)/2}$. It is invariant under $T_{\theta, \sigma}x = \theta + \sigma x$, and $T_{\theta, \sigma}^{-1}x = (x - \theta)/\sigma$, where $-\infty < x, \theta < \infty$ and $0 < \sigma < \infty$. We see that $\Phi(\theta, \sigma; \theta', \sigma') = (\theta' + \theta\sigma'; \sigma\sigma')$, meaning that we find the Jacobian and fill in $\epsilon = (0, 1)$ to find that $\mu(\theta, \sigma) = \mu(0, 1) \left| \begin{array}{cc} \sigma & 0 \\ 0 & \sigma \end{array} \right|^{-1} = \mu(0, 1)\sigma^{-2}$. Note, this is not the product of the measure of the individual groups, which would happen if the group was Abelian, which it is not. ◀

Example 2.1.5. Lastly, consider a two-dimensional Gaussian:

$$f(x) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-x^\dagger \frac{1}{2C} x}, \quad (2.27)$$

where $x = (x_1, x_2)^T$ and $C = \text{diag}(\sigma_1, \sigma_2)$. Introducing T_ϕ as the two-dimensional rotation matrix seen before. Essentially this rotates $C \rightarrow T_\phi C T_\phi^\dagger$ when letting the group elements act on x . This symmetry goes into more geometrical arguments when finding the invariant measure, and can be found in [26]. We can also introduce triangular matrix $T_\gamma = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$ or $T_\theta = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$, where $\theta = (\alpha, \beta, \gamma)^T$. In the first case, $\Phi = \gamma + \gamma'$, so the invariant measure is the uniform one. In the second case, $\Phi = (\alpha\alpha', \beta\beta', \alpha\gamma' + \gamma\beta')^T$. This group is not Abelian, and contains the subgroup of scalings and of translations, and gives the invariant measure:

$$\mu(\theta) = \mu(\epsilon) \begin{vmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & \gamma & \alpha \end{vmatrix}^{-1} = \mu(\epsilon)(\alpha^2\beta)^{-1}. \quad (2.28)$$

◀

2.1.4 Jeffreys' Rule

Sir Harold Jeffreys proposed a formula that yields the measure μ of an invariant model p in a formulaic way only, without need of analyzing the symmetry group, or thus the multiplication function Φ . It even defines the prior for models that are not invariant under any group of transformations. Note that it does not mean the prior will be invariant, but the posterior will be, in the same way as shown before. Jeffreys proposed it as a generalization of his own findings in choosing least informative priors. His suggestion as improvement of Bayes' suggestion of the uniform prior was a prior $\pi \propto 1/\sigma$ if σ is continuous and positive. That way, we would say the same independent of the parameter being present as σ or σ^n in the model. This can however not describe other transformations.

The **Jeffreys prior** works such that, if you change variables before calculating a posterior with help of Jeffreys' prior, you will get the same result as changing variables in the posterior. This does not work on the level of the prior itself.

The prior is defined in terms of the **Fisher information matrix** I :

$$\pi_J(\theta) \propto \sqrt{\det(I(\theta))}; \quad I(\theta)_{i,j} = \mathbb{E} \left[\frac{\partial l(x|\theta)}{\partial \theta_i} \frac{\partial l(x|\theta)}{\partial \theta_j} \middle| \theta \right], \quad (2.29)$$

where \mathbb{E} denotes the expectation value, and l denotes the log-likelihood for θ , dependent on the model p . As is evident, it only uses the local properties of p (through l) with respect to θ , and not the global symmetry properties.

It can be shown that Jeffreys prior transforms as:

$$\pi_J(\theta) = \pi_J(T_\rho \rho) \left| \frac{\partial T_\rho \theta}{\partial \theta} \right|. \quad (2.30)$$

For more information and a proof you can check [26].

Jeffreys' prior can even be further generalized in the context of Reference Analysis, as found in [7], which we will lightly touch upon in the next section.

2.1.5 Reference priors

The focus of Reference Analysis is to use the information-theoretic concepts to produce an even clearer description of objective, non-informative priors, that will then be maximally dominated by the

data. This should be done while maximizing the missing information about the parameters. It has been shown that the principle explained beneath reduces to Jeffreys' prior under specific regularity conditions. The basic of this theory is the **Kullback-Leibler divergence**, or relative entropy or logarithmic divergence of a probability density \tilde{P} from its true probability density P , denoted by:

$$D(\tilde{P}||P) = \int P(x) \log \frac{P(x)}{\tilde{P}(x)} dx, \quad (2.31)$$

provided this is finite of course.

In [7] and [26], an argument is given why improper priors ($\int \pi(x)dx$ diverges) do still produce useful posteriors. This is needed not only in the case of reference priors, but in the case of Jeffreys' prior itself already, for it could also produce improper priors at first.

Having established this fact, we can define the **Shannon expected information** from one observation from a model $p(x|\theta)$ when the prior is $\pi(\theta)$:

$$I(\pi|p) = \int \int p(x|\theta) \pi(\theta) \log \frac{P(\theta|x)}{\pi(\theta)} dx d\theta, \quad (2.32)$$

where $P(\theta|x)$ is the posterior from Bayes' theorem. It can be seen that the amount of information expected depends on the sharpness of the prior. Taking more and more independent replications of the same experiment, we will eventually find a measure of missing information about θ associated to the prior $\pi(\theta)$. The reference prior is then a prior which maximizes that missing information. When the parameter space is continuous, the missing information typically diverges, so the definition for a prior having the property of maximizing missing information is if for any compact subset Θ_0 of the parameter space and $\tilde{\pi}$ for which $\int p(x|\theta)\tilde{\pi}(\theta)d\theta < \infty$ it holds that:

$$\lim_{k \rightarrow \infty} [I(\pi_0|p^k) - I(\tilde{\pi}_0|p^k)] \geq 0, \quad (2.33)$$

where p^k signifies us taking k repetitions of the experiment, and the subscript \cdot_0 signifies renormalized restrictions to Θ_0 .

The Jeffreys prior is the reference prior under regularity conditions guaranteeing asymptotic posterior normality, as proven by [7].

Reference priors are seen to be independent of sample size and consistent under reparametrizations, because the expected information is as well. Saying this, it is of course only useful to talk about a reference prior if the expected informations are finite for the replications of the experiment.

The solution for this is given by:

$$\begin{aligned} \pi(\theta) &= \lim_{k \rightarrow \infty} \frac{f_k(\theta)}{f_k(\theta_0)}; \\ f_k(\theta) &= \exp \left[\int p(x^k|\theta) \log P^*(\theta|x^k) dx^k \right]; \\ P^*(\theta|x^k) &= \frac{p(x^k|\theta)\pi^*(\theta)}{\int p(x^k|\theta)\pi^*(\theta)d\theta}, \end{aligned} \quad (2.34)$$

where $\pi^*(\theta)$ is a continuous strictly positive function such that P^* is proper and asymptotically consistent, θ_0 is an interior point of Θ_0 and x^k denotes the data generated by k independent repetitions of the same experiment. The choices of θ_0 and π^* do not influence the outcome.

In multiparameter models the priors are obtained sequentially by repeating the procedure keeping other parameters fixed, so we find first $\pi(\theta_1|\vec{\theta}_{i\neq 1})$, and then $\pi(\vec{\theta}) = \pi(\theta_n)\pi(\theta_{n-1}|\theta_n)\dots\pi(\theta_1|\vec{\theta}_{i\neq 1})$. This could depend on the ordering of the parameters, and then the ordering should prefer the ones that are perceived to be more important.

A great example of this principle in action is demonstrated in [27], where the reference prior (which is in this case argued to be the Jeffreys' prior) is found for the observable parameters in neutrino mass research, and then transformed to a prior on the three neutrino masses themselves.

In the next section we discuss some problems that can be encountered when working with priors even when you have already picked a least informative prior on a certain parameter. This is due to the concept of concentration of measure, which we will apply to random matrices.

2.2 Concentration of Measure

Because we want to infer the structure of the neutrino mass matrix to be able to say if neutrinos are Majorana or Dirac and Normal or Inverted, we need to put priors on the matrix elements. We will show why this amounts to ultimately informative priors by discussing a concept known as Concentration of Measure.

The simplest example of this concept is what happens with the distribution of the sum of n elements that are selected from a Uniform distribution between 0 and 1. This is displayed in the figure 2.1.

Because $[0, 1]$ is compact, the distribution of the n th sum scatters over the compact $[0, n]$. Due to the x_i being drawn independently, it is unlikely they simultaneously take extreme values, meaning that the tails of the distributions thin out.

Another, more general example describing the same principle is the law of large numbers, which is well-known and states that averages of independent random variables are, under some integrability conditions, close to their expectations with a high probability. Recently it was shown that this behaviour is shared by a large class of general functions of independent variables. The proof of the weak law of large numbers has been given dependent on inequalities, but also on convergence of characteristic functions. Both methods are nowadays used to prove more and more about the concentration phenomenon. The inequalities often account to predictions of the decrease of the tail probabilities, or upper bounds for the probability that a real-valued random variable differs from its expected value by more than a certain amount. We will discuss some necessary context, mostly for sums of independent variables, and then give an example of this phenomenon for random matrix theory.

2.2.1 Early Inequalities

The discussion below will follow the lines of [45].

Before concentration of measure became a well-studied theory, there were already some theorems leading up to proving the concept. These are still used and the basic ideas have only been extended to produce better bounds. The first is known as **Markov's inequality** for any nonnegative random variable X and $t > 0$, $X\mathbb{1}_{X\geq t} \geq t\mathbb{1}_{X\geq t}$, where $\mathbb{1}_C$ is 1 when the condition C is satisfied, and otherwise

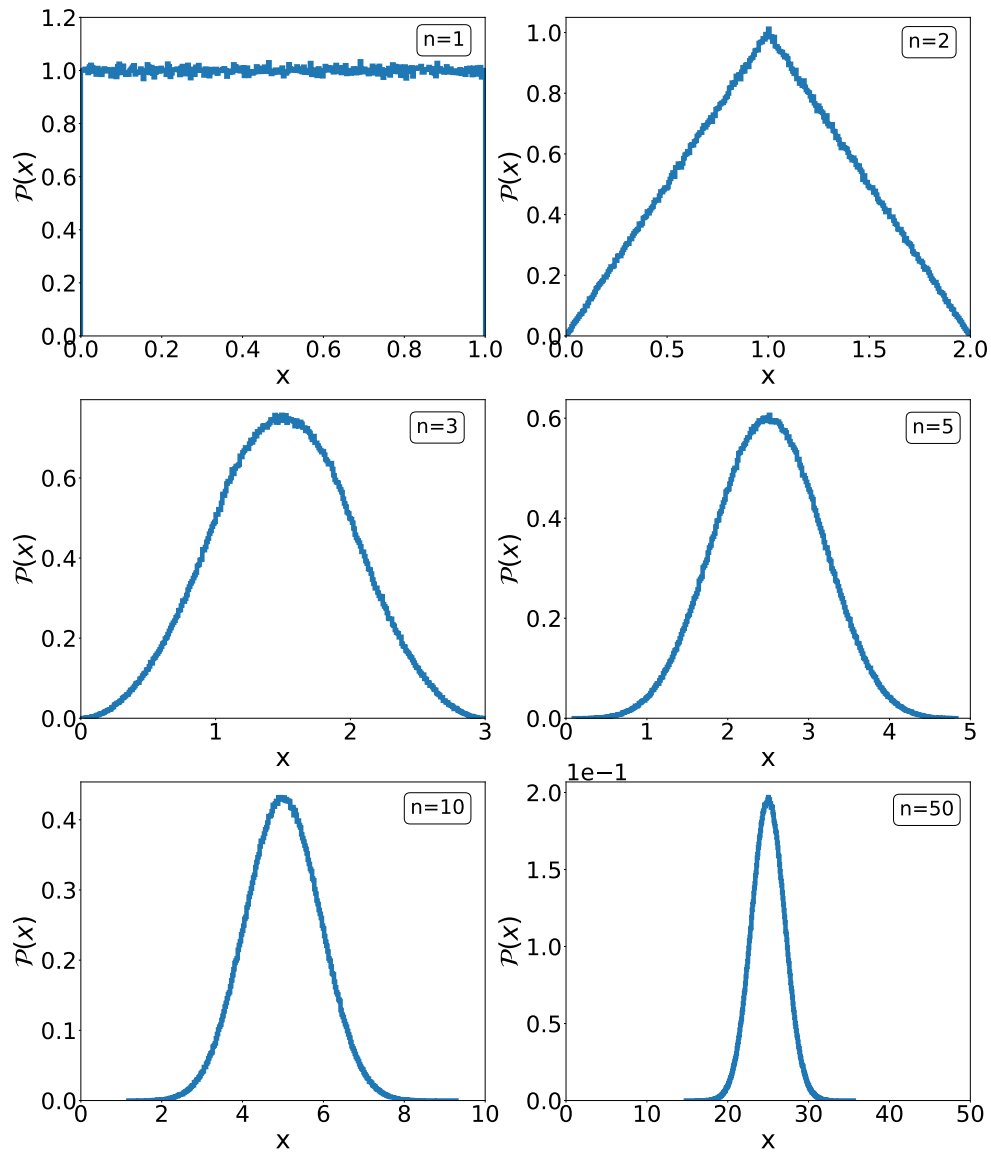


Figure 2.1: *Example of the phenomenon of Concentration of Measure. Each panel depicts the distribution of the sum $\sum_{i=1}^n x_i$, where each random number x_i was independently drawn with $x_i \sim \text{Uniform}[0, 1]$. The distributions are represented by histograms of equally big samples for each n .*

0. Taking the expectation values of both these sides gives $\mathbb{P}(X \geq t) \leq \mathbb{E}(X\mathbf{1}_{X \geq t})/t \leq \mathbb{E}X/t$, or, in other words, $\mathbb{P}(X \geq t) \leq \mathbb{E}X/t$. Of course this is only of interest if $\mathbb{E}X < \infty$, or thus if X is integrable. This can be extended by introducing a nondecreasing and nonnegative function ϕ , defined on a set containing the domain of X . This gives:

$$\mathbb{P}(X \geq t) = \mathbb{P}(\phi(X) \geq \phi(t)) \leq \frac{\mathbb{E}\phi(X)}{\phi(t)}. \quad (2.35)$$

This is known as **Chebyshev's inequality** in the case that $\phi = x^2$, and $Y = |X - \mathbb{E}X|$, giving:

$$\mathbb{P}(|X - \mathbb{E}X| \geq t) = \mathbb{P}(|X - \mathbb{E}X|^2 \geq t^2) \leq \frac{\mathbb{E}(|X - \mathbb{E}X|^2)}{t^2} = \frac{Var(X)}{t^2}. \quad (2.36)$$

When X is a sum of n independent variables X_i , we have $Var(X) = \sum_i Var(X_i)$, and Chebyshev's inequality gives that:

$$\mathbb{P}\left(\frac{1}{n} \left| \sum_{i=1}^n (X_i - \mathbb{E}X_i) \right| \geq t\right) \leq \frac{\sigma^2}{nt^2}, \quad (2.37)$$

where $\sigma^2 = \frac{1}{n} \sum_i Var(X_i)$. When $\phi = x^q$, these are called the **moment bounds**, and you can choose the optimal q for your bound. Using the Cauchy-Schwarz inequality ($|\mathbb{E}(XY)| \leq \sqrt{\mathbb{E}(X^2)\mathbb{E}(Y^2)}$ for two random variables with finite second moments) gives $\mathbb{P}(X - \mathbb{E}X \geq t) \leq Var(X)/(Var(X) + t^2)$. By instead taking $\phi(x) = e^{\lambda x}$, with λ a positive number, we get to work with the moment generating functions $F(\lambda) = \mathbb{E}e^{\lambda X}$ of the random variables. Then we can optimize, in λ , the bound obtained. This is easier to handle than moment bounds, especially in the case of sums of variables, for then:

$$\mathbb{E}e^{\lambda \sum_i (X_i - \mathbb{E}X_i)} = \prod_i \mathbb{E}e^{\lambda(X_i - \mathbb{E}X_i)}, \quad (2.38)$$

although moment bounds will generally result in sharper bounds. When $F(\lambda)$ exists for non-zero values of λ this will even lead to exponential bounds for the tail probabilities $\mathbb{P}(|X - \mathbb{E}X| \geq t)$, and we will now only focus on exponential bounds for $\mathbb{P}(X \geq t)$ where X is a **centered** random variable, equal to $Y - \mathbb{E}Y$ for some random variable Y .

Markov's inequality now implies that $\mathbb{P}(X \geq t) \leq e^{-\lambda t} \mathbb{E}e^{\lambda X}$. Finding the optimal λ is then done by defining $\psi_X(\lambda) = \log(\mathbb{E}e^{\lambda X})$ (known as the logarithmic moment-generating function) and $\psi_X^*(t) = \sup_{\lambda \geq 0} (\lambda t - \psi_X(\lambda))$ (known as the **Cramér transform** of X). By setting the derivative ψ_X' to zero, we find the optimal λ , and can give the so-called **Chernoff's inequality**:

$$\mathbb{P}(X \geq t) \leq e^{-\psi_X^*(t)} = e^{\psi_X(\lambda_t) - \lambda_t t}, \quad \lambda_t = (\psi_X')^{-1}(t), \quad (2.39)$$

where the \cdot^{-1} does not denote the $1/\cdot$ but the functional inverse.

Using the properties of this type of theory, classes of variables have been found for which bounds are easily calculated. Two important ones are known as the sub-Gaussian and the, somewhat more general, sub-gamma variables. A centered random variable is called **sub-Gaussian** with factor v when $\psi_X(\lambda) \leq \lambda^2 v/2$ for all $\lambda \in \mathbb{R}$. This will thus give an upper bound on the Cramér transform, and thus a bound on the tail probabilities. An important lemma known as Hoeffding's lemma states that all bounded centered variables are sub-Gaussian with factor $(b - a)^2/4$, where $[a, b]$ is the bounded interval in which the variable takes its values.

The bound given by sub-Gaussian should be $\lambda^2 v/(2(1 - c\lambda))$ for all λ such that $0 < l < 1/c$ when we

want to say a variable is **sub-gamma** on the right tail with factor v and scale parameter c . A variable is sub-gamma on the left tail when $-X$ is sub-gamma on the right tail.

The maximum of these functions can be bounded by $\mathbb{E} \max_i X_i \leq \sqrt{2v \log(N)}$ for N real-valued sub-Gaussian random variables X_i for which $\psi_{X_i} < \psi$, where ψ is a convex and continuously differentiable function such that $\psi(0) = \psi'(0) = 0$. This becomes $\sqrt{2v \log(N)} + c \log(N)$ for sub-Gamma variables.

Continuing on for sums of n variables specifically, defining $S = \sum_i (X_i - \mathbb{E}X_i)$ for X_i independent with finite mean, giving:

$$\psi_S = \sum_{i=1}^n \log \left(\mathbb{E} \left[e^{\lambda(X_i - \mathbb{E}X_i)} \right] \right), \quad (2.40)$$

which is going to be bounded. As a consequence of the already mentioned Hoeffding's lemma and Chernoff's inequality, we find **Hoeffding's inequality**:

$$\mathbb{P}(S \geq t) \leq \exp \left[-\frac{2t^2}{\sum_{i=1}^n (b_i - a_i)^2} \right], \quad (2.41)$$

where X_i takes values in $[a_i, b_i]$ almost surely⁵. This can equally well be done for $\mathbb{P}(S \leq -t)$. This seems to ignore knowledge of the variances, and we rewrite, using $\log(x) \leq x - 1$ for $x > 0$:

$$\psi_S(\lambda) = \sum_{i=1}^n \left[\log \mathbb{E} \left(e^{\lambda X_i} \right) - \lambda \mathbb{E}X_i \right] \leq \sum_{i=1}^n \mathbb{E} \left[e^{\lambda X_i} - \lambda X_i - 1 \right]. \quad (2.42)$$

Two inequalities, Bennett's and Bernstein's, can be derived from this under different integrability conditions. **Bennett's inequality** states that if the X_i have finite variance such that $X_i \leq b$ for $b > 0$ almost surely for all $i \leq n$, and $v = \sum \sigma_i^2$ where $\sigma_i^2 = \mathbb{E}(X_i^2)$, it holds that:

$$\mathbb{P}(S \geq t) \leq \exp \left(-\frac{v}{b^2} h \left(\frac{bt}{v} \right) \right); \quad h(x) = (1+x) \log(1+x) - x. \quad (2.43)$$

Bernstein's inequality needs stricter bounds and tightens this bound somewhat more. Let $\lambda \in (0, 1/c)$ and v as in Bennett's, then:

$$\mathbb{P}(S \geq t) \leq \exp \left[-\frac{t^2}{2(v + ct)} \right]. \quad (2.44)$$

Lastly, deriving concentration inequalities often involves association properties. Chebyshev's association property can be generalized to **Harris' inequality**, which states: For $f, g : \mathbb{R}^n \rightarrow \mathbb{R}$ non-decreasing functions (non-decreasing in each variable while keeping all others fixed) and $X = (X_1, \dots, X_n)$ independent real-valued random variables, then $\mathbb{E}(f(X)g(X)) \geq \mathbb{E}(f(X))\mathbb{E}(g(X))$ and if f is non-increasing and g nondecreasing we have similarly $\mathbb{E}(f(X)g(X)) \leq \mathbb{E}(f(X))\mathbb{E}(g(X))$.

In the next section we will give a first attempt for concentration inequalities for more general functions of independent random variables.

2.2.2 Efron-Stein Inequality

We will present the Efron-Stein inequality for bounding the variance of complex functions of independent random variables, and will then use this to present better bounds for tail probabilities. Once we have

⁵In probability theory, an occurrence is almost surely when the probability of it happening is 1.

this, we can give our first application to random matrices.

We will write $X = f(Y_1, \dots, Y_n)$, where $f : A^n \rightarrow \mathbb{R}$ where A is some measurable space and the Y_i are independent random variables taking values in A . We write $\mathbb{E}_i X = \mathbb{E}(X | Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_n)$, and it can then be proven that:

$$\text{Var}(X) \leq \sum_i \mathbb{E} [(X - \mathbb{E}_i X)^2]. \quad (2.45)$$

This proof is dependent on elementary properties of conditional expectations, and can be found in [45]. Let now Y'_1, \dots, Y'_n form an independent copy of Y_1, \dots, Y_n and write $X'_i = f(Y_1, \dots, Y'_i, \dots, Y_n)$. The **Efron-Stein inequality** now states that:

$$\text{Var}(X) \leq \frac{1}{2} \sum_{i=1}^n \mathbb{E} [(X - X'_i)^2], \quad (2.46)$$

which follows from the equation above, using that if X and Y are independent and identically distributed, then $\text{Var}(X) = \frac{1}{2} \mathbb{E}(X - Y)^2$, and therefore:

$$\mathbb{E}_i [(X - \mathbb{E}_i X)^2] = \frac{1}{2} \mathbb{E}_i [(X - X'_i)^2]. \quad (2.47)$$

Note that when $X = \sum_i Y_i$, the inequality above becomes an equality, thus in some sense, this inequality can not be improved. The equality also means that sums are the least concentrated functions of independent random variables.

Recalling that for any random variable it holds that $\text{Var}(X) \leq \mathbb{E}[(X - a)^2]$ for any $a \in \mathbb{R}$, so we can also write:

$$\mathbb{E}_i [(X - \mathbb{E}_i X)^2] \leq \mathbb{E}_i [(X - X_i)^2], X_i = f_i(Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_n), \quad (2.48)$$

for arbitrary functions $f_i : A^{n-1} \rightarrow \mathbb{R}$, thus also:

$$\text{Var}(X) \leq \sum_{i=1}^n \mathbb{E} [(X - X_i)^2]. \quad (2.49)$$

The Efron-Stein inequality and its deviates can be used for special classes of functions f , giving more useful results. The first is for functions with the bounded differences property, which a function has if for some nonnegative constants c_i it holds that:

$$\sup_{x_1, \dots, x_n, x'_i} |f(x_1, \dots, x_n) - f(x_1, \dots, x_{i-1}, x'_i, x_{i+1}, \dots, x_n)| \leq c_i. \quad (2.50)$$

Thus, changing the i -th variable while keeping the others fixed should not allow the function to vary more than c_i . This is also known as the **Lipschitz hypothesis**. Now the following is implied due to Efron-Stein:

$$\text{Var}(X) \leq \frac{1}{2} \sum_{i=1}^n c_i^2. \quad (2.51)$$

This is also known as **McDiarmid's inequality**. It is interesting to note that this serves as a "tensorization" of Hoeffding's lemma.

For self-bounding functions this becomes even simpler. A non-negative function is called self-bounding

if there exist $f_i : A^{n-1} \rightarrow \mathbb{R}$ such that both:

$$\begin{aligned} 0 &\leq f(x_1, \dots, x_n) - f_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \leq 1 \\ \sum_{i=1}^n [f(x_1, \dots, x_n) - f_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)] &\leq f(x_1, \dots, x_n) \\ \Rightarrow \sum_{i=1}^n [f(x_1, \dots, x_n) - f_i(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)]^2 &\leq f(x_1, \dots, x_n). \end{aligned} \quad (2.52)$$

This last in turn implies that for f self-bounded it holds that $\text{Var}(X) \leq \mathbb{E}X$.

We can now take a look at the first example.

Example 2.2.1. Let A be a symmetric real matrix whose entries $X_{i,j}$ are independent random variables with absolute value bounded by 1. Let $Z = \lambda_1$ denote the largest eigenvalue of A . If $v = (v_1, \dots, v_n)$ is an eigenvector corresponding to λ_1 with $\|v\| = 1$, it holds that $\lambda_1 = v^T A v = \sup_{u: \|u\|=1} u^T A u$.

We can use the Efron-Stein inequality for the symmetric matrix $A'_{i,j}$ obtained by replacing $X_{i,j}$ from A with an independent copy $X'_{i,j}$ keeping the rest fixed, implying a $Z'_{i,j}$ denoting the largest eigenvalue of the new matrix.

Then, using the property for the largest eigenvalue ($x_+ = \max(x, 0)$):

$$\begin{aligned} (Z - Z'_{i,j})_+ &\leq (v^T A v - v^T A'_{i,j} v) \mathbb{1}_{Z > Z'_{i,j}} = (v^T (A - A'_{i,j}) v) \mathbb{1}_{Z > Z'_{i,j}} \\ &\leq 2(v_i v_j (X_{i,j} - X'_{i,j}))_+ \leq 4|v_i v_j|, \end{aligned} \quad (2.53)$$

and therefore:

$$\sum_{1 \leq i \leq j \leq n} (Z - Z'_{i,j})_+^2 \leq \sum_{1 \leq i \leq j \leq n} 16|v_i v_j|^2 \leq 16 \left(\sum_{i=1}^n v_i^2 \right)^2 = 16. \quad (2.54)$$

Using Efron-Stein after now taking the expectation of both sides, we thus find $\text{Var}(Z) \leq 16$. Note that we only used independence and boundedness of the entries, they do not even have to have the same distribution. This is analogous for the smallest eigenvalue. The variance is thus bounded by a constant regardless of the size of the matrix. \blacktriangleleft

We already introduced the Lipschitz hypothesis, and continue with likewise definitions by defining a function f to be called **Lipschitz continuous** if there exists some $K \in \mathbb{R}$ such that for all x, y it holds that $|f(x) - f(y)| \leq K|x - y|$. Talagrand proved for n independent, complex variables $X = (X_1, \dots, X_n)$, with $|X_i| \leq M$ for all $1 \leq i \leq n$ for some constant $M > 0$. If f is a Lipschitz convex function (in all parameters), then for any t one has:

$$\mathbb{P}(|f(X) - \mathbb{E}f(X)| \geq tK) \leq C e^{-ct^2}, \quad (2.55)$$

where C and c are positive absolute constants. A proof can be found on [1].

Some theorems that are quite alike are the convex Poincaré inequality and the Gaussian Poincaré inequality. The first is less restricted than the second. For the first we need $X = (X_1, \dots, X_n)$ to be independent random variables taking values in $[0, 1]$, and $f : [0, 1]^n \rightarrow \mathbb{R}$ a separately convex function whose partial derivatives exist. Then $\text{Var}(f(X)) \leq \mathbb{E}(\|\nabla f(X)\|^2)$. A function is separately convex if it is convex in each variable when keeping the others fixed. Note that convexity implies Lipschitz continuity. We can apply this to random matrices again.

Example 2.2.2. Let A be an $m \times n$ matrix with entries $X_{i,j}$, being independent random variables taking values in $[0, 1]$. The largest singular value is defined as the square root of the largest eigenvalue of the symmetric $n \times n$ matrix $A^T A$, so we define:

$$Z = \sqrt{\lambda_1(A^T A)} = \sqrt{\sup_{u \in \mathbb{R}^n: \|u\|=1} u^T A^T A u} = \sup_{u \in \mathbb{R}^n: \|u\|=1} \|Au\|. \quad (2.56)$$

For each u , $\|Au\|$ is a convex function of the mn -dimensional vector formed by the $X_{i,j}$, the supremum of which is convex, so Z is convex.

An inequality known as **Lidskii's inequality** from linear algebra states for two symmetric $n \times n$ matrices A and B , with their non-increasing eigenvalues λ_i ($i \leq n$) that:

$$\sum_{i=1}^n (\lambda_i(A) - \lambda_i(B))^2 \leq \sum_{i,j=1}^n (a_{i,j} - b_{i,j})^2. \quad (2.57)$$

In this case, for the singular values for A with $x_{i,j}$ and B with $y_{i,j}$, $m \times n$ dimensional matrices, this implies:

$$\begin{aligned} (s_1(A) - s_1(B))^2 &\leq \sum_{i=1}^n (s_i(A) - s_i(B))^2 \leq \sum_{i=1}^n s_i(A - B)^2 \\ &= \text{tr}((A - B)^T (A - B)) = \sum_{i=1}^m \sum_{j=1}^n (x_{i,j} - y_{i,j})^2, \end{aligned} \quad (2.58)$$

meaning the largest singular value is Lipschitz with $K = 1$, giving $\text{Var}(Z) \leq 1$. ◀

In the next section we will introduce the entropy into the mix again, which can give even better bounds.

2.2.3 Information inequalities and the Entropy method

Ledoux [29] developed the entropy method, based on the logarithmic Sobolev inequalities, to derive exponential tail analogues of the Efron-Stein inequality. We here stick to discrete random variables for now, and comment on the generalization afterwards. The general argument needs mathematics outside the scope of this thesis.

Important is the Shannon entropy from equation (2.3), and the relative entropy of two distributions P and Q :

$$D(P||Q) = \sum_x p(x) \log \frac{p(x)}{q(x)}. \quad (2.59)$$

Note that this is always nonnegative (since $\log(x) \leq x - 1$ for $x > 0$) and zero only if $P = Q$. This implies that $H(X) \geq H(X|Y) := H(X, Y) - H(Y)$, so conditioning reduces entropy. An important result is **Han's inequality**:

$$H(X_1, \dots, X_n) \leq \frac{1}{n-1} \sum_{i=1}^n H(X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n), \quad (2.60)$$

and Han's inequality for relative entropies following from this. Therefore A needs to be a countable set, and P and Q probability distributions on A^n such that $P = P_1 \times \dots \times P_n$ is a product measure. A superscript $\cdot^{(i)}$ denotes leaving out the i -th component, meaning for P and Q that it denotes the

marginalization over the i -th component, which we thus sum out. Then it holds:

$$\begin{aligned} D(Q\|P) &\geq \frac{1}{n-1} \sum_{i=1}^n D(Q^{(i)}\|P^{(i)}) \\ D(Q\|P) &\leq \sum_{i=1}^n \left(D(Q\|P) - D(Q^{(i)}\|P^{(i)}) \right). \end{aligned} \tag{2.61}$$

This is essential in deriving the sub-additivity of the entropy, which is the basis of the entropy method. We can rewrite Efron-Steins inequality as: $Var(X) \leq \sum_i \mathbb{E} [\mathbb{E}_i(X^2) - (\mathbb{E}_i X)^2]$, or, for $\Phi = x^2$:

$$\mathbb{E}\Phi(X) - \Phi(\mathbb{E}X) \leq \sum_{i=1}^n \mathbb{E}[\mathbb{E}_i\Phi(X) - \Phi(\mathbb{E}_i X)], \tag{2.62}$$

which appears to be true for a large class of convex functions Φ . We are specifically considering $\Phi = x \log x$. Then the L.H.S. above is often called the **entropy** of X , and denoted by $\text{Ent}(X)$. We can introduce $\text{Ent}_i(X) = \mathbb{E}_i\Phi(X) - \Phi(\mathbb{E}_i X)$, and the theorem of the sub-additivity, otherwise called the tensorization of the entropy, then states: For $\Phi(x) = x \log(x)$ for $x > 0$ and $Y = f(X_1, \dots, X_n)$ where the X_i take values in A and f is positively valued on A^n , we have the expression in equation (2.62) for Y , or otherwise:

$$\text{Ent}(Y) \leq \mathbb{E} \left[\sum_{i=1}^n \text{Ent}_i(Y) \right]. \tag{2.63}$$

The main idea is now to apply this to the positive $Y = e^{\lambda X}$. Then, denoting the moment generating function of X by $F(\lambda) = \mathbb{E}e^{\lambda X}$, the L.H.S. of that inequality becomes:

$$\lambda \mathbb{E}[X e^{\lambda X}] - \mathbb{E}e^{\lambda X} \log \mathbb{E}e^{\lambda X} = \lambda F'(\lambda) - F(\lambda) \log F(\lambda). \tag{2.64}$$

Now we can derive upper bounds for the derivative of $F(\lambda)$, and use Chernoff's inequality to find bounds for the tail probabilities. To do this in a convenient way, we need further bounds for the R.H.S. of the sub-additivity theorem. The inequalities we will use are closely related to the so-called logarithmic Sobolev inequalities.

Herbst's argument

A name that should be mentioned when working within this theory is Herbst, who gave a generalization argument for the logarithmic Sobolev inequalities, which first only worked for Bernoulli or Gaussian random variables. We will not present these first results, but continue to state the results of Herbst's argument applied to the original Sobolev inequalities. If X is an integrable random variable such that, for some $v > 0$ we have for all $\lambda > 0$ that $\text{Ent}(e^{\lambda X})/\mathbb{E}(e^{\lambda X}) \leq \frac{\lambda^2 v}{2}$, then it holds as well that $\log(\mathbb{E}e^{\lambda(X-\mathbb{E}X)}) \leq \frac{\lambda^2 v}{2}$.

This can already be used to proof a better, exponential, version of the bounded differences inequality from above. When we denote f having the bounded differences property and $v = \frac{1}{4} \sum_i c_i^2$ and $X = f(Y_1, \dots, Y_n)$ where Y_i are independent, then $\mathbb{P}(X - \mathbb{E}X > t) \leq e^{-t^2/2v}$. The proof of this combines the sub-additivity of the entropy, Hoeffding's lemma and Herbst's argument, and finishes with Markov's inequality. It can be found in [45].

This already allows us to look at the next example:

Example 2.2.3. Consider a Hermitian $n \times n$ random matrix $H = (H_{i,j})$ such that the vectors $(H_i)_{1 \leq i \leq n}$, where $H_i = (H_{i,j})_{1 \leq j \leq i}$, are independent. We introduce a probability measure which we will need to discuss a phenomenon which occurs when taking the eigenvalue decomposition of a matrix, done in the next section. This is the so-called spectral measure L_H , which gives mass r/n to an eigenvalue of H with multiplicity r . Given a bounded function g with a specific boundedness property: the total variation should be smaller than 1. The total variation is:

$$\|g\|_{TV} = \sup_{n=1,2,\dots} \sup_{x_1 < \dots < x_n} \sum_{i=1}^{n-1} |g(x_{i+1}) - g(x_i)|. \quad (2.65)$$

We are interested in the concentration of the integral $X = \int g dL_H = \int F_H dg$.

Defining $(H(x))_{i,j} = x_{i,j}$ for $1 \leq j \leq i \leq n$, where $x = (x_1, \dots, x_n)$ such that $x_i \in \mathbb{C}^{i-1} \times \mathbb{R}$ for all i . Then $f(x) = \int g dL_{H(x)}$ has the bounded differences property, which follows from the Cauchy interlacing theorem. It results in $\|F_A - F_B\|_\infty \leq \text{rank}(A - B)/n$, where F_A is the distribution function related to the spectral measure L_A . Integrating by parts gives for every x and x' : $|f(x) - f(x')| \leq \text{rank}(H(x) - H(x'))/n$. If x' only differs from x in one coordinate, the matrix we take the rank of has all zero entries except for one row and one column, so its rank is ≤ 2 . Thus $c_i = 2/n$ suffices for all i . Therefore, using the bound found before this example, $\mathbb{P}(|X - \mathbb{E}X| \geq t) \leq 2e^{-nt^2/2}$ for all $t > 0$.

This illustrates a concentration of spectral measure. \blacktriangleleft

Denote again $X_i = f_i(Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_n)$ as we did for the Efron-Stein discussion, then one of the generalized Sobolev inequalities states that the L.H.S. from equation (2.64) can be bounded from above by:

$$\sum_{i=1}^n \mathbb{E} \left[e^{\lambda X} \psi(-\lambda(X - X_i)) \right], \quad (2.66)$$

where $\psi(x) = e^x - x - 1$. The proof makes use of the fact that $\mathbb{E}(Y \log(Y)) - \mathbb{E}Y \log \mathbb{E}Y \leq \mathbb{E}(Y \log Y - Y \log u - Y + u)$ for any $u > 0$. Proofs for both can be found in [45].

This already allows us to prove for X that $\sum_i (X - X_i)^2 \leq v$ almost surely for some $v > 0$, then $\mathbb{P}(X - \mathbb{E}X > t) \leq e^{-t^2/2v}$, generalizing the result we had for functions with the bounded differences property.

This can be applied to eigenvalues again:

Example 2.2.4. The results from example 2.2.1 can now be improved, for we have proven $\sum_i (Z - Z_{i,j})^2 \leq 16$, which can be applied to the result given above to find $\mathbb{P}(Z - \mathbb{E}Z > t) \leq e^{-t^2/32}$, because we thus have already shown $v = 16$ works in this case. \blacktriangleleft

As we did before, the result can be specifically applied to $Z = f(X_1, \dots, X_n)$, where X_i take values in $[0, 1]$ and f is separately convex and Lipschitz. Then: $\mathbb{P}(Z - \mathbb{E}Z > t) \leq e^{-t^2/2}$, from the fact that $\sum_i (Z - Z_i)^2 \leq \|\nabla(f(X))\|^2 \leq 1$, from the Lipschitz property and our previous statements. This applies to the largest singular value of a random matrix as done in example 2.2.2.

Isoperimetry and Measures

It is interesting to note that to connect these probabilistic concentration inequalities with more set-theoretical measure theory, the functions observed will mainly be distances on sets, and the variables will represent a random set of points in the space observed. The most important result in this

context is the Convex Distance inequality, proven by Talagrand, stating that for any subset $A \subseteq \mathcal{X}^n$ where \mathcal{X} is any measurable space, and $t > 0$:

$$\mathbb{P}(A)\mathbb{P}(d_T(X, A) \geq t) \leq e^{-t^2/4}. \quad (2.67)$$

Here $d_T(x, A) = \sup_{\alpha \in [0, \infty)^n: \|\alpha\|=1} d_\alpha(x, A)$, X a random variable taking values in \mathcal{X} , $d_\alpha(x, A) = \inf_{y \in A} d_\alpha(x, y) = \inf_{y \in A} \sum_{i: x_i \neq y_i} \alpha_i$, where α is a vector of nonnegative numbers, the weighted Hamming distance.

In the next section we will dive further into the theory of random matrices and explain some phenomena that happen there due to the concentration of measure phenomenon.

2.3 Random Matrices

A **random matrix** is simply a matrix with random variables as entries, as already seen in the examples of the last section. We will study these, for they will form the basis of our discussion in the next chapter, which focuses on how to find the fitting model for explaining the current results of the mass of neutrinos. We consider complex matrices mainly, but will denote symmetric matrices with \mathcal{S}_n the set of $n \times n$ symmetric matrices, and \mathcal{H}_n the set of $n \times n$ hermitian matrices. The standard **Lebesgue measures** on \mathcal{S}_n and \mathcal{H}_n can be written as products of the Lebesgue measures on the linearly independent entries of a matrix $S = (S_{ij})$, $H = (H_{ij})$:

$$\begin{aligned} dS &= \prod_{1 \leq i < j \leq n} dS_{ij} \prod_{i=1}^n dS_{ii}; \\ dH &= \prod_{1 \leq i < j \leq n} d(\operatorname{Re}H_{ij})d(\operatorname{Im}H_{ij}) \prod_{i=1}^n dH_{ii}. \end{aligned} \quad (2.68)$$

We will write the n eigenvalues for $M \in \mathcal{H}_n$ in decreasing order $\lambda_1^M \geq \dots \geq \lambda_n^M$. The **spectral measure** for M is the probability measure:

$$L_M = \frac{1}{n} \sum_{i=1}^n \delta_{\lambda_i^M}, \quad (2.69)$$

giving Dirac mass $1/n$ to each eigenvalue.

In the next section we look at results about some special ensembles of matrices with Gaussian distributions as entries.

2.3.1 Gaussian Ensembles

The simplest examples are **Gaussian ensembles** of random matrices. In the **Gaussian Orthogonal Ensemble** we observe $M \in \mathcal{S}_n$ with independent centered Gaussian random variables X_{ij} and Y_i :

$$M_{ij} = \begin{cases} X_{ij} & \text{if } 1 \leq i < j \leq n \\ X_{ji} & \text{if } 1 \leq j < i \leq n; \\ Y_i & \text{if } 1 \leq i = j \leq n \end{cases} \quad \mathbb{E}[X_{ij}^2] = \frac{\sigma^2}{n}; \quad \mathbb{E}[Y_i^2] = \frac{2\sigma^2}{n}. \quad (2.70)$$

The choice of $1/n$ is to leave the spectrum bounded when $n \rightarrow \infty$, and the difference in scaling is motivated by the fact that the resulting probability measure should stay invariant under conjugation by orthogonal matrices, otherwise it would not be the Orthogonal ensemble⁶. This means that:

$$dM \exp \left[-\frac{n}{2\sigma^2} \left(2 \sum_{i=1}^n M_{ii}^2 + \sum_{1 \leq i < j \leq n} M_{ij}^2 \right) \right] = dM \exp \left[-\frac{n}{2\sigma^2} \text{Tr} M^2 \right] \quad (2.72)$$

should stay invariant. The measure dM is invariant under conjugation by an orthogonal matrix B : $M \rightarrow B^{-1}MB$, and the trace of the square is as well, thus we need the scaling as given to allow this property.

The **Gaussian Unitary Ensemble** has the same property, but for Hermitian matrices conjugated by unitary matrices. We therefore need independent centered Gaussian random variables X_{ij} , \tilde{X}_{ij} and Y_i such that:

$$M_{ij} = \begin{cases} X_{ij} + i\tilde{X}_{ij} & \text{if } 1 \leq i < j \leq n \\ X_{ji} + i\tilde{X}_{ji} & \text{if } 1 \leq j < i \leq n; \\ Y_i & \text{if } 1 \leq i = j \leq n \end{cases} \quad \mathbb{E}[X_{ij}^2] = \mathbb{E}[\tilde{X}_{ij}^2] = \frac{\sigma^2}{2n}; \quad \mathbb{E}[Y_i^2] = \frac{\sigma^2}{n}. \quad (2.73)$$

This gives the probability measure $dM \exp \left[-\frac{n}{\sigma^2} \text{Tr} M^2 \right]$.

Concentration results

Using these results, it seems the spectral measure concentrates around the semi circular density, which is $\phi(x) = (2/\pi)\sqrt{1-x^2}\mathbf{1}_{x \in [-1,1]}$. In other words, if L_n denotes the spectral measure, for any function f defined over \mathbb{R} : $L_n f = (1/n) \sum_i f(\lambda_i^{M_n})$ of a random $n \times n$ matrix from the Gaussian Unitary ensemble, then it converges weakly in probability to the semi-circular distribution. Thus for all $\epsilon > 0$:

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\sup_{f \in \mathcal{B}} \left| L_n f - \int f(x)\phi(x)dx \right| \right] = 0, \quad (2.74)$$

where \mathcal{B} is the set of Lipschitz functions $f : \mathbb{R} \rightarrow [-1, 1]$. This is of course not of interest in our case of 3×3 matrices, but demonstrates a first look at the fact that eigenvalues of a random matrix are not as randomly distributed as one might think, of which more is said in the next section.

It can also be shown for the largest eigenvalue $Z = \lambda_1^{M_n}$, where M_n is $n \times n$ from the Gaussian Unitary Ensemble gives for all $0 \leq t \leq 1$:

$$\mathbb{P}[Z \geq 1 + t] \leq \frac{1}{2\sqrt{t}} e^{-nt^{3/2}}. \quad (2.75)$$

⁶If we fill a matrix M_n of size n with entries of size $\mathcal{O}(1)$, we have:

$$\text{Tr} M_n^2 = \sum_{i,j=1}^n |[M_n]_{ij}|^2 = \sum_{i,j=1}^n [\lambda_i^{M_n}]^2 \sim \mathcal{O}(n^2), \quad (2.71)$$

since the first expression gives it as a sum of n^2 terms of order 1. This means that roughly, the eigenvalues should be of order \sqrt{n} from the second expression, thus filling a matrix with entries of $\mathcal{O}(n^{-1/2})$ or random variables with variance $1/n$ gives a bounded spectrum.

Clearly this is not about deviations from the mean, for $\mathbb{E}Z \neq 1$ per se, but from the convergence of the spectral density we find that $\liminf_{n \rightarrow \infty} \mathbb{E}Z \geq 1$, thus $\lim_{n \rightarrow \infty} \mathbb{E}Z = 1$. There even exists a universal constant $K > 0$ such that for all n , $|\mathbb{E}Z - 1| \leq Kn^{-2/3}$.

2.4 Eigenvalue Repulsion

This discussion combines info from [45] and [2].

We would like to find the joint distribution of eigenvalues of a random matrix to see how the eigenvalues are related to each other when working in random matrix ensembles. It is simplest to do this for a hermitian (or symmetric) one. This needs to be done by transformation of variables in integrals of the form $\int dM f(M)$, so we need its Jacobian. This is somewhat outside the scope of this thesis, but we will explain shortly the main concepts. Further on in this section we will discuss the 'force' felt by eigenvalue repulsion from a different point of view.

Consider first a symmetric matrix and let \mathcal{O}_n be the set of orthogonal $n \times n$ matrices, having $O^T O = I_n$ for all $O \in \mathcal{O}_n$. A symmetric matrix can be diagonalized by an orthogonal one, but that map is not injective, for vectors in eigenspaces could be permuted to give the same diagonalization. Thus we only observe matrices with pairwise distinct eigenvalues⁷, but still permutations with ± 1 can then be done. Therefore, the invertible map is, with \mathbb{R}_Δ^n the set of decreasing vectors (x_1, \dots, x_n) with $x_1 > \dots > x_n$:

$$M : (\mathcal{O}_n / \{\pm 1\}^n) \times (\mathbb{R}_\Delta^n) \rightarrow (\mathcal{S}_n)_\Delta : (O, \lambda_1, \dots, \lambda_n) \mapsto O \text{diag}(\lambda_1, \dots, \lambda_n) O^{-1}. \quad (2.76)$$

This happens to be even an \mathcal{C}^∞ diffeomorphism, thus an admissible change of variables, when we choose coordinates $(\omega_{ij})_{1 \leq i < j \leq n}$ of an antisymmetric matrix parametrizing an orthogonal via $O = e^\omega$ when in the vicinity of $I_n \in \mathcal{O}_n$. In \mathcal{S}_n our coordinates are the matrix entries. Then:

$$dM = 2^{-n} \prod_{1 \leq i < j \leq n} d\omega_{ij} \prod_{i=1}^n d\lambda_i \mathcal{J}(\lambda, \omega); \quad \mathcal{J}(\lambda, \omega) = \left| \det \begin{bmatrix} \frac{\partial M_{ij}}{\partial \omega_{kl}} & \frac{\partial M_{ij}}{\partial \lambda_m} \end{bmatrix} \right|, \quad (2.77)$$

where the 2^{-n} comes from the quotient by $\{\pm 1\}^n$ and $1 \leq i < j \leq n$, $1 \leq k < l \leq n$ and $1 \leq m \leq n$. Because the Lebesgue measure is invariant under conjugation of M by an orthogonal matrix, we can evaluate the derivatives at $\omega = 0$ and find $dM_{ij} = d\omega_{ij}(\lambda_i - \lambda_j) + d\lambda_i \delta_{ij}$. Therefore the Jacobian matrix is diagonal⁸, thus $\mathcal{J}(\lambda, 0) = \prod_{1 \leq i < j \leq n} |\lambda_j - \lambda_i|$.

This can be repeated for Hermitian matrices, giving $\mathcal{J}(\lambda, 0) = \prod_{1 \leq i < j \leq n} |\lambda_j - \lambda_i|^2$.

This is where we introduce the Haar measure again, for these results have to hold for any point in \mathcal{S}_n or \mathcal{H}_n , not only around I_n . When $d\mu(O)$ is the Haar measure on $\mathcal{O}_n / \{\pm 1\}^n$, we get for constants $c_{\beta, n} > 0$:

$$dM = c_{\beta, n} d\mu(O) \prod_{i=1}^n \lambda_i \prod_{1 \leq i < j \leq n} |\lambda_j - \lambda_i|^\beta. \quad (2.78)$$

As a result, summarizing, if M is a random symmetric or hermitian matrix whose probability density function of entries is $dMF(M)$ and f is invariant under conjugation with an orthogonal or unitary

⁷The Lebesgue measure of the complement of the open dense subset $(\mathcal{S}_n)_\Delta$ of \mathcal{S}_n is 0, making this harmless for our goal.

⁸In the first block for $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$ the elements $(i, j) = (k, l)$ are $(\lambda_i - \lambda_j)$, and in the second block they are 1.

matrix, then $F(M)$ is actually a function $f(\lambda_1, \dots, \lambda_n)$ of eigenvalues only, and the joint probability density function of the eigenvalues of M is proportional to:

$$C_{n,\beta}^{-1} \prod_{1 \leq i < j \leq n} |\Delta(\lambda_1, \dots, \lambda_n)|^\beta f(\lambda_1, \dots, \lambda_n); \quad \Delta(\lambda_1, \dots, \lambda_n) = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i), \quad (2.79)$$

and $C_{n,\beta}$ such that the integral of it against the Lebesgue measure over \mathbb{R}^n evaluates to 1. Because of the factor Δ^β the probability that two eigenvalues are close to each other is small, so they seem to usually repel each other.

In the next section we will look at a more analytical way to show this same property.

2.4.1 Analytical Method

We already know that the eigenvalue maps $A \mapsto \lambda_i^A$ are Lipschitz continuous on Hermitian matrices from previous sections. We can, as shown, obtain better regularity when eigenvalues are not repeated. This actually seems to be rare⁹, and we define a **simple spectrum** if it has no repeated eigenvalues.

We want to see how far the spectrum of A can 'move' due to perturbations that could shift the eigenvalues. To do this we first observe that when A has a simple spectrum, the zeros of its characteristic polynomial ($\lambda \mapsto \det(A - \lambda I)$) are thus simple (the polynomial has nonzero derivatives at those zeros). Using the inverse function theorem¹⁰, we see that $A \mapsto \lambda_i^A$ are smooth on the region where A has simple spectrum. Eigenvectors u_i^A are determined by $(A - \lambda_i^A I)u_i^A = 0$ and $u_i^{A*} u_i^A = 1$, which gives that $A \mapsto u_i^A$ is also smooth.

We of course do not know how to differentiate a complete set, the spectrum, so we work with the eigenvalues itself and take $\lambda_j = \lambda_j(t)$ of $A = A(t)$ where A varies smoothly with t .

We assume that the eigenvalues were distinct, and it is not unreasonable to then assume $\lambda_i(t)$ to be distinct. An application of the implicit function theorem allows one to make $\lambda_i(t)$ smooth in t as well, which can be done for the eigenvectors $u_i(t)$ as well.

Observe the basis of \mathbb{C}^n given by the eigenvectors $\{u_i\}_{1 \leq i \leq n}$, and let $\{w_i\}_{1 \leq i \leq n}$ be its dual basis, giving for all vectors v :

$$v = \sum_{j=1}^n (w_j^* v) u_j; \quad w_j^* u_k = \delta_{jk}. \quad (2.80)$$

Combining this with the eigenvector equations $Au_k = \lambda_k u_k$ we find and differentiate the following equations:

$$w_k^* A = \lambda_k w_k^* \Rightarrow \dot{A}u_k + A\dot{u}_k = \dot{\lambda}_k u_k + \lambda_k \dot{u}_k. \quad (2.81)$$

Taking inner products of this with the dual vector w_k we find $\dot{\lambda}_k = w_k^* \dot{A}u_k$, the first variation formula for the eigenvalues.

If A is normal, $\{u_i\}_{1 \leq i \leq n}$ can be made orthonormal, and the basis and its dual are identical. This then gives $|\dot{\lambda}_k| \leq \|\dot{A}\|_{op} = \sup_{x \in \mathbb{C}^n: \|x\|=1} |\dot{A}x|$, thus the change of each eigenvalue does not exceed the change of the matrix itself.

⁹This was used in the previous section to state that $(\mathcal{S}_n)_\Delta$ is open and dense in \mathcal{S}_n , but we can count dimensions finding that the space of Hermitian matrices with at least one repeated eigenvalue has codimension 3 in the space of all Hermitian matrices. This already suggests a certain repulsion takes place.

¹⁰If f is continuously differentiable with nonzero derivative at a , then it is invertible in a neighbourhood of a , the inverse is continuously differentiable and the derivative of the inverse at $b = f(a)$ equals $1/f'(a)$.

If we instead take the inner product with w_j for $j \neq k$, we obtain:

$$w_j^* \dot{A}u_k + (\lambda_j - \lambda_k)w_j^* \dot{u}_k = 0. \quad (2.82)$$

Using this with the fact that every vector can be written in terms of the basis and its dual, we get:

$$\dot{u}_k = \sum_{j \neq k} \frac{w_j^* \dot{A}u_k}{\lambda_k - \lambda_j} u_j + c_k u_k; \quad \dot{w}_k^* = \sum_{j \neq k} \frac{w_k^* \dot{A}u_j}{\lambda_k - \lambda_j} w_j^* - c_k w_k^*. \quad (2.83)$$

These can be used to find second variation formulas for the eigenvalues, deriving the first variation formula above:

$$\begin{aligned} \ddot{\lambda}_k &= \dot{w}_k^* \dot{A}u_k + w_k^* \ddot{A}u_k + w_k^* \dot{A}\dot{u}_k \\ \Rightarrow \ddot{\lambda}_k &= u_k^* \ddot{A}u_k + 2 \sum_{j \neq k} \frac{|u_k^* \dot{A}u_j|^2}{\lambda_k - \lambda_j}. \end{aligned} \quad (2.84)$$

The term at the right here can be interpreted as a sum of forces working on λ_k . The 'acceleration' of the matrix providing one of them, while the other eigenvalues provide a repulsive force.

In this chapter, we summarized many mathematical results that are of interest when considering priors in Lagrangians and random matrices. The concept of a prior and the choice for a least informative prior were introduced. The influence of symmetries in choosing these priors was shown. The concept of Concentration of Measure was introduced and a summary of the most important results until now was given. This seems to be a problem when working with priors in Lagrangians and Random Matrices. Lastly, general results about symmetric and hermitian random matrices were introduced. Most important of these is the concept of eigenvalue repulsion, which is another type of concentration taking place when decomposing a matrix. In the next chapter, we will provide a clearer discussion of our specific context and the goal and results of the research.

Chapter 3

Methods and Results

In the introduction we mentioned that we want to discern between the models for neutrino masses per statistical (Bayesian) methods. In the first chapter we then introduced the possible models describing neutrino masses and wanting to determine if they have Dirac or Majorana masses and if their hierarchy is Normal or Inverted, thus giving physics beyond the SM. In chapter 2 we showed mathematical concepts that play a role when we need to put priors on the matrix elements of the mass matrix from the Lagrangian to compare models.

In this chapter, we will first discuss the available data and the results of previous research on Bayesian model comparison and experimental results in neutrino physics. In the context thereof, we discuss our own methods and results afterwards.

3.1 Previous Results and Phenomenology

First we have to discuss previous results before introducing the specifics of our experiments. This will include some thoughts on experimental techniques when observing neutrino masses, which links back to Chapter 1, and it will involve prior comments and results on statistical comparison for models, linking back to Chapter 2.

3.1.1 Experimental and Phenomenological Details

The current data on neutrino observations have been presented and summarized quite often, for instance in [18] and [15]. This data often combines multiple sources of measurements, including solar, atmospheric, reactor and accelerator neutrino experiments.

Solar neutrinos are electron neutrinos that are produced in the pp and CNO chains in the sun by which the solar energy is generated. These oscillate into neutrinos of other flavors in neutrino oscillations, and we do not only receive the produced electron neutrinos on earth. Important experiments of this type include the chlorine experiment giving the first detection, SAGE, GALLEX/GNO (ended in 2003), (Super)-Kamiokande, SNO (Sudbury Neutrino Observatory) and Borexino. These are all used in the determination of the values presented below. Different experiments produced different results, which meant that the energy of the produced neutrinos had something to do with the rate of oscillation, as can be seen in Chapter 1.

Atmospheric neutrinos are created due to cosmic rays interacting with atmospheric elements

like nitrogen and oxygen in the Earth's atmosphere. These reactions produce pions and kaons that decay into electron and muon neutrinos and antineutrinos. These were first detected in 1960 by underground experiments in South Africa and the Kolar Gold Field experiment in India, but in our results the data from IceCube/DeepCore (an under-ice experiment on the south-pole) was used.

Reactor neutrinos are produced in nuclear reactors in beams with $E_\nu \sim \text{MeV}$, so electrons are the only charged leptons that can be produced in the neutrino interaction within the detector. If the neutrinos oscillated, this interaction would thus not take place, meaning these are disappearance experiments. Smaller values of Δm^2 can be observed due to the lower neutrino beam energy. Results from short or intermediate baselines from CHOOZ, Palo Verde and Double-Chooz was used, as well as long baseline data from KamLAND.

Accelerator neutrinos are mostly produced by pion and kaon decays, and contain both electron- and muon neutrinos and antineutrinos, and the final composition of the beam is determined by the decaying pions. These beams are similar in nature to the atmospheric neutrinos. The first experiment having such a long baseline, which is needed to observe the Δm_{ij}^2 involved, was K2K. Currently MINOS, T2K and NO ν A are used. Also shorter baselines can be used, but they can only observe bigger Δm_{ij}^2 values.

As already touched upon above, the different experiments result in the different mass splittings Δm_{ij}^2 , due to the energy levels of the neutrino rays involved. The energy is partly dependent on the process of production, but also dependent on the distance until the ray hits the detector. Therefore, solar and reactor experiments are connected to $\Delta m_{\odot}^2 = \Delta m_{21}^2$, the smaller of the two splittings, and atmospheric and accelerator experiments are connected to $\Delta m_{atm}^2 = \Delta m_{3i}^2$, where i is dependent on the hierarchy (normal or inverted). More specifications and references about the experimental values we use here can be found in [18].

The sum of neutrino masses is mostly constrained from Cosmological probes, including the Lyman-alpha forest power spectrum, as well as the cosmic microwave background power spectrum. Current constraints are mostly dominated by Planck data [12], [13], [37] and [35].

Because, as stated, the neutrino oscillations are the most readily available data, we can only observe Δm_{ij}^2 -values as well as mixing angles between the flavor eigenstates from experiments. These last we observe as values that we marginalize over, for they do not contribute to the underlying model structure that should discern between Dirac or Majorana masses. The values we take are[27]:

$$\begin{aligned}
 \Delta m_{21}^2 &= (7.50 \pm 0.18) \times 10^{-5} \text{ eV}^2 \\
 \Delta m_{31}^2 &= (2.524 \pm 0.04) \times 10^{-3} \text{ eV}^2 \text{ (Normal)} \\
 \Delta m_{32}^2 &= (-2.514 \pm 0.04) \times 10^{-3} \text{ eV}^2 \text{ (Inverted)} \\
 \sum_{H,M,L} m_\nu &< 0.12 \quad 95\% \text{ credible region.}
 \end{aligned} \tag{3.1}$$

In this chapter H, M and L will signify the High, Middle and Low mass neutrino, and 1, 2 and 3 will signify the three fixed mass eigenstates as their mixings of the charge eigenstates. This structure can be found in Figure 3.1, where we also specify to only use one of the Δm_{3i}^2 values for $i = 1, 2$ given the ordering we test for. The normal ordering is named as such, for there the lightest neutrino contains

the biggest contribution of the flavour eigenstate of the electron neutrino, and the electron is the lightest lepton.

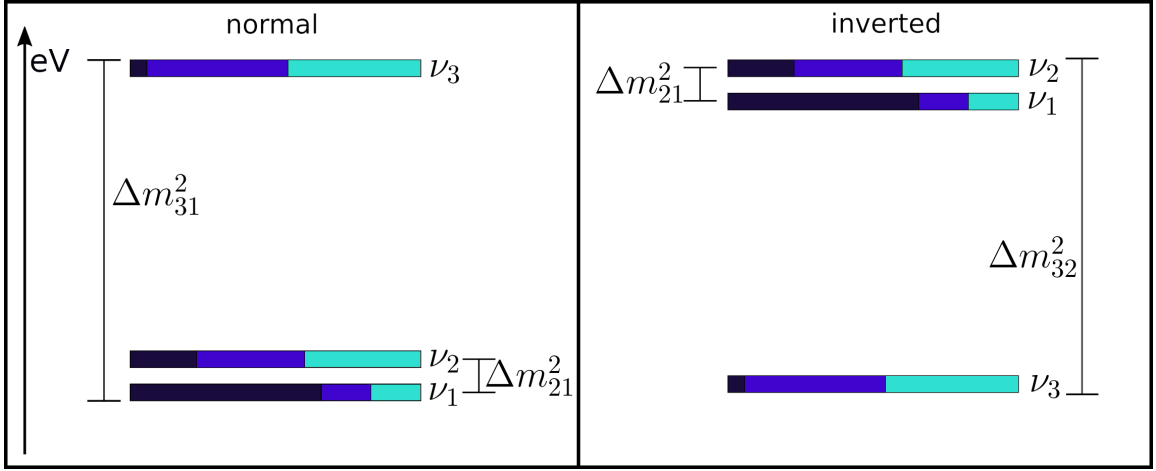


Figure 3.1: *The structure of the neutrino hierarchy used in this chapter, coinciding with the phenomenology from [39]. ν_1, ν_2 and ν_3 signify the mass eigenstates, and the colours signify the flavour content of each mass eigenstate. The light colour is ν_τ , the darker is ν_μ and black signifies ν_e . Δm_{ij}^2 signify the two predicted mass splittings between the eigenvalues present in the neutrino mass matrix.*

The standard deviations of these quantities should be constrained better within due time with the new neutrino mass experiments proposed and currently performed and some of the previously named experiments still operable. From CMB and galaxy clustering measurements as well as 3D weak lensing surveys the sum on the neutrino masses could be bounded more sharply. In [39] we find current laboratory experiments that are performed to further improve bounds on the other quantities.

It should be noted that the mass matrix from the Lagrangian is not directly observable as has been implicit in the preceding discussion. The observable masses that result in the mass splittings are the singular values of the mass matrix. Therefore, we introduce the singular value decomposition of a matrix S . If S is a complex $m \times n$ matrix, then the singular value decomposition of S exists and is a factorization of the form:

$$S = U\bar{S}V^\dagger, \quad (3.2)$$

where U and V are unitary matrices (and V^\dagger is the hermitian conjugate of V), respectively $m \times m$ and $n \times n$ with complex entries, and \bar{S} is a diagonal $m \times n$ matrix with non-negative real numbers on the diagonal (and zeros everywhere else, including rows or columns without a diagonal element when starting in the upper left corner). These diagonal elements are the singular values of S , and we list them in decreasing order. In our case, we introduce a 3×3 neutrino mass matrix M , and its singular value decomposition $M = USV^\dagger$. Of course, the discussion about random matrices in the preceding chapter has been focused mostly on eigenvalues, not singular values. Eigenvalue decompositions can only be performed for diagonalizable matrices, and it could happen that eigenvalues overlap (although it is of course not probable given the discussion about eigenvalue repulsion). It happens so that the singular values are the square roots of the non-zero eigenvalues of $M^\dagger M$ or MM^\dagger , for we can write:

$$M^\dagger M = VS^\dagger U^\dagger USV^\dagger = V(S^\dagger S)V^\dagger; MM^\dagger = USV^\dagger VS^\dagger U^\dagger = U(SS^\dagger)U^\dagger, \quad (3.3)$$

where we use that U and V are unitary. Because S only contains real values and M is square in our case, $S^\dagger S = S^2$, so $MM^\dagger U = US^2$, so each column u_i of U is an eigenvector of MM^\dagger with eigenvalue s_i^2 , and each column of V is in the same sense an eigenvector of $M^\dagger M$.

Another interesting property of S is that $Tr(MM^\dagger) = Tr(S^2)$, which can be seen from the equation above as well, for:

$$Tr(MM^\dagger) = Tr(US^2U^\dagger) = Tr(S^2U^\dagger U) = Tr(S^2) \Rightarrow \sum_{i,j=1}^3 |m_{ij}|^2 = \sum_{\nu=L,M,H} m_\nu^2, \quad (3.4)$$

where m_{ij} signify the matrix elements of $M = (m_{ij})_{i,j=1,2,3}$.

To further introduce the context of our research we present previous results on statistical analysis in the next paragraph.

3.1.2 Statistical Analysis

How to choose priors and perform statistical analysis for research in neutrino mass physics has been a hot topic for the past few years. The focus has been on priors to determine the hierarchy [24] and to find a good upper limit for the sum of the neutrino masses (refer to citations [9], [10], [24-31] from [32]). Many of these papers do discern between the three possible models, but do not touch upon considering evidences for the Dirac or Majorana (or Seesaw) models. They marginalize over these options and often just regard evidences for the Normal or Inverted hypothesis. Recently, an experimental paper was published which proposes an experimental search for distinction between Dirac and Majorana masses, but this should of course be able to be done statistically as well with current or future data as explained in the last section [5].

Using the concept of reference priors from Chapter 2, [27] constructed a least informative prior on the neutrino masses themselves, as found by transformation from a least informative prior on the two mass splittings Δm_{21}^2 and Δm_{3i}^2 and the sum of neutrino masses Σ . In this case, they argue asymptotic posterior normality and use Jeffreys prior, which is the product of the Jeffreys priors for all three parameters separate due to independence of experiments, to find a constant prior for all three parameters. This is then transformed to a prior for the matrix elements from the parametrization:

$$\begin{aligned} \Delta m_{21}^2 &= m_M^2 - m_L^2 \text{ Normal;} \\ &= m_H^2 - m_M^2 \text{ Inverted;} \\ \Delta m_{31}^2 &= m_H^2 - \frac{1}{2}(m_M^2 + m_L^2) \text{ Normal;} \\ \Delta m_{32}^2 &= \frac{1}{2}(m_M^2 + m_H^2) - m_L^2 \text{ Inverted;} \\ \Sigma &= m_H + m_M + m_L, \end{aligned} \quad (3.5)$$

but we here redefine these last two by not considering the 'mean' of the two closer masses, but actually defining them as the split between the highest and lowest mass, so:

$$\begin{aligned} \Delta m_{31}^2 &= m_H^2 - m_L^2 \text{ Normal;} \\ \Delta m_{32}^2 &= m_L^2 - m_H^2 \text{ Inverted,} \end{aligned} \quad (3.6)$$

The resulting prior was $\pi(m_L, m_M, m_H) = 4(m_L m_M + m_L m_H + m_M m_H)$, independent of hierarchy. In [44], the hierarchy was studied using, in their own words, a different choice and dimensionality

of the prior. They concluded that the normal ordering was highly preferred with odds of 42:1 when assuming a prior that is logarithmically flat. [43] proved that this result was dominated by choice of prior, for it a priori favored a hierarchical mass spectrum, which is easier in the normal than the inverted mass ordering. Additionally, [16] then argued that more applications of the logarithmic prior have resulted in published constraints that are way tighter than they are for other choices for priors, as well as highly dependent on the choice of prior boundaries. In [20] a method was shown to marginalize over possible cosmological models to get more general bounds on Σ , relaxing the upper bound to 0.35eV when not explicitly assuming a Λ CDM universe.

In [25] the so-called Anarchy hypothesis is proposed, where the view of the mass matrix having an ordered, regulated pattern was challenged. They proposed that mass matrices having random entries (though only real-valued) could just as well describe the current data. The only constraint was that the entries should all be of $\mathcal{O}(1)$, and the theory was further explored in [24]. This is the basis of the discussion in [32], who want to put priors on the individual matrix elements of the neutrino mass matrix to derive a prior distribution for Σ , which would depend on the physical origin of the neutrino mass. Therefore, this could be used to discern between models. They want to argue this way that the flat Jeffreys prior on Σ does not arise from base-independent priors on the individual matrix elements. They implement a probability on the matrix using the Haar measure for the entire Lagrangian term, and transform this such that the chosen prior probability distribution on the neutrinos (independent of that on the leptons) will be only dependent on singular values. Then they transform to a prior on Σ and the Δm_{ij}^2 , which they use to perform statistical analysis. Their analysis is performed using flat priors on the matrix elements (both the Real and Imaginary parts), and results in a preference for the Normal hierarchy with odds of 130:1, 19:1 and 470:1 in the Dirac, Majorana and Seesaw cases respectively.

We, however, expect that the concepts of concentration of measure and eigenvalue repulsion affected these results significantly, meaning that they are physically irrelevant. Furthermore, the odds found in [32] are as specific as in [44], and multiple sources confirm that the choice of priors dominated those results.

The problem is that, to study the physical origin of neutrino masses, one does need to put priors directly on the neutrino mass matrix, for only then the possible degeneracy can be shown. What we mean by this is the fact that Dirac models should then have 18 independent parameters, and Majorana models 12, so when simulations of the Dirac model show preference for dependent parameters, this would indicate a degeneracy in the model.

We use slightly updated values than [32], namely those from [18] (also used in [27]) to perform simulations to try to produce ensembles of matrices that are allowed within the given constraints. In the next section we explain the course we will take in our research.

3.2 Methods

We will try to analyze the prospects for statistical model determination of neutrino mass matrices when putting the priors directly on the matrix elements. This is needed when one wants to discern between the Dirac or Majorana mass models. We will demonstrate this by generating samples of random matrices for all the possible models, and find how their observable quantities are distributed. These include Σ , m_H , m_M , m_L , Δm_{21}^2 and Δm_{3i}^2 , where i depends on the hierarchy in question (1 for Normal, 2 for Inverted). The generation of these samples will be performed in `python` using `numpy.random`

methods¹, and the singular value decomposition will be taken with the `numpy.linalg` package. The priors we take for both the real and imaginary parts of the elements are independent and uniform, as argued by [32], in the range $[-w, w]$, with $w > 0$. For the Dirac model, we get 18 independent parameters (9 independent matrix entries each having an independent real and imaginary part), but in the Majorana case we expect the matrix to be symmetric, so we have 12 independent parameters (6 independent matrix entries having independent real and imaginary parts).

We alert to the influence of the concentration of measure phenomenon in this case, for the singular values are weighted sums of the individual matrix elements, and are thus sums of independent random variables as discussed in the section about concentration of measure. The other operations (taking the square root of an eigenvalue or the norm of a complex number) are all Lipschitz functions, so they do not negate concentration of measure effects. To check the importance of the specific decomposition in this phenomenon, we check the matrix elements for each base transformation matrix for the mass matrices generated.

We then analyze if the range of the chosen prior has an influence on the range of the observable quantities produced, to check for stability of our algorithm. It is then expected that when we have 18 random variables of order n , for example uniformly distributed on $[0, n]$, that the maximum value taken of the trace of the SVD, which is shown to be the sum of the squares of the matrix elements, would be of order $18n^2$. The sum of the squares should as well be independent under permutations of the matrix elements, and we try to find a specific matrix satisfying this theorized upper bound. We check if this actually occurs, or if the concentration of measure phenomenon indeed restricts the range of this newly assumed prior distribution on the sum of squared neutrino masses. It is quite predictable as stated above that the concentration phenomenon occurs, as is also seen in figure 2.1, where for $n = 50$, the range should be $[0, 50]$, but the actual histogram of the generated sample is largely constrained between $[10, 40]$ as shown.

The anarchy principle states that these random matrices have to produce the same results independent of basis, thus we also test the distribution of the matrix entries of the base transformation matrices produced from the SVD algorithm, to check for the stability of the concentration of measure phenomenon once more.

For prior ranges $[-w, w]$, where w can range between $[0.01, 1]$ to still produce a sample of matrices with mean Σ between $[0.06, 0.12]$ eV, as should hold as found in [15], we produce cornerplots showing the correlation between the measurable quantities and physical parameters. This is done for one specific $w = 0.02$, and marginalized over the choice of w .

To get evidences for the possible models from the known experimental values, we will need to assume one 'true' mass matrix, which we will randomly find by simulating a sample and picking one that is compatible with current data. This we want to do for the Normal and Inverted, and Dirac and Majorana cases separately. Then we use that mass matrix to explore the parameter space around it with the package `emcee` (when transforming the 18 parameters into an 18-dimensional vector) to find a sample of 'physical' mass matrices. These we can bin such that we can find possible correlations between the matrix elements. This way we should also be able to find forecast plots for the likelihoods

¹This package implements random number generators that have been tested a multitude of times and proven to be the best pseudorandom methods, so they should not affect our discussion significantly.

of Σ and the Δm_{ij}^2 from the sample of 'physical' mass matrices. This we want to do for different values of the standard deviations on the parameters constraining the parameter space. That way we can predict the impact of future research on the results of this method of model comparison.

The produced likelihoods could be compared with the priors gained from random generated matrices to see if they agree. Lastly, following the course of [27], it should be possible to plot posteriors for our 'fake data' and find evidences. These could be used to conclude if putting priors on individual matrix elements is a viable method of producing priors on the observable parameters. As is of course clear, we expect this analysis to result in the conclusion that the application of prior distributions for matrix elements produces a highly informative prior on the observable parameters.

The results of this research are shown in the next section.

3.3 Results and Discussion

As discussed in the last section, we start by finding the distribution of the observable quantities $\Sigma, m_H, m_M, m_L, \Delta m_{ij}^2$ for all possible models. The figures 3.2a and 3.2b below show the various histograms for the discussed quantities for $w = 0.23$, but this value should not affect the results, for those are independent of w , which would only alter the range of the x -axis.

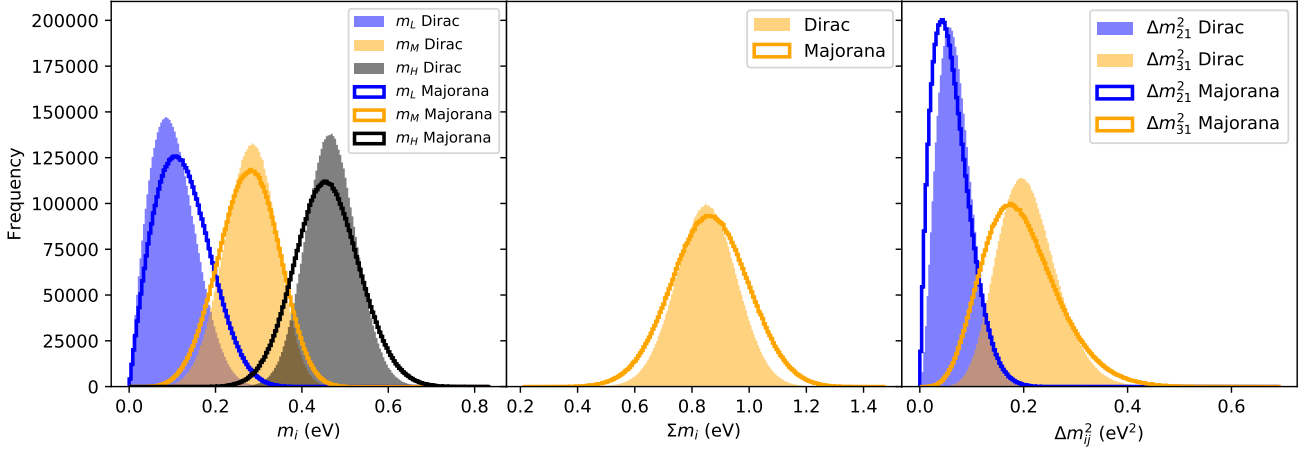
In these figures, it is already apparent what kind of predicted principles play a role in these calculations. The principle of Eigenvalue Repulsion is clearly demonstrated in the first panel, where the three (L, M and H) masses have three clearly separate peaks. We expect the overlap in the tails to be mostly due to matrices with overall shifted singular values, for in the third panel, we see that the squared mass splittings are often quite strong. Note that the experimental values for the squared mass splittings are of way smaller magnitudes than the peaks of these distributions. The concentration of measure is also apparent in the left panel, and even more clearly in the middle panel. It is even clear that the Dirac model is more strongly concentrated than the Majorana model, due to the difference in amount of independent parameters.

These results have been tested for stability under change of the w -parameter and change of prior choice. Scaling of the w -parameter only changes the magnitude of the matrix elements, and thus increases the range of Σ, m_i and Δm_{ij}^2 in approximately the same way as seen in Figure 2.1. A normal distribution with mean $\mu = 0$ and variance $\sigma = 0.23$ did not alter these results qualitatively, as did a log-linear distribution (as introduced in [32] as well).

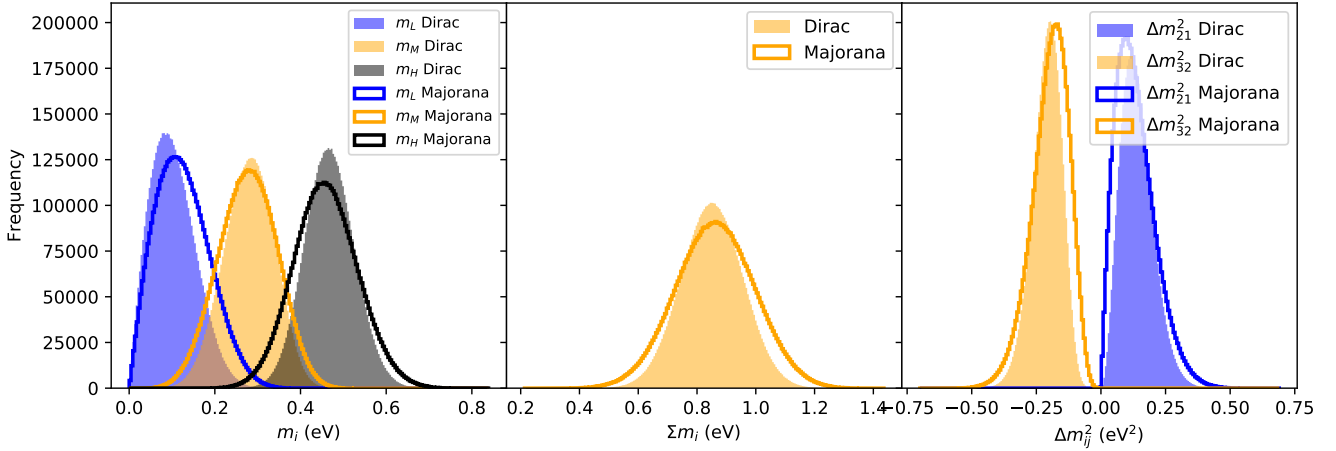
We then started by checking which types of matrices got closest to reaching the expected maximum of $\sum_{i,j=1,2,3} m_{ij}^2$, which is 18 for the Dirac model, and found it would need extreme entries, in the sense that all matrix elements should have real and imaginary parts either 0 or 1.

This we concluded because it seemed that only taking matrices for which $\Sigma > z\sqrt{18}$ (chosen value because 18 is the maximal $\sum_{i=L,M,H} m_i^2$, and we want to upper bound Σ) seemed to produce a distribution that peaked sharp at 0 and 1 for the matrix elements. It had a concentration of $\sum_{i,j=1,2,3} m_{ij}^2$ around 9.75, so not 18. Which is lucky, because the constant $1+i$ -matrix gave $Tr(S^2) = 18$ and $Tr(S) = \sqrt{18} \approx 4.24$, but has two eigenvalues equal to 0, which is not physical.

We did try to find a theoretical upper bound for Σ , but did not find a good method, thus tried finding it computationally by using random generation and again, extreme entries. This is possible due to the fact that the base transformation matrix entries were found to all be distributed via the same sampling distribution, so they should not be the main influence on the trace of S in $USV^\dagger = M$. The highest we were able to get for $w = 1$ without 0 singular values was around the order of $\Sigma \sim 5.4$ for



(a) Normal hierarchy, $\Delta m_{21}^2 = m_M^2 - m_L^2$ and $\Delta m_{31}^2 = m_H^2 - m_L^2$.



(b) Inverted hierarchy, $\Delta m_{21}^2 = m_H^2 - m_M^2$ and $\Delta m_{32}^2 = m_L^2 - m_H^2$.

Figure 3.2: Sampling distributions of m_H, m_M, m_L, Σ and Δm_{ij}^2 for both the Normal (a) and Inverted (b) hierarchies. Random matrices $M = (m_{ij})_{i,j=1,2,3}$ were generated with $\text{Re}(m_{ij}) \sim \text{Im}(m_{ij}) \sim \text{Unif}[-w, w]$, for $w = 0.23$ specifically. The lines show the results for the Majorana model, and the filled histograms those of the Dirac model. For visibility, these are not normalized. The sampling size was 5000000.

the matrix:

$$M = \begin{pmatrix} 1 & 0 & 1+i \\ i & 1+i & 1+i \\ 1+i & 1+i & 0 \end{pmatrix}, \quad (3.7)$$

independent of permutations of the matrix elements. This is of course not satisfying, so we would want a theoretical upper bound for this, which should be able to be derived in the same way as derivations in concentration of measure for singular values of Gaussian Ensembles of matrices, but then for Uniform Ensembles.

The SVD-algorithm used produced base transformation matrices with orthonormal columns for all sampled matrices. Also, as stated, the sampling distribution of the entries in U and V^\dagger were identical, and did not change qualitatively when using a normal or log-linear prior on the elements of M .

The next figures 3.3 and 3.4 show 2D histograms portraying the explained correlations between the measurable quantities and physical parameters. This only in the Dirac Normal case for $w = 0.02$ and marginalized over $w \in [0.01, 1]$ the other cases can be found in the Appendix.

In these figures we see of course the same principle as in figures 3.2 occur on the basis of eigenvalue repulsion and concentration of measure in the bottom panels. We also see the strong dependence of Δm_{ij}^2 on one of the singular values in the bottom middle panels. It is also apparent that the two mass splittings are not independent of each other in the middle panel, especially emphasizing the presence of an effect due to eigenvalue repulsion. This would be along the same lines of the advice given in [16], who warned for unexpected effects of careless choices for priors.

We then ran simulations to find possible candidate matrices to serve as 'true' mass matrices for each of the four models considered in this thesis. We did this by randomly generating mass matrices with real and imaginary parts distributed like $\text{Unif}[-w, w]$. This seemed to not give results in the Inverted hierarchy model, for this seemed to not produce any matrices compatible with the experimental values found for the observables in the Inverted model. This leads us to only consider the Normal hierarchy here. This observation is not unexpected, seeing as how [32] found a very conclusive disfavor of the inverted hierarchy by working with matrices this way. The ones we found for the Normal hierarchy are:

$$M_{Dirac} = \begin{pmatrix} -0.00217233 + 0.00661112i & 0.00942989 + 0.0202946i & 0.00455078 + 0.00099768i \\ -0.01536176 - 0.01971274i & -0.00170237 - 0.02307217i & 0.00775354 - 0.00563322i \\ -0.01839779 + 0.00501415i & -0.02282788 - 0.02031838i & 0.01247925 + 0.01020482i \end{pmatrix};$$

$$M_{Majorana} = \begin{pmatrix} -0.01215611 + 0.02033776i & -0.00509162 + 0.00667866i & 0.00496916 + 0.01937143i \\ -0.00509162 + 0.00667866i & -0.00731963 + 0.00765739i & 0.00801066 + 0.02054418i \\ 0.00496916 + 0.01937143i & 0.00801066 + 0.02054418i & 0.01010924 + 0.01413701i \end{pmatrix}, \quad (3.8)$$

which will thus be assumed to be the 'true' underlying mass matrix of the theory. Their observable parameters are equal to:

$$\begin{aligned} \Delta m_{21}^2 &= 7.55 \cdot 10^{-5} \text{ eV}^2; & \Delta m_{31}^2 &= 0.0026 \text{ eV}^2; & \Sigma &= 0.087 \text{ eV}; \\ \Delta m_{21}^2 &= 7.55 \cdot 10^{-5} \text{ eV}^2; & \Delta m_{31}^2 &= 0.0025 \text{ eV}^2; & \Sigma &= 0.074 \text{ eV}. \end{aligned} \quad (3.9)$$

We can produce prospects for the likelihoods with these values, where we take the current values of the standard deviation σ on these parameters, and produce a set of possible values of σ for when future research is performed. These are shown in A.7.

We then tried constructing an MCMC-algorithm (Markov Chain Monte Carlo) for exploring the

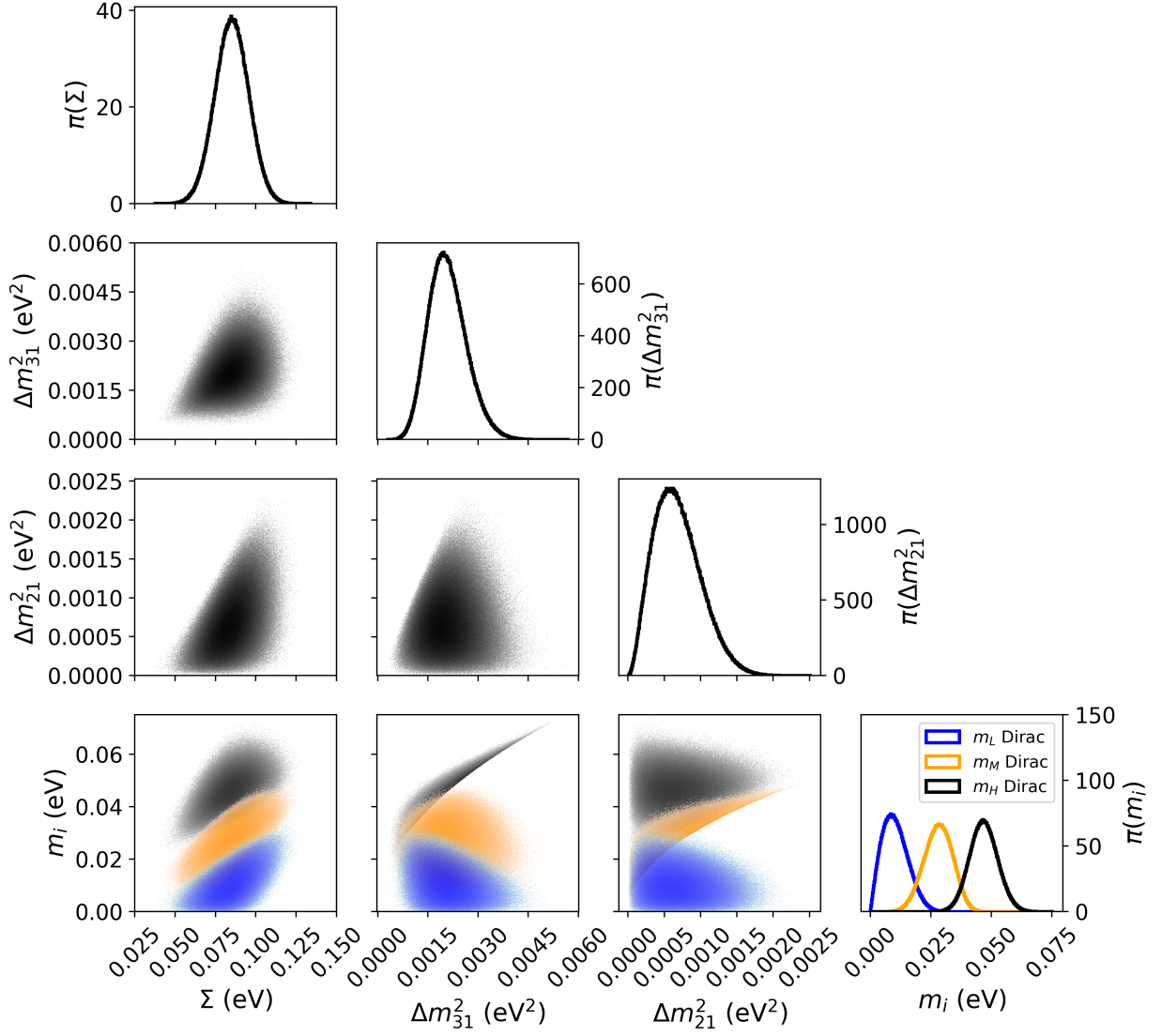


Figure 3.3: Sampling distributions of m_H, m_M, m_L, Σ and Δm_{ij}^2 for the Normal Dirac model for $w = 0.02$ and 2000000 sampled matrices. The curves show the sampling distributions of the single parameters.

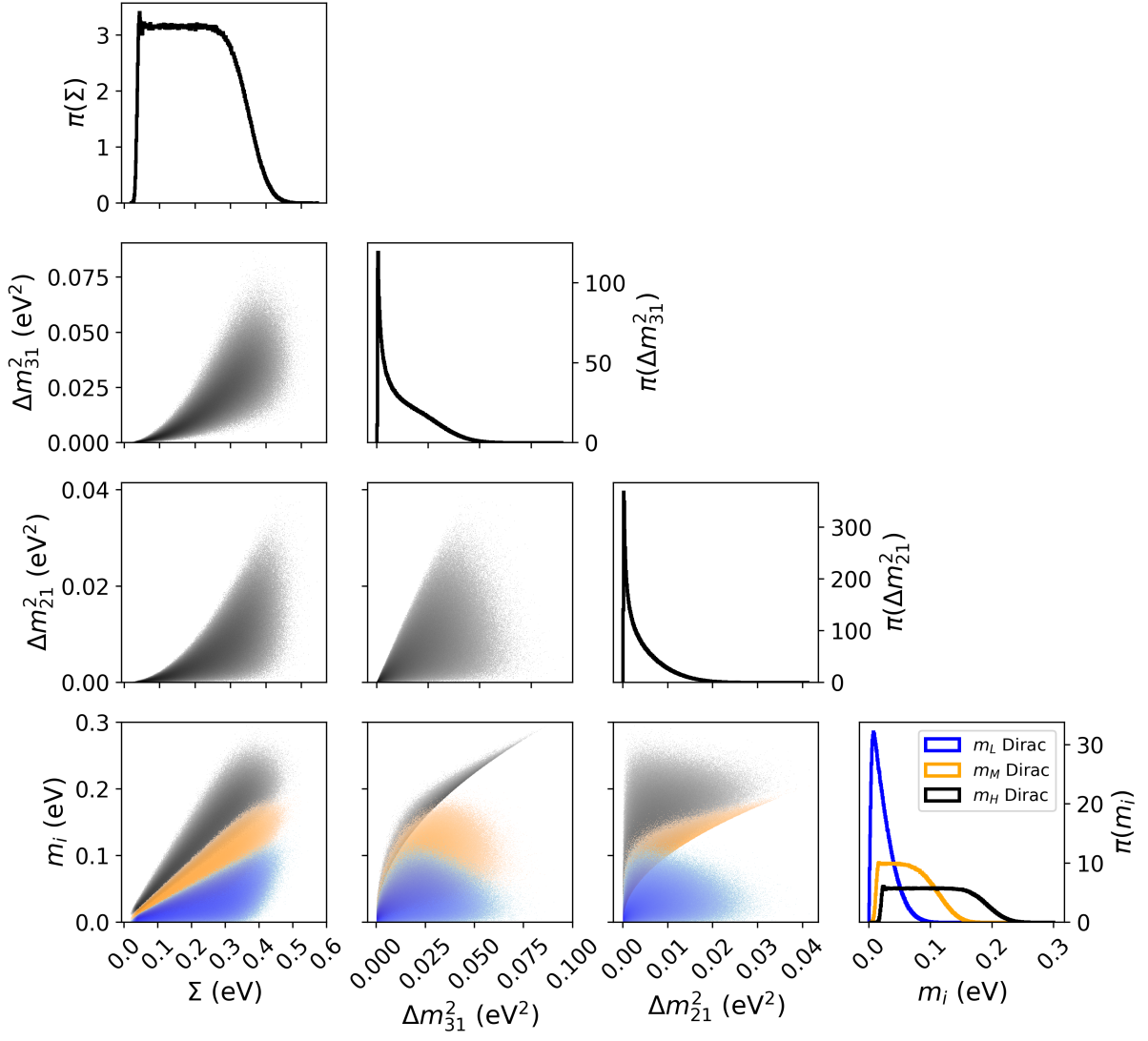


Figure 3.4: Sampling distributions of m_H, m_M, m_L, Σ and Δm_{ij}^2 for the Normal Dirac model for $w \in [0.01, 1]$ and 150000 sampled matrices for each w with steps of 0.003. The curves show the sampling distributions of the single parameters.

parameter space around the 'true' mass matrices, and thus construct sampling priors for the matrix elements, and check for possible correlations between the elements. We however encountered technical problems with this process, and were not able to finish it. These included the occurrence of blobs within the parameter space, most likely due to local minima from which the algorithm could not escape. Also the correlation length of the chains could not be calculated by `emcee` itself, and it seemed to not be able to escape a 'burnin' period. We implemented a prior constraining all real and imaginary parts to $[-.2, .2]$, because we do not want to constrain too tightly, but saw that most viable mass matrices were within this range. We calculate a Gaussian likelihood for the Δm_{ij}^2 and their experimental σ 's, and for Σ a truncated one between $[0.06, 1]$. We use a Gaussian Move recipe for determining new steps for the walker, for stretch moves do not work in such high dimensional parameter spaces. Another indication of problems in the 'black box' of the `emcee` sampler was that the acceptance fraction seemed to be (and stay, even for high numbers) dependent on the number of steps taken.

This entire discussion is based on the assumption that the Λ CDM model holds for our universe, for that assumption produces the used upper bounds on the sum of neutrino masses. This model need not be true in reality however, but it is currently the most well-tested and accepted model. We claimed that priors heavily dependent on boundary values are not preferred, and when future experimental research produces sharper bounds, these could be considered to again become useful. Of course an alteration of bounds does not have impact on the formulaic structure of the non-informative Jeffrey's prior, and only restricts the parameter space. Therefore, it is still preferable to have a prior heavily dependent on bounds, for we will never exactly know the true underlying bound, thus our experimental uncertainties can play a big role.

We did not discuss the Seesaw model in this thesis, for it would be a generalization of the methods produced to even more parameters, for M_D would have to be a Dirac mass matrix, and M_R a Majorana one, so 18 and 12 independent parameters respectively. So a working method here would be simple to extend, although working with that many parameters could result in degeneracies, for we only consider three observable parameters, for the discussion was independent of mixing angles.

Additionally, we did not discuss the Inverted model 'true' mass matrix, due to the fact that we could not find a randomly generated one fitting the observed parameters that would have to hold for this model. Looking at the cornerplots in figures A.1 and A.5 and figure 3.2b, this could be explained by the eigenvalue splitting prohibiting the combination of Δm_{\odot}^2 and Δm_{atm}^2 we observe. In the cornerplots we see in the 2D-histogram of these a big prohibited area, and in 3.2b we see the distribution being very thin at 0 for both, while both Δm_{ij}^2 should be of the order of 10^{-3} and 10^{-5} respectively.

In this Chapter we presented a summary of previous research performed in the field of neutrino mass experiments, and the practice of applying Bayesian statistical methods for model comparison in the field. We then presented our methods and hypotheses, and the actual results.

Conclusion

In this thesis we hypothesized how mathematical concepts known as the concentration of measure phenomenon and eigenvalue splitting can significantly alter the results of Bayesian analysis, especially in the context of neutrino mass physics.

To argue this, in Chapter 1 we described how symmetry principles and groups can be used in constructing Lagrangians, and how mass terms for neutrinos enter in physics beyond the SM. The most important results here were the general forms of the Dirac (3×3 complex matrix, 18 free parameters), Majorana (3×3 complex symmetric matrix, 12 free parameters) and Seesaw models (combination of both) for the mass matrix, and the distinction between the Normal and Inverted models ($m_3 \gg m_2 > m_1$ or $m_2 > m_1 \gg m_3$). These were the results in the last section.

In Chapter 2 we introduced the principles of least informative priors, eigenvalue splitting and concentration of measure, and showed that they play a role in random matrices. The most important results here are the construction of the Jeffrey's prior and the Reference prior from symmetries of the theory, which implies the importance of knowing the symmetries of a Lagrangian which were discussed in Chapter 1. Furthermore, the demonstration of the principle of concentration of measure for Lipschitz functions in the case of singular values of random matrices (examples throughout Chapter 2) are of importance for the analysis of neutrino physics. Lastly, the illustration of eigenvalue repulsion shows how this could play a role in the huge odds found in [32] for comparing the Normal and Inverted models.

In the third chapter, we showed general sampling distributions for the observables in neutrino mass matrix theory, and highlighted where the principles from Chapter 2 influenced the distributions. This should also be seen as a general warning when working closer to the model with priors, and not directly putting priors on the observables.

We presented the fact that our methodology was not able to find a 'true' mass matrix for the Inverted hierarchy, and predict this to be due to eigenvalue splitting. We state the implementation of MCMC that we used and how we found this to not work ideally yet.

Limitations of our study were pointed out at the end of Chapter 3, and future research should work to get around these.

We would advise further research to try and perfect the MCMC-analysis we introduced, to be able to follow up with prospect plots of the prior, which can be compared to the prospect likelihoods we show in figure A.7. Then the evidences for the models can be calculated, and the Seesaw model could also be considered.

To continue along this path, it would be instructive if a more theoretical proof can be found that the 'true' mass matrix for the inverted hierarchy models can not be found due to eigenvalue splitting, or to find an actual 'true' mass matrix.

It would help for this cause to create a more general theory of random matrices for uniform priors on

the matrix elements in stead of Gaussian priors as is done in the Gaussian ensembles introduced. A second path we advise to explore is to try and find the Jeffrey's prior on the matrix elements (both real and imaginary parts separately) themselves and repeat the steps from [27] for comparison of these two methods. Important for doing this is the discussion about symmetries in this thesis, linking Chapters 1 and 2 together.

Furthermore we of course advise to continue with experiments to constrain the observable parameters more tightly, for it would lead to more conclusive statistical answers, especially when working with as much independent parameters in our model as we do here.

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I would like to thank my parents for their support, and my father especially for reading through subjects almost completely unknown to him to point out spelling errors and inconsistencies.

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Appendix A

Additional Figures and Derivations

We present, for the corresponding chapters or section, the references to the appendix here.

A.1 Chapter 1, Gauge Symmetry

The structure of the Lagrangian can also be based on gauge symmetries, which are different from global symmetries discussed before. A gauge transformation is a transformation leaving the Lagrangian invariant just like a symmetry transformation, so it again does not change physics¹. Symmetry transformations do not always leave the solutions to the equations of motion invariant, while gauge symmetries relate identical states which describe exactly the same physics. Another difference is that gauge symmetries can depend on the exact coordinates, and can thus be interpreted as local symmetries, whereas the symmetry transformations produced global symmetries. We will first introduce some examples for this concept.

Example A.1.1. Consider the Lagrangian with a Mexican hat potential:

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi + \mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2, \quad (\text{A.1})$$

which has a ring of minima, namely where $|\phi|^2 = \mu^2/(2\lambda)$, meaning that any constant ϕ with the given modulus is a solution to the resulting equation of motion:

$$\square \phi - \phi^\dagger (\mu^2 - 2\lambda \phi^\dagger \phi) = 0. \quad (\text{A.2})$$

These are not invariant under $U(1)$ phase rotations, which does form a symmetry of the Lagrangian. Here $U(n)$ is the **unitary group**, the group of $n \times n$ unitary matrices. $U(1)$ specifically corresponds to all complex numbers with absolute value equal to 1.

It is however apparent that any solution of the equations of motion will be mapped onto another under such a transformation. ◀

Example A.1.2. Consider a planet moving around a star. This system is invariant under spatial rotations around said star, the symmetries of the $SO(3)$ group ($SO(n)$ is the **special orthogonal group**, containing $n \times n$ orthogonal matrices with determinant 1, also called the rotation group). There are thus three conserved charges, for this group is three-dimensional, corresponding to the three

¹For example for scattering experiments at colliders this statement translates into 'transformed initial states lead to transformed final states', for transforming before or after the experiment should make no difference in the result.

components of angular momentum. The solutions are ellipses in a plane, so they are not invariant under the elements of $SO(3)$. Not even rotations in the plane itself leave the solution invariant. But still all these transformations result in other solutions of the equations of motion. ◀

Example A.1.3. The most commonly known example of a theory that has gauge symmetries is Electrodynamics, where solutions for Maxwell's equations are determined up to a Gauge². When using the formulation with \vec{E} , \vec{B} , ρ and \vec{j} , thus writing the Maxwell equations:

$$\begin{aligned}\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} &= 0; & \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} &= \vec{j}; & \vec{\nabla} \cdot \vec{E} &= \rho\end{aligned}\tag{A.3}$$

The first two can be solved in terms of potentials ϕ and \vec{A} , when defining:

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}; \quad \vec{B} = \vec{\nabla} \times \vec{A}.\tag{A.4}$$

In doing this, we reduce the six components of \vec{E} and \vec{B} to four independent components in ϕ and \vec{A} . Now we find that the correspondence between these potentials and the physical fields is not unique, for a transformation of the form:

$$\vec{A}' = \vec{A} + \vec{\nabla}\chi; \quad \phi' = \phi - \frac{\partial \chi}{\partial t},\tag{A.5}$$

where χ is a scalar field, allows \vec{A}' and ψ' to result in the same \vec{E} and \vec{B} . This is thus an example of a gauge transformation. It is a symmetry of the theory, but different from the global symmetries.

It is a local symmetry because χ can depend on space and time. Note that it also relates physically indiscernible configurations, since solutions of the equations of motion of \vec{E} and \vec{B} are invariant. In other words, the potentials ϕ and \vec{A} still contain too many degrees of freedom, and the gauge transformation quantifies the redundancy of the description in terms of these potentials. Physical states and observables have to be invariant under gauge transformations. ◀

The easiest way to come up with a gauge symmetry is to start from a global one, and demanding invariance of the Lagrangian under local transformations. Simply changing the transformation parameter to a function depending on spacetime is not enough. We often need to add another field, the gauge field, to be added for the Lagrangian to stay invariant. Again we present an illustrative example.

Example A.1.4. Recall the $U(1)$ symmetry of the Lagrangian for the free charged fermion field ψ :

$$\mathcal{L}_0 = \partial_\mu \bar{\psi} \partial^\mu \psi - m^2 \bar{\psi} \psi = \bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi,\tag{A.6}$$

and the transformation $\psi \rightarrow \psi' = e^{i\alpha} \psi$ given³. When $\alpha \rightarrow \alpha(x)$, the potential part is still invariant, but the kinetic part now produces derivatives of $\alpha(x)$, giving:

$$\delta \mathcal{L} = i\partial_\mu \alpha (\partial^\mu \bar{\psi} \psi - \bar{\psi} \partial^\mu \psi) = -\partial_\mu \alpha j^\mu,\tag{A.7}$$

²You could thus compare the practice of 'choosing/fixing' a Gauge as the choosing of a value for the integration constant when integrating an equation. It is determined by boundary conditions you yourself can impose on the physics by common sense or observational knowledge.

³For a clearer explanation of why we can write the Lagrangian as such, see [42] Chapter 10.2

To restore the invariance of the Lagrangian, we have to add the gauge field with a gauge transformation for the potentials $A^\mu = (\psi, \vec{A})$, just as in the case of electromagnetic potentials. To incorporate it more illustratively we try to write:

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x), \quad (\text{A.8})$$

which we combine into a **covariant derivative**:

$$\partial_\mu \psi \rightarrow D_\mu \psi = (\partial_\mu - ieA_\mu) \psi, \quad (\text{A.9})$$

so that:

$$\begin{aligned} D_\mu \psi \rightarrow (D_\mu \psi)' &= (\partial_\mu - ieA'_\mu) \psi' \\ &= \partial_\mu (e^{i\alpha(x)} \psi) - ie \left(A_\mu(x) + \frac{1}{e} \partial_\mu \alpha(x) \right) e^{i\alpha(x)} \psi = e^{i\alpha(x)} D_\mu \psi, \end{aligned} \quad (\text{A.10})$$

so we can construct an invariant Lagrangian as:

$$\mathcal{L} = \bar{\psi} (i\gamma^\mu D_\mu - m) \psi = \mathcal{L}_0 + e\bar{\psi} \gamma^\mu \psi A_\mu = \mathcal{L}_0 + \mathcal{L}_{int}, \quad (\text{A.11})$$

where we have thus introduced an interaction term, which is representative of interaction of the vector field A_μ with the electromagnetic current $j_\mu = e\bar{\psi} \gamma_\mu \psi$.

The vector field A_μ itself is not yet a dynamical field now, since it misses its own kinetic term in the Lagrangian. This can easily be added by thinking about the symmetries it should contain. It should be gauge invariant and contain derivatives up to second order, so we first check the commutator of two covariant derivatives:

$$\begin{aligned} F_{\mu\nu} &= \frac{1}{e} [D_\mu, D_\nu] = \frac{1}{e} ([\partial_\mu, \partial_\nu] - [\partial_\mu, ieA_\nu] - [ieA_\mu, \partial_\nu] - e^2 [A_\mu, A_\nu]) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu, \end{aligned} \quad (\text{A.12})$$

which looks exactly like the field strength tensor we know from electrodynamics. Note that:

$$\delta F_{\mu\nu} = \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu = \frac{1}{e} (\partial_\mu \partial_\nu - \partial_\nu - \partial_\mu) \alpha(x) = 0, \quad (\text{A.13})$$

so only the antisymmetry in μ and ν ensures the gauge invariance, thus we need to square this field strength tensor to find:

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (\text{A.14})$$

which makes A_μ become the photon field obeying Maxwell's equations. \blacktriangleleft

This is an example of the gauge principle, which is a recipe how to obtain an interaction term from a free Lagrangian which is symmetric with respect to a continuous symmetry.

A.2 Chapter 1, Neutrino Oscillations

Consider two flavor eigenstates as mix of two mass eigenstates:

$$\begin{pmatrix} \nu_e \\ \nu_\mu \end{pmatrix} = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix}. \quad (\text{A.15})$$

Now letting the system be in state $|\nu_e\rangle$ at time $t = 0$, we can find the probability $|\langle\nu_\mu|\nu_e(t)\rangle|^2$ of detection of a muon neutrino. This we do after assuming the mass states evolve as $|\nu_j(t)\rangle = e^{iE_j t}|\nu_j(0)\rangle$.

$$\begin{aligned}
|\nu_\mu\rangle &= -\sin(\theta)|\nu_1\rangle + \cos(\theta)|\nu_2\rangle \Rightarrow \langle\nu_\mu| = -\sin(\theta)\langle\nu_1| + \cos(\theta)\langle\nu_2|. \\
|\nu_e(t)\rangle &= \cos(\theta)e^{iE_1 t}|\nu_1(0)\rangle + \sin(\theta)e^{iE_2 t}|\nu_2(0)\rangle. \\
\langle\nu_\mu|\nu_e(t)\rangle &= -\sin\theta\cos\theta e^{iE_1 t}\langle\nu_1|\nu_1\rangle + \cos^2\theta e^{iE_1 t}\langle\nu_2|\nu_2\rangle - \sin^2\theta e^{iE_2 t}\langle\nu_1|\nu_2\rangle + \cos\theta\sin\theta e^{iE_2 t}\langle\nu_2|\nu_2\rangle \\
&= \cos(\theta)\sin(\theta)(e^{iE_2 t} - e^{iE_1 t}) = \frac{1}{2}\sin(2\theta)(e^{iE_2 t} - e^{iE_1 t}) \\
|\langle\nu_\mu|\nu_e(t)\rangle|^2 &= \frac{1}{4}\sin^2(2\theta)(e^{iE_2 t} - e^{iE_1 t})(e^{-iE_2 t} - e^{-iE_1 t}) \\
&= \frac{1}{4}\sin^2(2\theta)\left(e^{i(E_1-E_1)t} + e^{i(E_2-E_2)t} - e^{i(E_1-E_2)t} - e^{i(E_2-E_1)t}\right) \\
&= \frac{1}{4}\sin^2(2\theta)(2 - 2\cos[(E_1 - E_2)t]) \\
&= \frac{1}{2}\sin^2(2\theta)2\sin^2\left(\frac{E_1 - E_2}{2}t\right) = \sin^2(2\theta)\sin^2\left(\frac{E_1 - E_2}{2}t\right).
\end{aligned} \tag{A.16}$$

Therefore the probability $P_{e\rightarrow\mu}(t)$ of observing a muon neutrino at time t is:

$$P_{e\rightarrow\mu}(t) = |\langle\nu_\mu|\nu_e(t)\rangle|^2 = \sin^2(2\theta)\sin^2\left(\frac{E_1 - E_2}{2}t\right). \tag{A.17}$$

Now we can simply state the probability of observing an electron neutrino:

$$P_{e\rightarrow e}(t) = |\langle\nu_e|\nu_e(t)\rangle|^2 = 1 - P_{e\rightarrow\mu} = 1 - \sin^2(2\theta)\sin^2\left(\frac{E_1 - E_2}{2}t\right). \tag{A.18}$$

to shorten this expression somewhat, we can write, assuming both mass eigenstates to propagate with the same momentum:

$$E_i = \sqrt{p^2 + m_i^2} = p\sqrt{1 + \frac{m_i^2}{p^2}} \approx p + \frac{m_i^2}{2p} \Rightarrow E_1 - E_2 = \frac{m_1^2 - m_2^2}{2p} \approx \frac{\Delta m^2}{2E_\nu}. \tag{A.19}$$

This gives that:

$$\begin{aligned}
P_{e\rightarrow e} &= 1 - \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2}{4E}t\right) \\
P_{e\rightarrow\mu} &= \sin^2(2\theta)\sin^2\left(\frac{\Delta m^2}{4E}t\right).
\end{aligned} \tag{A.20}$$

Here we see how the Δm_{ij}^2 plays a role in oscillations, and why we can observe this quantity, while we can not observe the mass of the neutrinos directly.

We can define a quantity known as the **oscillation length** as the length of half a period (π), so defined such that:

$$P_{e\rightarrow\mu} = \sin^2(2\theta)\sin^2(\pi L/L_{osc}); \quad L_{osc} = \frac{4\pi E}{\Delta m^2}. \tag{A.21}$$

This entire discussion can be generalized for three eigenstates, resulting in the occurrence of two different squared mass splittings.

A.3 Chapter 3, Figures alike figure 3.3

In Chapter 3 we refer to the Appendix for cornerplots for the Inverted Dirac and Majorana and Normal Majorana models.

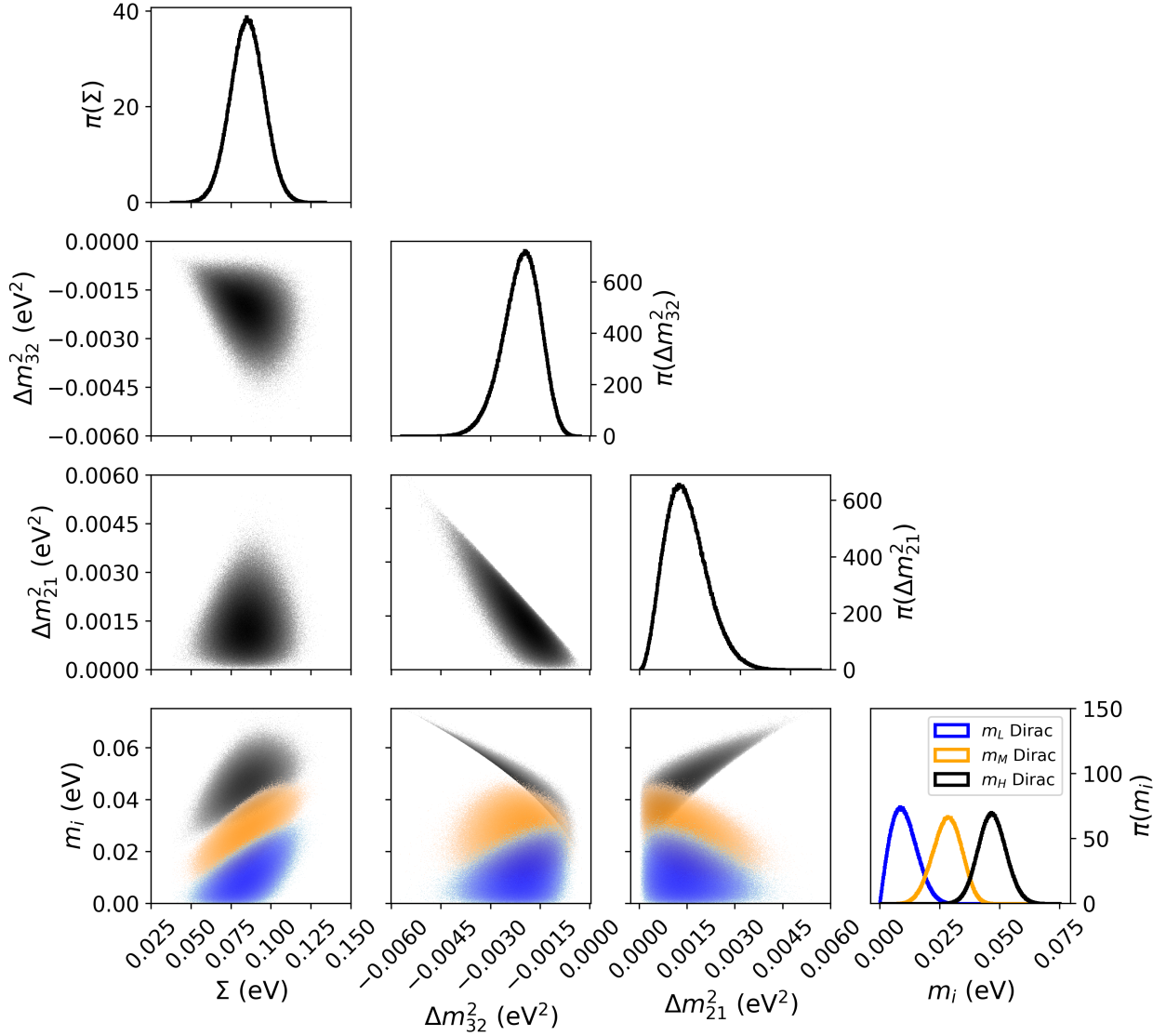


Figure A.1: Sampling distributions of m_H, m_M, m_L, Σ and Δm_{ij}^2 for the Inverted Dirac model for $w = 0.02$ and 2000000 sampled matrices.

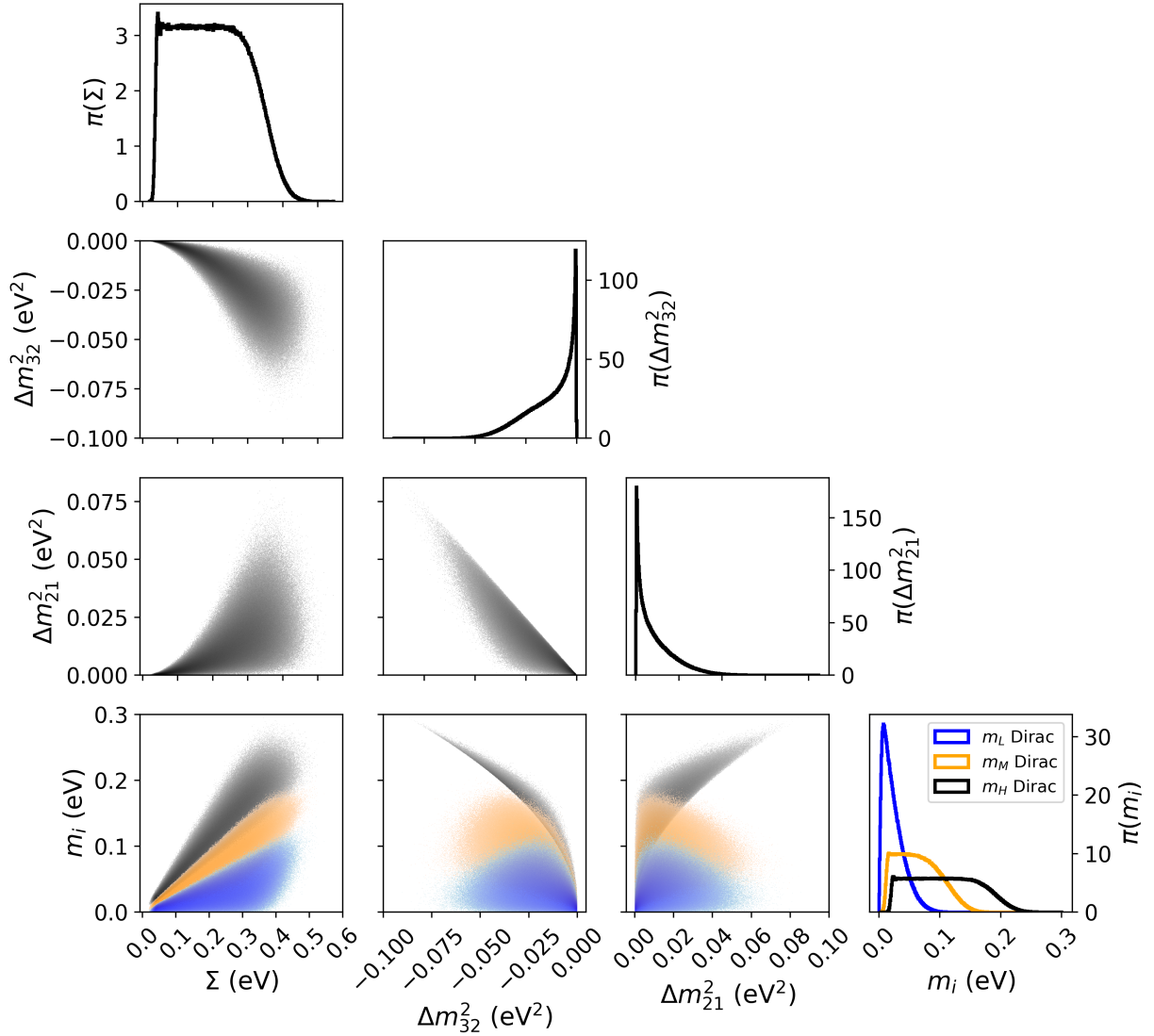


Figure A.2: Sampling distributions of m_H, m_M, m_L, Σ and Δm_{ij}^2 for the Inverted Dirac model for $w \in [0.01, 1]$ and 150000 sampled matrices for each w with steps of 0.003.

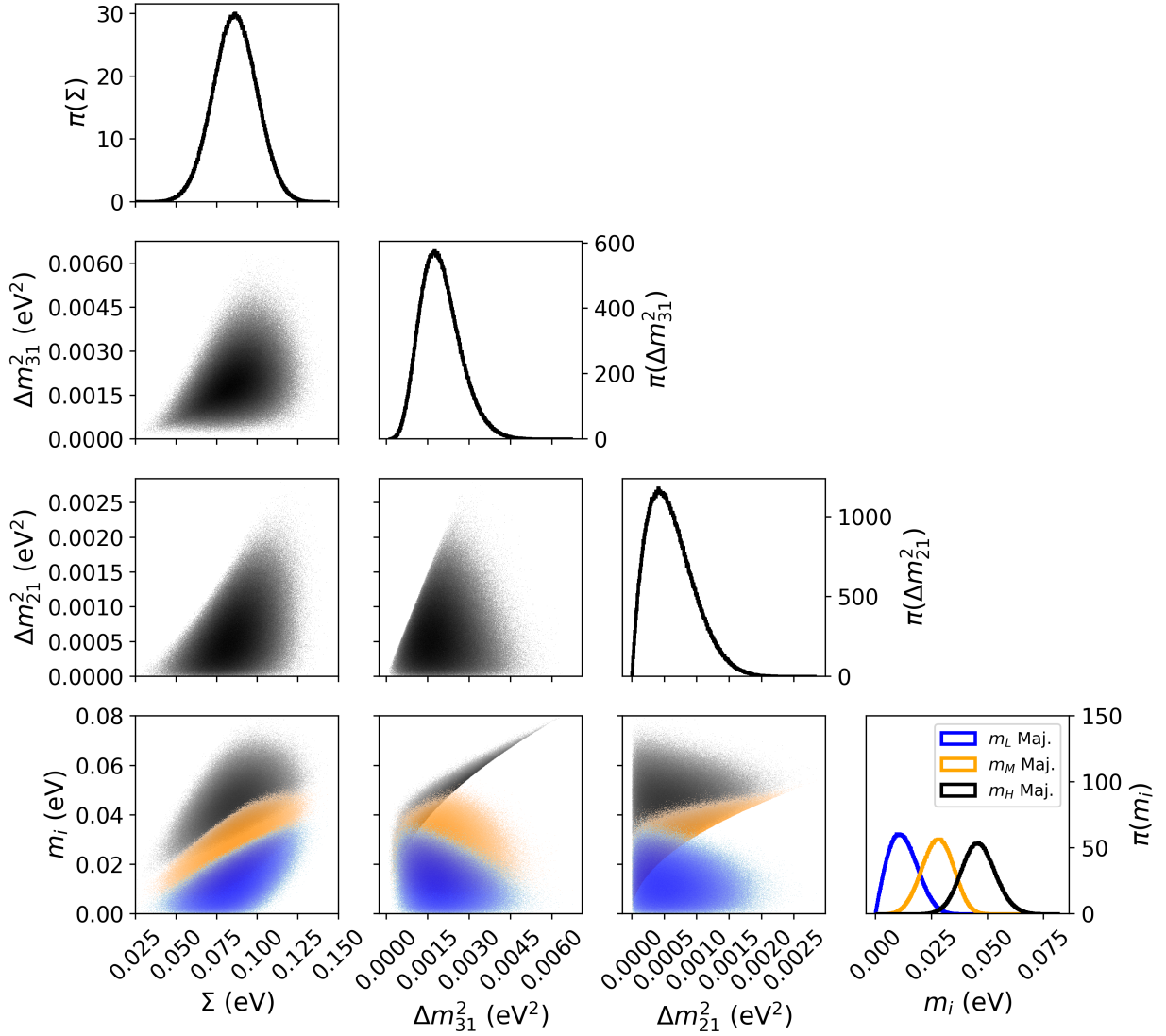


Figure A.3: Sampling distributions of m_H, m_M, m_L, Σ and Δm_{ij}^2 for the Normal Majorana model for $w = 0.02$ and 2000000 sampled matrices.

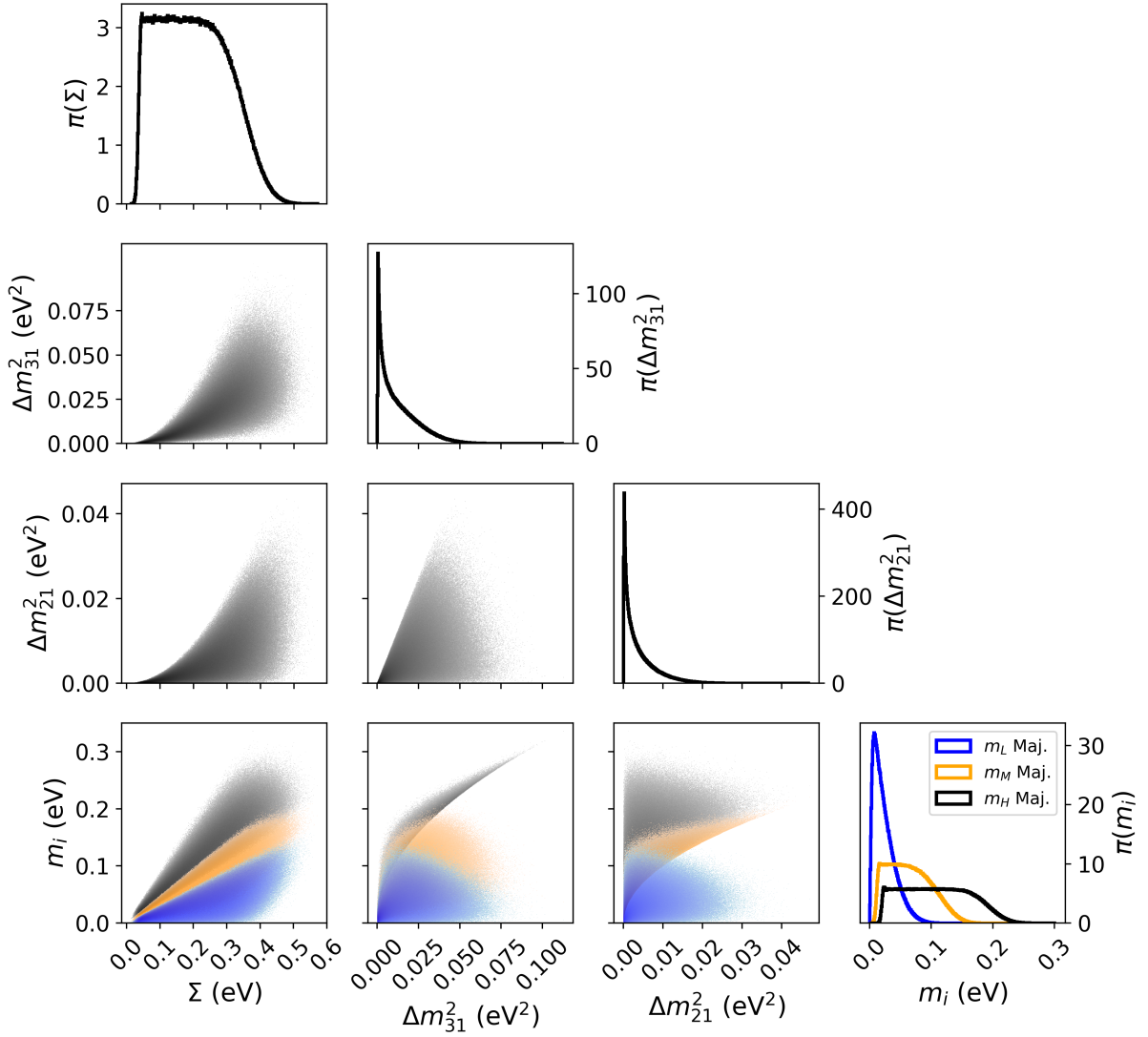


Figure A.4: Sampling distributions of m_H, m_M, m_L, Σ and Δm_{ij}^2 for the Normal Majorana model for $w \in [0.01, 1]$ and 150000 sampled matrices for each w with steps of 0.003.

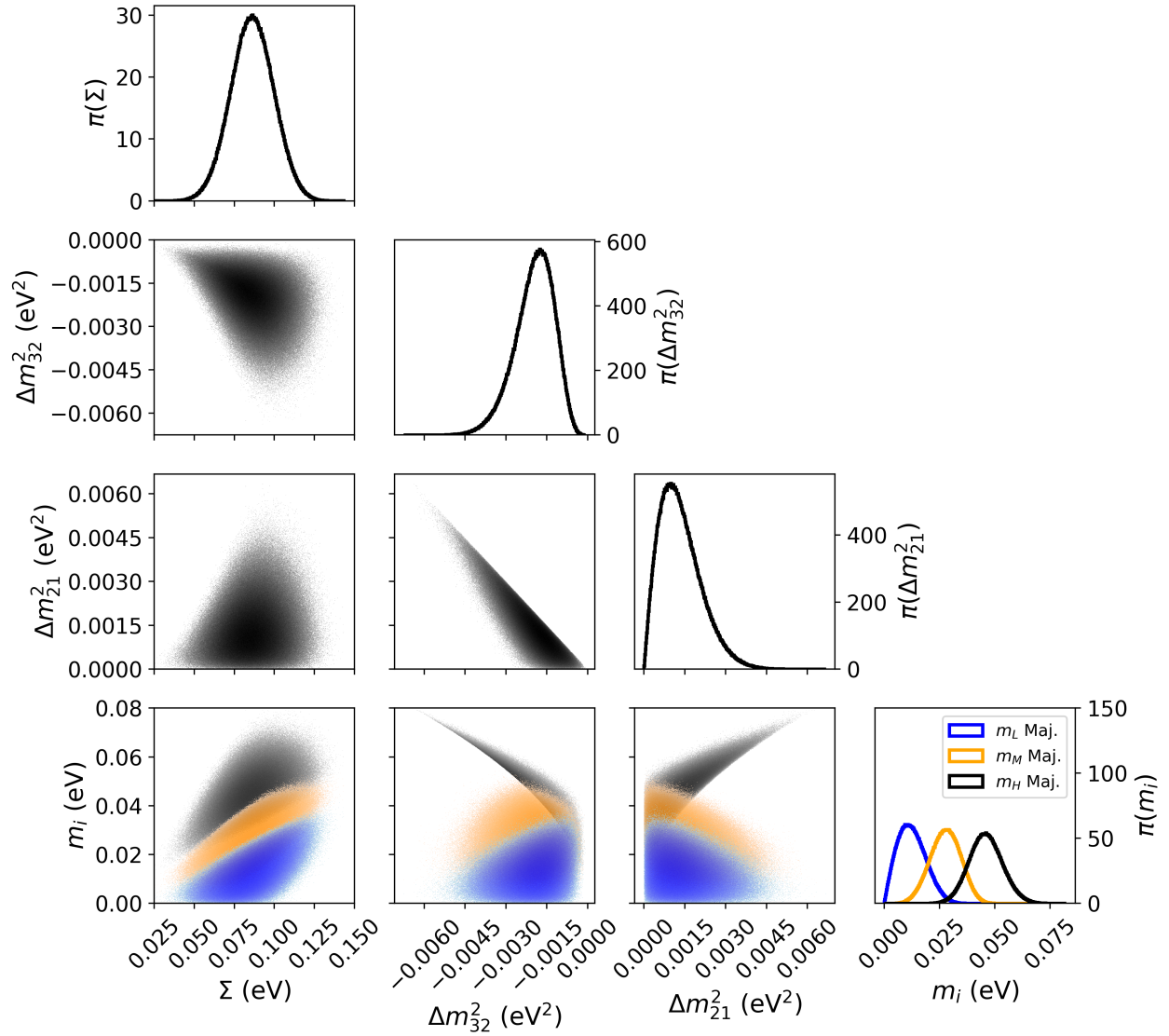


Figure A.5: Sampling distributions of m_H, m_M, m_L, Σ and Δm_{ij}^2 for the Inverted Majorana model for $w = 0.02$ and 2000000 sampled matrices.

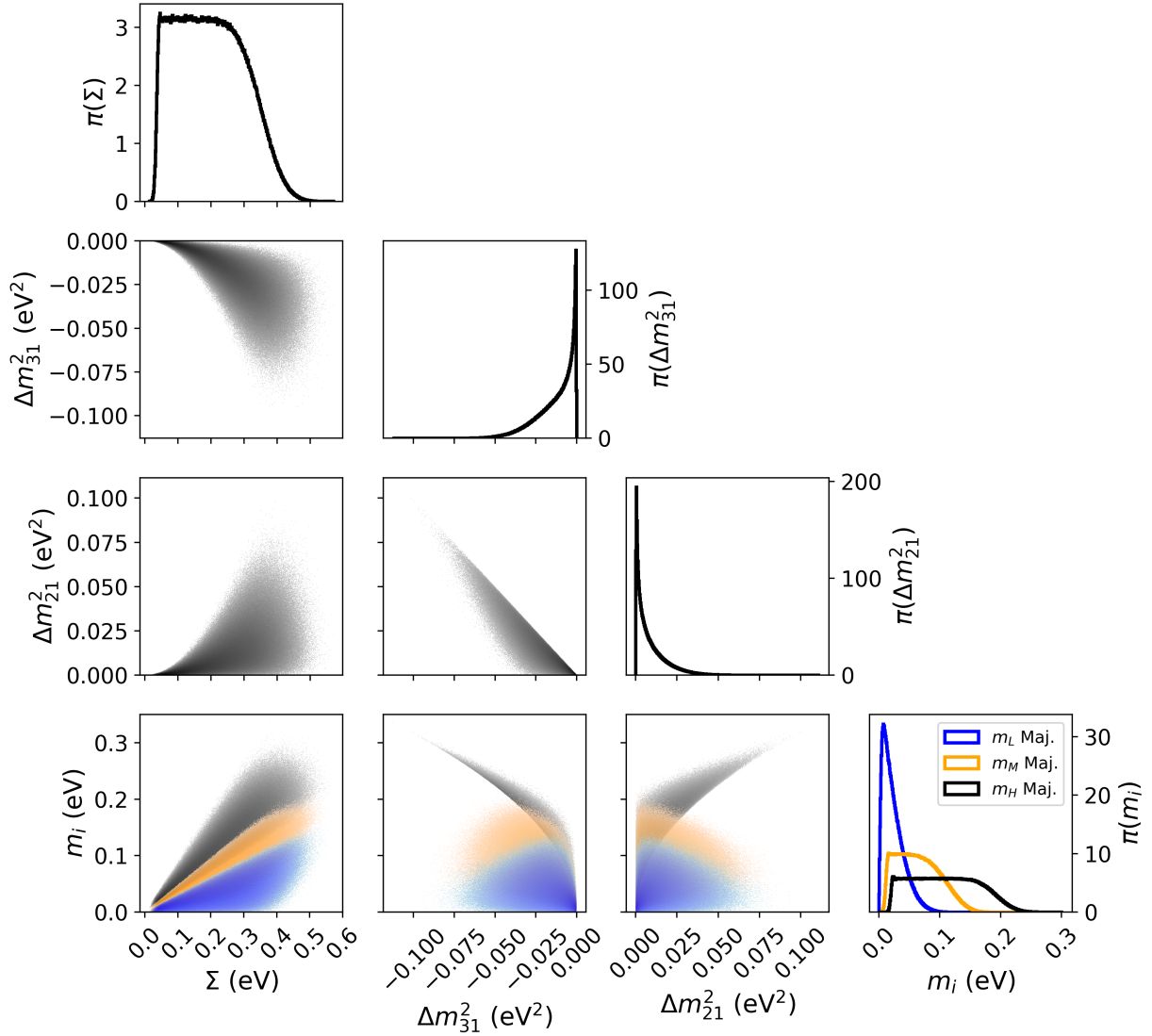


Figure A.6: Sampling distributions of m_H, m_M, m_L, Σ and Δm_{ij}^2 for the Inverted Majorana model for $w \in [0.01, 1]$ and 150000 sampled matrices for each w with steps of 0.003.

A.4 Chapter 3, Forecast likelihoods

We present two forecast likelihood plots for different values of the experimental standard deviation on the observable parameters.

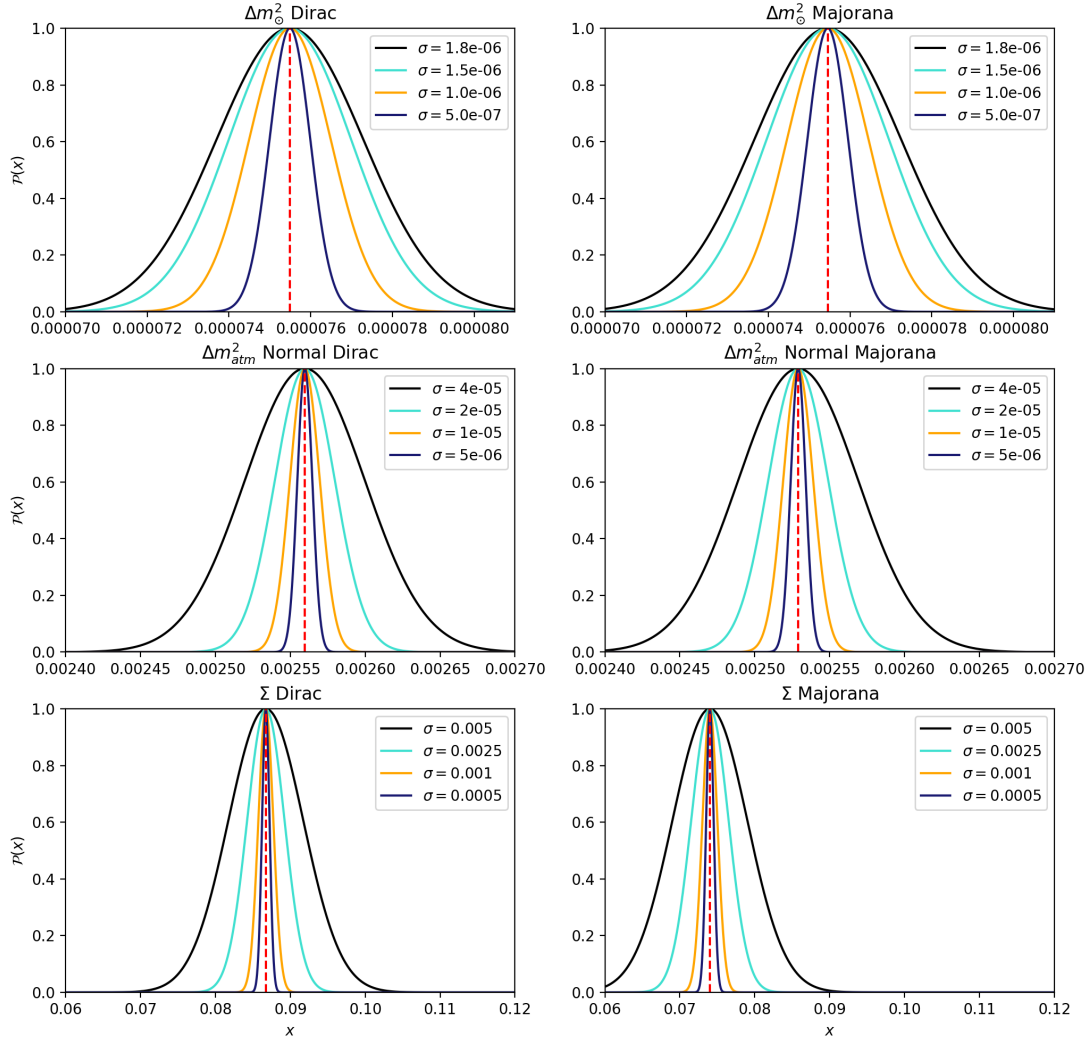


Figure A.7: Forecast likelihoods for the three observables ($\Delta m_{\odot}^2 = \Delta m_{21}^2$, $\Delta m_{atm}^2 = \Delta m_{31}^2$ and $\Sigma = \sum m_{\nu}$) in the normal model, for the 'true' mass matrices found in Chapter 3. The black lines portray current experimental standard deviations, and the others portray results for better experimental constraints. For Σ we use truncated normal distributions, and the fact that $\Sigma < 0.12$ is a 95% interval to find these sigmas.